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GENERAL MIXED MULTIVALUED MILDLY NONLINEAR VARIATIONAL INEQUALITIES IN H-SPACES

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Abstract - A general mixed multivalued mildly nonlinear variational inequality is considered. We establish an existence for (GMMMNVI) by replacing convexity assumptions with merely topological properties.

1. INTRODUCTION

In the years 1983-1985, Horvath [2] obtained minimax inequalities by replacing convexity assumptions with merely topological properties.

In this paper, we consider a general mixed multivalued mildly nonlinear variational inequality, introduced and studied by Siddiqi *et al* [3], and prove the existence of its solution without convexity.

Let X and Y be two real Banach spaces. Let $K \subset X$ be a nonempty closed convex subset of X, $g: K \to K$ be a continuous mapping, $T, A: K \to 2^{L(X,Y)}$ be the multivalued mappings, where L(X,Y) is the space of all linear continuous mappings from X into Y.

Consider the general mixed multivalued mildly nonlinear variational inequality (GMMMNVI). Find $u \in K$, $x \in T(u)$ and $y \in A(u)$ such that $g(u) \in K$ and

$$<x-y, g(v)-g(u)> + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } g(v) \in K, \qquad \ldots (1.1)$$

where $b: K \times K \to R$ is a nonlinear form.

2. EXISTENCE THEORY

First, we give definitions for the existence theorem for (GMMMNVI), that can be found in [1].

Definition 2.1. Let X be a topological space and $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X, indexed by the finite subsets of X.

A subset $D \subset X$ is called weakly H-convex if, for every finite subset $A \subset D$, it results that $\Gamma_A \cap D$ is nonempty and contractible. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space ;

A subset $K \subset X$ is called H-compact if there exists a compact, weakly H-convex set $D \subset X$ such that $K \cup A \subset D$ for every finite subset $A \subset X$;

A multifunction $G : X \to 2^X$, is called H-KKM mapping if

$$\Gamma_A \subset \bigcup_{x \in A} G(x), \text{ for every finite subset } A \subset X.$$

Lemma 2.1. [1]. Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to 2^X$, an H-KKM multifunction such that

(a) For each $x \in X$, $F(x)$ is compactly closed, that is $B \cap F(x)$ is closed in B, for every compact set $B \subset X$;

(b) There are a compact set $L \subset X$ and an H-compact set $K \subset X$ such that, for each weakly H-convex set D with $K \subset D \subset X$, we have

$$\bigcap_{x \in D} \{F(x) \cap D\} \subset L$$

Then

$$\bigcap_{x \in X} F(x) \neq \phi.$$

Now, we state and prove the main Theorem of this paper.

Theorem 2.1. Let $(X, \{\Gamma_k\})$ be an H-Banach space and Y be an ordered Banach space. Assume that

1^0. $T, A : X \to 2^{L(X,Y)}$ be the compact valued, continuous multivalued mappings ;

2^0. $g : X \to X$ is a continuous mapping ;

3^0. $b : X \times X \to R$ is a continuous, nonlinear form ;

4^0. $$\text{for each } v \in X,\ B_v = \left\{ \begin{array}{l} u \in X : \exists x \in T(v), y \in A(v) \text{ such that} \\ < x - y, g(v) - g(u) > + b(u, g(v)) - b(u, g(u)) < 0 \end{array} \right\}$$
is H-convex or empty;

5^0. there exists a compact set $L \subset X$ and an H-compact set $E \subset X$ such that for every weakly H-convex set D with $E \subset D \subset X$

$\{v \in D : \exists\, x \in T(v), y \in A(v)$ such that

$<x-y, g(v)-g(u)> + b(u,g(v)) - b(u,g(u)) \geq 0$, for all $u \in D\} \subset L$.

Then (GMMMNVI) is solvable.

Proof : Let

$$F(u)=\left\{\begin{array}{l} v\in X:\exists x\in T(v), y\in A(v) \text{ such that} \\ \qquad\qquad < x-y, g(v)-g(u) > + b(u,g(v)) - b(u,g(u)) \geq 0 \end{array}\right\}, \quad u\in X$$

First we prove that F is an H-KKM mapping and the conditions (a) and (b) of Lemma 2.1 hold.

Suppose that F is not an H-KKM mapping. Then there exists a finite subset $K\subset X$ such that $\Gamma_K \not\subset \bigcup_{u\in k} F(u)$. Thus there exists $z\in\Gamma_K$ such that

$$z\notin F(u), \text{ for all } u\in K,$$

that is

$$< x-y, g(z)-g(u) > + b(u,g(z)) - b(u,g(u)) < o, \text{ for all } u\in K, x\in T(z), y\in A(z).$$

By assumption 4^0, $K\subset B_Z$ and $\Gamma_K \subset B_Z$, since B_Z is H-convex. Therefore $z\in B_Z$, that is there exists $x\in T(z)$, $y\in A(z)$ such that

$$< x-y, g(z)-g(z) > + b(z,g(z)) - b(z,g(z)) < 0,$$

which is not possible. Thus $\Gamma_k \subset \bigcup_{u\in k} F(u)$ for every finite subset K of X, so that F is an H-KKM mapping.

Next, we prove that for every $u\in X$, F(u) is closed. Indeed, let $\{v_n\}$ be a sequence in F(u) such that $v_n \to v_o \in X$, $g(v_n)\to g(v_o)\in X$. Since $v_n \in F(u)$ for all n, there exists $x_n\in T(v)$ and $y_n \in A(v)$ such that

$$<x_n - y_n, g(v_n) - g(u) > + b(u, g(v_n)) - b(u,g(u)) \geq 0.$$

Since T(v) and A(v) are compact, without loss of generality, we can assume that there exists $x_0\in T(v)$ and $y_0\in A(v)$ such that $x_n\to x_0$ and $y_n\to y_0$. Now since $<.,.>$, $b(.,.)$ are continuous and $x_n\to x_0$, $y_n\to y_0$, $v_n\to v_0$, $g(v_n)\to g(v_0)$, we have

$$<x_n - y_n, g(v_n) - g(u) > + b(u, g(v_n)) - b(u,g(u))$$
$$\to < x_0 - y_0, g(v_0) - g(u) > + b(u,g(v_0)) - b(u,g(u)) \geq 0.$$

Therefore $v_o\in F(u)$ and so F(u) is closed for every $u\in X$, that is, the condition (a) of the Lemma 2.1 holds. It is easy to see that the assumption 5^0 of the Theorem 2.1 is same one with the condition (b) of the Lemma 2.1.

Then by Lemma 2.1

$$\bigcap_{u\in X} F(u)\neq\phi,$$

consequently, there exists $u_0\in X$, $x_0\in T(u_0)$, $y_0\in A(u_0)$ such that $g(u_0)\in X$ such that

$$< x_0 - y_0, g(v) - g(u_0) > + b(u_0, g(v)) - b(u_0, g(u_0)) \geq 0, \quad \text{for all } g(v) \in X.$$

This completes the proof.

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