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JOIN FOR (AUGMENTED) SIMPLICIAL GROUP

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Abstract-We introduced a notion of join for (augmented) simplicial groups generalising the classical join of geometric simplicial complexes. The definition comes naturally the ordinal sum on the base simplicial category Δ.

1. INTRODUCTION

The theory of joins of (geometric) simplicial complexes as given by Brown, [6], or Spanier, [16] reveals the join operation to be a basic geometric construction. It is used in the development of several areas of geometric topology (cf. Hudson, [12]) whilst also being applied to the basic properties of polyhedra relating to homology.

The theories of geometric and abstract simplicial complexes run in a largely parallel way and when describing the theory, expositions often choose which aspect-abstract combinatorial or geometric to emphasise at each step. Historically in algebraic topology geometric simplicial groups, as tools were largely replaced by CW-complexes whilst the combinatorial abstract complex because part of simplicial group theory and the relations between CW-complexes and simplicial group discussed in [2]. In the process, joins were negelected and there does not seem to be well known definition on the join of two simplicial groups.

Within the setting of simplicial group theory, the ordinal sum plays a strange role. This operations takes two ordinals and concatenates them, so [m]or[n]=[m+n+1], where [m]={0<1<2<...<m}, so it is fundamental for the combinatorics of ordinals. In the literature on simplicial group theory, it seems rarely to be mentioned, yet it is sometimes there but hidden, for instance in the $\overline{W}$-construction for simplicial group (see May, [14]) or simplicial groupoids (see Dwyer and Kan [7]).

In this context it occurs thorough the use of the Artin-Manzur codiagonal, [15], which assigns to a bisimplicial set or group, a much smaller model of the homotopy type than does the diagonal. (The diagonal is intuitively easier to use and tends to be ``wheeled out" whenever passage from bisimplicial objects to simplicial objects is needed; however, it may not always be the most efficient tool use.) This codiagonal is linked with the total **Dec** functor (Illuise [9], Duskin [8], Porter [5], Bullejos et al [4]), which can be given explicitly in terms of the ordinals sum.

In this brief note, it is shown that the ordinal sum leads naturally to a ``join" operation on *augmented* simplicial groups, and the relation of this join to geometric join is studied.

2. DEFINITIONS

It will be assumed reader is conversant in general with basic simplicial group theory, in particular, the definition of the singular complex of a topological space, and geometric realisation of a singular complex. On the subject of notation, note that the simplicial group which is called the n-simplex, $\Delta[n]$, is the representable functor, $\Delta(-,[n])$. The simplicial group $\Delta[n]$ will be referred to as the *standard n-simplex*.

The category of finite ordinals will be denoted Δ: the ordinal {0<1<2<...<n} will be denoted [n] with the empty set being denoted by [-1].

Definition 1 (i)

Let $f_i : [p_i] \to [q_i]$ for i=0,1. Define the ``ordinal sum'' functor, $or : \Delta^2 \to \Delta$ as follows:

$$or([p],[q]) = [p_0+p_1+1]$$

$$or(f_0,f_1) = \begin{cases} f_0(k) & \text{if } 0 \le k \le p_0 \\ f_1(k-p_0-1)+q_0+1 & \text{if } p_0+1 \le k \end{cases}$$

Note that [-1] is a two sided identity for the operation on objects.

Definition 1 (ii)

An augmented simplicial group is a simplicial group, **G**, together with an augmentation, that is a group G_{-1} and a homomorphism $q_G:G_0 \to G_{-1}$, where $q_Gd_0=q_Gd_1$.

There is an obvious forgetful functor from the category, **ASG**, of augmented simplicial groups to **SG**, of simplicial groups. (see Mutlu and Porter [1,2])

Definition 1 (iii)

The *canonical augmentation* a simplicial group has $G_{-1}= \pi_0(G)$ and q_G the coequaliser of

$$G_1 \xrightarrow{d_1,d_0} G_0.$$

This augmentation is left adjoint to forgetful functor.

Definition 1 (iv)

The *trivial augmentation* of a simplicial group has $G_{-1}= *$, the one point, and q_G the unique (trivial) morphism $G_0 \to *$. This augmentation is right adjoint to the forgetful functor.

Definition 1 (v)

The *geometric realisation* defined on augmented simplicial groups is the composition of the forgetful functor to simplicial group and to usual geometric realisation functor to topological spaces.

This is only reasonable definition of a geometric realisation on augmented simplicial groups, as the codomain of the augmentation is, in some sense, the image of the empty set.

Definition 1 (vi)

The *singular complex functor* from toplological spaces to augmented simplicial groups is the composition of the normal singular complex functor, which is right adjoint to the geometric realisation functor, with the trivial augmentation functor, right adjoint to the forgetful functor.

It is automatic that the two functors so defined are adjoint.

3. COMBINATORIAL JOIN

The following is our proposed definition for a join of augmented simplicial groups.

Definition 2

Let the join of two augmented simplicial group G and H be denoted $G \odot H$. The group of n simplicies, $(G\odot H)_n$ is:

$$\bigcup_{i=-1}^{n} G_{n-1-i} \times H_i$$

the face maps are given by:

$$d_i^n = \begin{cases} (d_i^p g, h) & \text{if } 0 \le i \le p \\ (g, d_{i-p-1}^{n-p-1} h) & \text{if } p \le i \le n-1 \end{cases}$$

where $(g,h) \in G_p \times H_{n-p-1}$, and d_0^0 is augmentation of G, or H; lastly, the degeneracies are:

$$s_i^n = \begin{cases} (s_i^p g, h) & \text{if } 0 \le i \le p \\ (g, s_{i-p-1}^{n-p-2} h) & \text{if } p \le i \le n-1 \end{cases}$$

where $(g,h) \in G_p \times H_{n-p-2}$.

There is also a coend definition for $\odot$:

$$G \odot H \cong \int^{p,q} (G_p \times H_p) \cdot \Delta([p] or [q]).$$

Remark

It is trivial to prove that $\Delta[n] \odot \Delta[m] \cong \Delta[n+m+1]$. It is also true that $\odot$ is an associative operation, but the simplest proof requires a number of constructions and results associated with the join which are not of immediate interest here.

4. TOPOLOGICAL JOIN

The following definition is a generalisation of the concept of join for two suitable subspaces of a vector space. The *topological join* thus defined is discussed in some detail in chapter 5, section 7 of [6]. Results proved there will be used here without proof: the notation for this section is largely taken from there. We work within the category of compactly generated spaces.

Definition 3

Consider two topological spaces **U** and **V**, and construct a group of 4-tuples (r,u,s,v), where $u\in$ **U**, $v\in$ **V**, $r,s\in[0,1]$ and r+s=1: in the case that r=0, the u will be ignored, in the case that s=0, the v will be ignored. This group will be suggestively called $\mathbf{U}*\mathbf{V}$.

There are obvious projections from this group of 4-tuples:
$p_U:\mathbf{U}*\mathbf{V}\to\mathbf{U}$, $p_V:\mathbf{U}*\mathbf{V}\to\mathbf{V}$, $p_r:\mathbf{U}*\mathbf{V}\to(0,1]$ and $p_s:\mathbf{U}*\mathbf{V}\to(0,1]$ which are termed the *coordinate functions* of $\mathbf{U}*\mathbf{V}$. The first two are obviously defined, the last two take a point (r,u,s,v) to r and s respectively.

The *topological join* of **U** and **V** is defined as the group $\mathbf{U}*\mathbf{V}$ together with initial topology with respect to the *coordinate functions*. Thus a functions with codomain $\mathbf{U}*\mathbf{V}\to\mathbf{U}$ is continuous function if and only if its composite with each of the coordinate functions are continuous.

To compare the combinatorial and topological join operators, we will need more precision on the construction of geometric realisation. There are a number of different constructive definitions of geometric realisation. The process is essentially the following:

(i) take one copy of Δ^n for each non-degenerate n-simplex of G ; and then
(ii) glue them all together using the face and degeneracy maps of the simplicial group G.

Explicitly we have:

Let **G** be a simplicial group. Define RG by:

$$RG=\cup_{n\in N}\cup_{g\in G_n}\Delta_g^n$$

Define an equivalence relation on RG as generated by the following relation: writing (**p**, g) for $(p_0,\ldots,p_m)\in\Delta_g^n$ and (**q**, g) for $(q_0,\ldots,q_m)\in$ then $(\mathbf{p},g)\sim(\mathbf{q},g)$ if either

$$d_ig=h \text{ and } \delta_i(q_0,\ldots,q_n)=(p_0,\ldots,p_n) \text{ or}$$
$$s_ig=h \text{ and } \sigma_i(q_0,\ldots,q_n)=(p_0,\ldots,p_n)$$

where the δ_i and σ_i are he continuous maps given by face inclusion and folding in the usual way. Then $|G|\cong RG/\sim$ where RG/~ has the identification topology.

Propositions 4.1 $\Delta^p*\Delta^q\cong\Delta^{p+q+1}$.

Proof Consider the vector space $\mathbf{R}^{p+q+1}$ and the two compact convex subgroups

$$G=\{(g_0,g_1,\ldots,g_p,0,\ldots,0):\sum_{i=1}^{p}g_i=1\}$$

$$H=\{(0,\ldots,0,h_0,h_1,\ldots,h_q):\sum_{i=1}^{p}h_i=1\}.$$

First note that $G\cong\Delta^p$ and $H\cong\Delta^q$. Furthermore, it is clear that no two lines in the group U={rg+(1-r)h : $0\le r\le(p+q+1)$, $g\in G$, $h\in H$ } intersect expect at end points. Thus $G*H$=U. However, U is the subgroup of $\mathbf{R}^{p+q+1}$ given by

$$\{(rg_0,\ldots,rg_p,\ (1-r)h_0,\ldots(1-r)h_q\ :\ \sum_{i=0}^{p} rg_i + \sum_{j=0}^{q}(1-r)h_j = 1\}.$$

That is, U is the affine (p+q+1)-simplex. Therefore $\Delta^p * \Delta^q \cong \Delta^{p+q+1}$. □

When we form $\Delta[p] \odot \Delta[q]$, we obtain, on varying p and q, a bisimplicial object in **SG**. (In general if **C** is category, a cosimplicial object in **C** is a functor from Δ to **C**, whilst a bisimplicial object is a functor from $\Delta \times \Delta$ to **C**.) Similarly $\Delta^p * \Delta^q$ is a bisimplicial space.

Lemma 4.2 There is a natural isomorphism

$$|\Delta^p| * |\Delta^q| \cong |\Delta[p] \odot \Delta[q]|$$

of bisimplicial spaces.

Proof Recall that $|\Delta[m]| = \Delta^m$. Since $\Delta[p] \odot \Delta[q] \cong \Delta([p] or [q]) = \Delta[p+q+1]$, the isomorphism exits for each pair (p, q). Now $\{\Delta^n\}_{n \in N}$ has an obvious cosimplicial structure, and the isomorphism is easily seen to be an isomorphism of bicosimplicial spaces. □

Theorem 4.3 Let G and H be trivially augmented simplicial groups. Then

$$|G \odot H| \cong |G| * |H|.$$

Proof

(The following is a direct geometric proof: we will comment later on the categorical aspects.) Recall that

$$|G| * |H| = \left\{ \begin{array}{ll} r[(p_0,\ldots,p_m)]_x & s.t.\ \sum_{i=0}^{m} p_i = 1,\ \sum_{i=0}^{m} q_i = 1 \\ +\, s[(q_0,\ldots,q_m)]_y & g \in G_m,\ h \in H_n,\ r+s=1 \\ & p_i, q_i, r, s \geq 0 \\ & and\ [-]\ denotes\ equivalence\ class \end{array} \right\}$$

It should also be noted that if r = 0, the point from $|G|$ is ignored and similarly, if s = 0, the point from $|H|$ is ignored.

Define a map f: $|G| * |H| \to |G \odot H|$

$$f(r[(p_0,\ldots,p_m)_g] + s[(q_0,\ldots,q_n)_h]) \mapsto [(rp_0,\ldots,rp_m,\ sq_0,\ldots,sq_n)_{g,h}].$$

The function f is well defined since if r = 0, the point g is ignored, similarly if s = 0. This means that for any h, it must be true that $(0,\ldots,0,q_0,\ldots,q_n)_{(g,h)} \sim (0,\ldots,q_0,\ldots,q_n)_{(g',h)}$ for all g, $g' \in G$. This will be true exactly when the augmentation of both G and H are trivial as was required. A moment's thought then will show that the function f respects the relation and so is well defied. Continuity is also trivial to check. The obvious inverse function is also continuous under the definition of the topology on $|G| * |H|$. Thus the two spaces are homemorphic. □

Remarks

(i) It may seem slightly contrived that the condition ``trivially augmented'' should be need, however consider the following example:

Let $G = \Delta[0] \cup \Delta[0]$ together with the *canonical* augmentation, and consider G⊙G. The results is the disjoint union of four unit intervals - that is, $\Delta[1] \cup \Delta[1] \cup \Delta[1] \cup \Delta[1]$: Ideally, the result should be homotopically equivalent to a 1-sphere.

(ii) The Theorem above is in fact a simple consequence of a categorical argument that shows a different aspect of the necessity for having a trivial augmentation.

The simplicial complex functor to augmented simplicial groups needs to specify an augmentation, and for the functor to be right adjoint to the geometric realisation functor, the augmentation must be trivial one (since the trivial augmentation is right adjoint to the forgetful functor from augmented simplicial groups to simplicial groups.) Thus the condition `trivially augmented' merely requires that the augmented simplicial groups be related to the geometric realisation functor upon which the theorem depends. The results is now seen to depend just on left adjoints interacting nicely with the coends in the geometric realisation and join functors.

5. SIMPLICIAL SPHERES

Recall (from [6]) that

$$S^p * S^q \cong S^{p+q+1}.$$

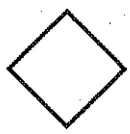
This essentially says that the n-sphere in the category of topological spaces is the join of n+1 copies of the 0-sphere.

The are several simplicial models for the n-sphere. For instance, Gabriel and Zisman, [13], p.26, define the simplicial circle, Ω, to be the coequaliser of the pair of morphism

$$\Delta[0] \xrightarrow{\delta_1, \delta_0} \Delta[1],$$

and the suspension of a pointed simplicial group G to be $\Omega \wedge G$. This gives an n-sphere as being $\Lambda^n \Omega$, obtained from the n-cube $\Delta^n[1]$ by collapsing the `boundary' of the cube to a point. Other authors form a simplicial sphere by collapsing the boundary $\partial\Delta[n]$ of n-simplex to a point.

The join operation suggests another form. Consider the simplicial group formed as the disjoint union of two copies of $\Delta[0]$ and augmented trivially. This will be denoted by S^0 and will be referred to as the simplicial 0-sphere. then $S^0 \odot S^0$ has four non-degenerate 1-simplices connected to each other in a ``diamond'' as below:



Define the simplicial n-sphere, $S^n \in$ **obASG**, as follows:

$$S^n = S^0 \odot \ldots \odot S^0.$$

It is clear from the definition of combinatorial join and of the simplicial 0-sphere that the simplicial n-sphere is a triangulation of the topological n-sphere. In fact, Theorem 4.3 gives explicitly that

$$|S^n| \cong S^n.$$

Moreover this model clearly satisfies

$$S^p \odot S^q \cong S^{p+q+1}$$

unlike the other models. Thus if we write $\sum^n = \Delta[n]/\partial\Delta[n]$ then $\sum^p \odot \sum^q$ has one degenerate simplex in each of the dimensions 1, p+1,q+1 and p+q+1, and two non-degenerate simplicies in dimension 0 and so `looks' totally unlike $\sum^{p+q+1}$.

The combinatorial join forms part of a closed monoidal structure on the category of augmented simplicial groups, **ASG**. (The `internal hom' is given by

$$[G,H]_n = \mathbf{ASG}(G, \mathbf{Dec}^{n+1}H),$$

where **Dec** is the decalage functor (see Duskin,[8]). It is therefore possible to define augmented analogues to the loop space construction that are compatible with the join.

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