THE BOUNDARY INTEGRAL EQUATION METHOD FOR ELASTOPLASTIC STRESS ANALYSIS OF FRICTIONAL CONTACT PROBLEMS

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Abstract: For two-dimensional elastoplastic stress analysis of frictional contact problems an isoparametric quadratic formulation of the boundary integral equation (BIE) method is presented with emphasis on clarity at the expense of efficiency. The overall system of equation is obtained by using multi-domain approach in which the contact conditions are taken into account to couple the different system of equation for each body in contact. Several types of contact interface conditions are covered including Infinite friction (stick), frictionless and Coulomb friction slip. In order to include plastic deformation in the analysis, the initial strain formulation and von Mises yield criterion are used. In order to evaluate internal stress and strain rates the differentiation of the displacement in elementwise manner is adopted with its compulsory modeling of partial (where plastic deformation is expected) domain. Elastic-perfectly plastic, work-hardening or mixed hardening material behaviour can be assumed. An efficient and robust incremental-iterative process is proposed to find the proper contact conditions. For the numerical implementation, isoparametric three-noded line elements are used to represent the boundary and eight-noded quadrilateral domain cell are used for partially modelling interior domain. The proposed algorithm is applied to the hertz-like indentation problem and the results are, where appropriate, compared with the presented results in the literature to confirm its correctness and robustness.

1. INTRODUCTION

The Boundary Integral Equation (BIE) method is well established as accurate numerical tool particularly well suited for linear elastic problems. Due to its high resolution of stresses on the surface, the BIE approach has been shown to be well suited to problems involving stress concentration, fracture mechanics and contact analysis. However, its extension to non-linear problems including material and geometric non-linearity is not widespread and is under-developed when compared to the Finite Element (FE). In many non-linear BIE formulations the interior of the solution domain has to be discretized, thus losing the main BIE advantage of surface-only modeling. Another difficulty encountered in the non-linear BIE formulations is the accurate evaluation of strongly-singular integral functions. Because of the unknown contact area and -in the presence of frictional contact- the unknown contact conditions, elastic contact problems are inherently non-linear. By including plastic deformation, the numerical analysis gets yet more complicated. In most engineering applications, severe deformation may take place in the contact area, thus producing rapidly changing stress rates in solution domain. It is evident that the application of the BIE for the analysis of the contact problems is efficient and suitable. For elastic and thermo-elastic contact problems, the application of the BIE method has been widely discussed (see for example, references [1-4]). To cover elastoplastic contact analysis, the BIE formulation would require a carefully designed robust numerical algorithm to incorporate nested iterations and load increments that are capable of monitoring contact

developments as well as marching the solution along the elastoplastic material path. Introducing frictional stick-slip behaviour would further complicate the numerical algorithms. In such processes two different iterative procedures incorporate in order to find the proper contact conditions for every load increment. One is to calculate the plastic deformations. The other is to find the correct contact area. Some algorithms using fixed load increments are presented [4-6]. Alcantud *et al* [7] presented the extension of the load scaling approach to plasticity, in which linear interpolation are employed to find a permissible load increment and thus the non-linearity of plasticity are not considered. Kong *et al* [8] presented a similar mathematical approach in order to treat friction. Huesmann *et al* [9] proposed an automatic incremental technique, in which contact condition change at only one node at the end of the increment, for elasto-plastic contact problems including friction.

The presented algorithm in this paper, based on the use of the initial strain rates, incorporates the nested iterations to find the accurate strain rates and the proper contact conditions and load increments. The calculated strain rates at the previous load step are used as the initial prediction of plastic strain rates for progressive kind of plastic contact deformation. It should be worth mentioning that it is practical and efficient to reduce the load increments to minimum and to use the average values for stress and strain rates instead of initial values.

2. GOVERNING EQUATIONS

By considering a two-dimensional isotropic homogeneous body which has the boundary S and the domain A during time-independent plastic deformation, the Navier rate equation for plane strain and plane stress can be written as follows:

$$\frac{\partial^2 \dot{u}_i}{\partial x_j \partial x_j} + \frac{1}{(1-2\nu)} \frac{\partial^2 \dot{u}_j}{\partial x_i \partial x_j} - \left(k_1 \frac{\partial \dot{\epsilon}_{ij}^p}{\partial x_i} + 2 \frac{\partial \dot{\epsilon}_{ij}^p}{\partial x_j} \right) = \frac{-\dot{f}_i}{\mu} \quad (1)$$

Where u_i is the displacement rate, $\dot{f}_i$ is the body-force rate, $\dot{\epsilon}_{ij}^p$ is the plastic strain rate and the parameter k_1 is either 0 (for plane strain) or $2\nu/(1-2\nu)$ (for plane stress).

For a material obeying the von Mises yield criterion the plastic strain rates are given by the following incremental elastoplastic flow rules:

$$\dot{\epsilon}_{ij}^p = \frac{9}{4} \left(\frac{\dot{S}_{kl} \dot{\sigma}_{kl}}{H} \right) \frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \quad (2a)$$

$$\dot{\epsilon}_{ij}^p = \frac{3}{2} \left(\frac{\dot{S}_{kl} \dot{\epsilon}_{kl}}{1 + H/3\mu} \right) \frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \quad (2b)$$

where $\dot{S}_{ij}$ and $\dot{\sigma}_{eq}$ denote the current deviatoric stress tensor and the equivalent stress respectively, $\dot{\epsilon}_{ij}$ is the total strain rate and H represents the plastic modules.

Axelsson and Samuelson [10] proposed that the plastic strain rate is decomposed into its isotropic and kinematic parts as follows:

$$d\epsilon_{ij}^{p(i)} = M d\epsilon_{ij}^p \quad (3a)$$

$$d\epsilon_{ij}^{p(k)} = (1 - M) d\epsilon_{ij}^p \quad (3b)$$

in which M is defined as the mixed hardening parameter which is equal to -1 for isotropic softening, 0 for kinematic hardening and 1 for isotropic hardening.

The slope of the stress-plastic strain curve in a uniaxial tensile test, H , is given as follows:

$$H = \frac{d\sigma_{eq}}{d\epsilon_{eq}^{p(i)}} \quad (4)$$

3. THE INITIAL STRAIN BE APPROACH

3.1 Analytical Formulation

One way of the dealing with the plastic terms in the navier rate equations is to treat the derivatives of the plastic stress rates (see Henry and Banerjee [11]) or the plastic strain rates (see, for example, Mendelson [12]) as a kind of body force. Alternatively, by considering the plastic strain rates as initial strain rates then modifying Betti's reciprocal theorem to include plasticity, the pseudo-boundary integral equation for initial strain approach can be written as follows (see, for example, Lee [13]):

$$\begin{aligned} C_{ij}(P)\dot{u}_i(P) + \int T_{ij}(P, Q)\dot{u}_j(Q)dS(Q) &= \int U_{ij}(P, Q)\dot{t}_j(Q)dS(Q) \\ &+ \int U_{ij}(P, Q)\dot{f}_j(q)dA(q) + \int W_{kij}(P, q)\epsilon_{ij}^p(q)dA(q) \end{aligned} \quad (5)$$

In this expression U_{ij} and T_{ij} are fundamental displacements and tractions at x in the j th direction at the field point Q or q due to a unit load at load point P acting in i th direction. W_{kij} represents the third-order tensor for the stress at the field point due to a unit load at load point P . The tensors U_{ij} , T_{ij} , W_{kij} , derived from fundamental solution to Kelvin's problems in two dimensions, are given as follows:

$$U_{ij}(P, Q) = -\frac{1}{8\pi(1-\nu)\mu} \left\{ \delta_{ij}(3-4\nu) \ln r - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \right\} \quad (6)$$

$$T_{ij}(P, Q) = -\frac{(1-2\nu)}{4\pi(1-\nu)} \frac{1}{r} \left\{ (\delta_{ij} + \frac{2}{(1-2\nu)} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j}) \frac{\partial r}{\partial n} - \frac{\partial r}{\partial x_i} n_j + \frac{\partial r}{\partial x_j} n_i \right\} \quad (7)$$

$$W_{kij}(P, q) = -\frac{1}{4\pi(1-\nu)} \frac{1}{r} \left\{ (1-2\nu) \left(\delta_{jk} \frac{\partial r}{\partial x_i} + \delta_{ik} \frac{\partial r}{\partial x_j} - k \delta_{ij} \frac{\partial r}{\partial x_k} \right) + 2 \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \right\} \quad (8)$$

where the parameter k is either equal to 1 (plane stress) or $1/(1-2\nu)$ (plane strain). Finally, the free-term tensor, C_{ij} , is defined as follows:

$$C_{ij}(P) = \delta_{ij} + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon(P)} T_{ij}(P, Q) dS(Q) \quad (9)$$

3.2 NUMERICAL IMPLEMENTATION

3.2.1 Isoparametric Quadratic Elements

Since interior modeling is required, both boundary elements and domain cells are necessary to perform the integrals. Figure 1 shows a typical three-node line element and a typical eight-node quadrilateral domain cell. For the boundary element, the geometry can be described in terms of quadratic shape functions in a local coordinate axes system as follows:

$$x_i(\xi) = \sum_{c=1}^3 N_c(\xi) (x_i)_c \quad (10)$$

where N_c is the quadratic shape function and ξ is the local coordinate. similarly, the displacement and traction vectors can be expressed in terms of quadratic shape functions. For the interior cells, the geometry can be defined in terms of quadratic shape functions described in local coordinates, ξ_1 and ξ_2 as follows:

$$x_i(\xi_1, \xi_2) = \sum_{c=1}^8 N_c(\xi_1, \xi_2) (x_i)_c \quad (11)$$

Similary, the displacement rates can be expressed in the domain cells. Thretherefore, in discretized form, for each body in contact, the elastoplastic BI equation in the initial strain approach can be written as follows:

$$\begin{aligned} C_{ij} \dot{u}(P) + \sum_{m=1}^M \sum_{lc=1}^3 \dot{u}_j(Q) \int_{-1}^{+1} T_{ij}(P, Q) N_c(\xi) J(\xi) d\xi \\ = \sum_{m=1}^M \sum_{lc=1}^3 \dot{t}_j(Q) \int_{-1}^{+1} U_{ij}(P, Q) N_c(\xi) J(\xi) d\xi \\ + \sum_{m=1}^D \sum_{lc=1}^8 \dot{\varepsilon}_{ij}^p(q) \int_{-1}^{+1} \int_{-1}^{+1} W_{ijk}(P, q) N_c(\xi_1, \xi_2) J(\xi_1, \xi_2) d\xi_1 d\xi_2 \end{aligned} \quad (12)$$

where P denotes the node where the integration is performed, Q indicate the c^{th} node of a boundary element and q indicates the c^{th} node of a domain cell. Note that the jacobian of transformation, J , is used to transform the variables the from global to local axes.

Fig. 1: Isoparametric quadratic elements used in elastoplastic BE analysis
 (a) Three-noded boundary element
 (b) Eight-noded quadrilateral domain cell

The details of an efficient integration process for the discretized BI equation are presented by Gun and Becker[14]. It should be noted that the integration process is performed separately for each domain in contact. For each contacting domain, the linear algebraic equations obtained from the discretized integral BI equation can be form as follows:

$$[A][\dot{u}] = [B][\dot{t}] + [W][\dot{\epsilon}^p] \quad (13)$$

where the matrices $[A]$, $[B]$ and $[W]$ contain the integrals of the displacement, traction and domain kernel integrals, respectively. For two-dimensional problems, if the total number of boundary nodes is N and the total number of the domain cell points is H , then the solution matrices $[A]$ and $[B]$ will be a square matrices of size $2N \times 2N$, whereas the matrix $[W]$ will be a rectangular matrix of size $2N \times 2H$. All matrices are fully populated.

3.2.2 The Displacement Gradient Approach

By differentiating the interior displacements via the shape functions over each domain cell, and using the jacobian of transformation, the displacement differentials with respect to the local coordinates ξ_1 and ξ_2 can be obtained. To obtain the total strain rates, the strain-displacement relationship can be used as follows:

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) \quad (14)$$

In the plane stress problems the total strain rate in the third direction can be obtained by using $\dot{\epsilon}_{33} = 0$, i.e.

$$\dot{\epsilon}_{zz} = \frac{-\nu}{1-\nu} (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) - \frac{1-2\nu}{1-\nu} (\dot{\epsilon}_{xx}^p + \dot{\epsilon}_{yy}^p) \quad (15)$$

Using Hooke's law, the cell stress rates can be written as follows:

$$\dot{\sigma}_{ij} = 2\mu(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^p) + \frac{2\mu\nu}{1-2\nu} \delta_{ij} \dot{\epsilon}_{kk} \quad (16)$$

4. FORMULATION OF THE FRICTIONAL CONTACT PROBLEMS

In order to couple the different system of equation obtained from the discretized BI equation for each body in contact, the contact conditions have to be imposed on the contacting node pairs. To do this, as shown in figure 2, consider node a in domain A and node b in domain B to make a contacting node pair. It should be noted that the nodes outside the contact region have either prescribed displacement or prescribed traction rates. If the nodes in the contacting surface are not allowed to slip in any direction (stick contact), continuity of displacements and equilibrium conditions requires:

$$\text{Equilibrium : } u_x^a - u_x^b = x^b - x^a \quad \text{and} \quad u_y^a - u_y^b = y^b - y^a \quad (17)$$

$$\text{Compatibility : } t_x^a - t_x^b = 0 \quad \text{and} \quad t_y^a - t_y^b = 0 \quad (18)$$

If slip condition occurs, referring to the tangential and the normal components of traction and displacements (see fig.2), the following constraint conditions hold:

$$\text{Frictional law : } t_t^a = \pm \mu t_n^a \quad (19)$$

$$\text{Equilibrium : } t_n^b + t_n^a = 0 \quad \text{and} \quad t_t^b + t_t^a = 0 \quad (20)$$

$$\text{Compatibility : } u_n^a = u_n^b \quad \text{and} \quad u_t^a = u_t^b + \delta_t \quad (21)$$

Finally, in the presence of interference (clearance), δ_n , the following compatibility of displacements holds:

$$\text{Compatibility : } u_n^a - u_n^b = \delta_n \quad (22)$$

In the above-given expressions, the subscripts t and n indicate the tangential and the normal directions, respectively and δ_t represent the amount of slip in the tangential direction. The local components of traction and displacements rates can be transformed into global components by using the contact angle θ (see fig.2).

4.1 Contact Iterations

Contact problems are inherently non-linear by virtue of the unknown contact area and the presence of friction, the behaviour may be history dependent. For numerical analysis of such problems, a reasonable initial guess of the contact area must be used in the first iteration. Therefore the proper numerical algorithms for such problems require an iterative and / or incremental procedure.

One of the nested iterative procedures in the presented algorithm is an efficient automatic iterative scheme in order to perform the contact iterations without load incrementation. This iterative processes checks the possibilities which are overlap, tensile stress and frictional slip that may arise in determining the proper contact conditions. Since a contact area is

based on the compressive forces, contacting nodes with normal tensile stresses, which suggests that the contact area is too large for the given load, must be released from the contact in the next iteration. Overlapping nodes indicates that the contact area is too small for the given load. Therefore the associated elements must be included in the contact area for the next iteration in order to prevent any interpenetration of the bodies in contact. Finally, in the presence of Coulomb friction slip, which occurs when the absolute value of the ratio of the tangential to the normal traction exceeds the coefficient of friction, μ , the related nodes have to be allowed to slip in the next iteration. It should be worth mentioning that all node pairs are assumed to be a sticking stage for the first iteration. The details of such iterative scheme can be found in the text book by Becker[15].

Figure 2 : Local and tangential directions

4.2 Coupling the system of equations

In order to form the overall system of equations for two bodies in contact, the linear algebraic equations obtained from the discretized the BI equation for each contacting domain can be formed as follows:

$$\begin{bmatrix} [A]^{(a)} & 0 \\ 0 & [A]^{(b)} \end{bmatrix} \begin{bmatrix} [\dot{u}]^{(a)} \\ [\dot{u}]^{(b)} \end{bmatrix} = \begin{bmatrix} [B]^{(a)} & 0 \\ 0 & [B]^{(b)} \end{bmatrix} \begin{bmatrix} [\dot{t}]^{(a)} \\ [\dot{t}]^{(b)} \end{bmatrix} + \begin{bmatrix} [W]^{(a)} & 0 \\ 0 & [W]^{(b)} \end{bmatrix} \begin{bmatrix} [\dot{\varepsilon}]^{(a)} \\ [\dot{\varepsilon}]^{(b)} \end{bmatrix} \quad (23)$$

In order to arrive at the solution matrix, both prescribed displacement and prescribed traction are imposed on the system of equations obtained from the discretized the BI equation for each solution domain and the resulting matrices are coupled together according to the contact conditions, thus the following final form of the system of equation can be obtained.

$$[A^*] \dot{x} = [C^*] \quad (24)$$

In this expression the matrix $[A^*]$ is the final coefficients of the overall system of equations, the vector $[\dot{x}]$ includes the unknown traction and displacement rates and $[C^*]$ is the known vector.

Since the plastic deformation depends on the current state of the stresses and the nature of the contact problems, the equations must be solved in iterative manner. Therefore two different iterative procedures incorporate to find the correct plastic deformations and the correct contact conditions which give a reasonable load increment. Gaussian elimination can be used to solve the resulting set of simultaneous equations. The solution matrices should have roughly the same order of the magnitude. To do this, a reasonable scaling factor can be used. The stresses at the boundaries are obtained from the local displacement gradients and traction rates. The stress and strain rates at interior nodes are calculated by differentiating displacement rates, obtained from the BI equation for displacement rates, in elementwise manner.

5.APPLICATIONS

The BIE is used for the plane strain analysis of the frictional contact of an punch of height H_p and half-width W_p indenting the foundation of height H_f and half-width W_f . The subscripts p and f refer to the punch and foundation respectively. Boundary and loading conditions are given in figure 3. The relevant data for dimensions employed in this analysis are $H_p/W_p=2$, $W_f/W_p=4$, $H_f/W_p=4$. This problem was investigated by Zhong and Wu[16], and Bujang[17] who used a commercial finite element package, ABAQUS, in his work. In order to be able to compare BE results with Zhang and Wu's analysis and the ABAQUS results presented by Bujang [17], the same material properties and loading conditions were employed in the BE analysis. It should be noted that The overall dimensions are differ, but still in portion, with the model analyzed by Zhong and Wu[16]. In their analysis the yield stress of the foundation is 100 times greater than that of the punch ($\sigma_p=1.96 \cdot 10^6$). However, for ABAQUS analysis Bujang[17] swapped the values of σ_y in order to make σ_{yp} grater than σ_{yf} which is more in line with engineering application. The Poisson's ratio, ν , and the young's modulus E are assumed to both at 0.3 and $206 \cdot 10^9$ Pa respectively. The applied uniform loads are within the rage of 2.5 and 3 MPa with two different values of coefficient of friction, $\mu=0$ and $\mu=0.2$. The mesh design for the present BE analysis is shown in figure 4.

The foundation starts to plasticise at a load of 2.4 MPa. Therefore the results are plotted in terms of load ratios, r , which is the ratio of the applied load to 2.4 MPa. Figure 5 shows normal contact pressure at two different load rations versus the normalized contact interface for frictionless contact ($\mu=0$). The elastic-plastic curves by Zhong and Wu[15] for $r=1.04$ and $r=1.33$ indicate a smooth increase of pressure towards the end of the contact interface. While ABAQUS results did not give this trend The present BE results validate the Zhong and Wu's analysis.

Figure 6 shows the rate of the increase in tangential friction stresses at load rations, $r=1.04$, $r=1.08$, $r=1.1$ when $\mu=0.2$. The BE results give a same view of the relationship as that suggested by Zhong and Wu (*), but it is clear from the work of Bejung[17] that ABAQUS is not able to give results including part of the contact region sliding,

while the rest sticking. Therefore the presentation of the ABAQUS results was abandoned in figure 6. The dashed lines represent the BE elastic solutions in figure 6.

Figure 3: The indentation problem

Figure 4: BE mesh design for the indentation problem

Figure 5: Normal contact pressure distribution when $\mu = 0.0$.

Figure 6: shear stress distribution when $\mu = 0.2$

6. CONCLUSION

An efficient numerical algorithm, using an quadratic boundary integral formulation, to incorporate the nested iterations and load increments, that are capable of monitoring contact

developments as well as marching the solution along the elasto-plastic material path, is presented in order to cover multi-domain and contact problems including friction. For determining the proper contact conditions an automatic iterative scheme with / or without load increment can be used. However, in elasto-plastic analysis, because of the nature of the boundary integral equations, an automatic iterative scheme without load increment is suitable and efficient

REFERENCES

1. T. Andersson, *Boundary element in two-dimensional contact and friction*, Linköping University Report, Dept. of Mech. Eng., Linköping, 1982.
2. G. Karami, *A Boundary element method for two-dimensional contact problems*, Springer, Berlin 1989.
3. G. Karami and R. T. Fenner, *A two-dimensional BEM for thermo-elastic body forces Contact problems*, edited by C.A. Brebbia, W.L. Wendland and G. Kuhn, *Boundary Element IX, 2: Stress Analysis Applications*, Springer, Berlin Heidelberg, 417-437, 1987.
4. O. A. Olukoko, A. A. Becker and R. T. Fenner, *A new boundary element approach for contact problems with friction*, *Int. J. numer. Meth. Engng* **33**, 1513-1522, 1993
5. G. Karami, *Boundary element analysis of elasto-plastic contact problems*, *Comput., Struct.*, **41**, 927-935, 1991.
6. T. Tsuta and S. Yamaji, *Boundary element analysis of contact thermo-elastoplastic problems with creep and the numerical technique*. *Boundary Elements-Proc., 5th Int. Conf.*, Springer, Berlin, 1983.
7. M. Alcantud, M. Doblare, L. Gracia and J. Dominguez, *Analysis of the contact problem between elastoplastic 2-D bodies by means of the BEM*. *Boundary elements in mechanical and electrical engineering* (Edited by C. A. Brebbia and A. Chaudonet-Miranda). Computational Mechanics Publications, Southampton, 1990.
8. X.-A. Kong, A. Gakwaya, A. Cardou and L. Cloutier, *A unified BEM formulation for Elasto-plastic contact analysis by mathematical programming*. *Boundary elements XIV, 2, Stress Analysis and Computational Aspects* (Edited by C.A Brebbia, J. Dominguez and F. Paris). Computational Mechanics Publications, Southampton, 1992
9. A. Huesmann and G. Kuhn, *Automatic load incrementation technique for plane elastoplastic frictional contact problems using boundary element method*, *Computers and Structures*, **56**, 733-745, 1995.
10. K. Axelsson and A. Samuelsson, *Finite element analysis of elastic-plastic materials displaying mixed hardening*, *Int. J. Numer. Math. Engng.*, **14**, 211-225, 1979
11. Henry, D. P. and Banerjee, P.K., *A new BEM formulation for two and three Dimensional Elasto-Plasticity Using Particular Integrals*, *Int. J. Numer. Meth. Engng.*, **26**, 2079-2096, 1998.
12. Mendelson, A., *Boundary Integral Methods in Elasticity and Plasticity*, Report NASA TND-7418, 1973.
13. K. H. Lee, *A boundary integral equation method for two dimensional elasto-plastic analysis*, PhD thesis, Imperial College, University of London, 1983.

14. H. Gun and A. A. Becker , A Review of boundary element quadratic formulations in Elastoplastic stress analysis problems, Advances in Computational Structural Mechanics (Edited by R.H.V.Topping), Civil-Comp Press , 4th International Conference on Computational Structures Technology, Edinburg, 1998.
15. A. A. Becker, *The boundary element method in engineering - a complete course*, London, McGraw-Hill, 1992.
16. W. X. Zhong and C.W. Wu, Elastic-plastic contacts using parametric quadratic Programing *Computational Methods in Contact Mechanics*, Computational Mechanics Publications, Southampton, 305-352 , 1993.
17. H. Bujang ,*Contact analysis with plasticity using finite elements*, Bsc., the University of Nottingham, Nottingham, 1995.
18. ABAQUS version 5.2, HKS inc., Rhode Island, 1994.