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SOLUTION OF BOUSSINESQ PROBLEM USING LIE SYMMETRIES

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Abstract- Firstly the Lie point symmetries of cylindrically symmetric homogeneous Navier equations are obtained. Using the symmetries the general class of similarity solutions are found. The subclass that also satisfies the non-homogeneous system of the medium subject to a singular force is determined. Substituting the subclass into the non-homogeneous system, a system of ordinary differential equations is obtained. The solution of the system satisfying the boundary conditions of Bousssinesq problem gives the exact solution.

1.INTRODUCTION

Sophus Lie introduced the concept of continuous groups (known as Lie groups) in order to unify and extend the methods of solutions of ordinary differential equations in the last few decades of the nineteenth century (see [1], [2]). He proved that if a differential equation remains invariant under a one-parameter Lie group of transformation then its order may be reduced by one. A symmetry group of a differential equation is the transformation group that transforms its solutions to other solutions. Such a group may be a point symmetry that depends on dependent and independent variables, but it may also be a contact symmetry that depends on first derivatives as well. Some solutions remain invariant under symmetry group transformations; these solutions are called *invariant* (or *similarity*) solutions. The invariant solutions may be used in finding the solutions of boundary value problems.

In the twentieth century the application of Lie groups to differential equations attracted the attention of quite a number of mathematicians. L. Ovsiannikov(see [3]), N. H. Ibragimov, (see [4], [5]). W. Bluman(see [6]), P. J. Olver(see [7]) are some of them. E. Cartan applied exterior forms to Lie groups(see [8]). Edelen extended the methods established by E. Cartan(see [9]). E. Şuhubi developed an algorithm and applied it to many problems of mechanics(see [10]).

In this work Lie groups of Navier equations are determined and the solution of Boussinesq problem is obtained by using its symmetries.

2. NAVIER EQUATIONS OF THE CLASSICAL ELASTICITY THEORY

As is known the Navier equation for the elastostatics of linear homogeneous isotropic media is given in the form (see [11], [12], [13], [14], [15])

$$(\lambda + \mu)\ \text{grad div}\, \boldsymbol{u} + \mu \Delta\, \boldsymbol{u} + \rho\, \boldsymbol{p} = \rho \ddot{\boldsymbol{u}} \tag{2.1}$$

where λ, μ are Lamé constants, $\boldsymbol{u}$ the displacement vector, ρ the mass density and $\boldsymbol{p}$ the body force for unit mass. The handled problem is Boussinesq problem for concentrated normal force on the boundary of half-space (see [12], [15]).

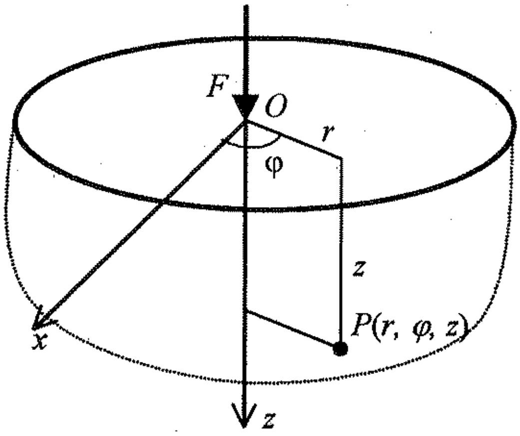

Figure 1

where F is a finite concentrated load at O in the positive z direction.

The system of equilibrium equations of a medium subject to such a singular force F acting at origin along z -axis becomes

$$
\begin{aligned}
&u_{rr}+\frac{1}{r}u_r-\frac{1}{r^2}u+w_{rz}+\kappa\left(u_{zz}-w_{rz}\right)=0 \\
&u_{zr}+w_{zz}+\frac{1}{r}u_z-\kappa\left(u_{rz}-w_{rr}+\frac{1}{r}u_z-\frac{1}{r}w_r\right)+\frac{F}{\rho c_0^2}\frac{\delta(r)}{2\pi r}\delta(z)=0
\end{aligned}
\tag{2.2}
$$

in cylindrical coordinate system (r, φ, z) where

$\kappa=\dfrac{1-2\nu}{2(1-\nu)}, c_o^2=\dfrac{\lambda+2\mu}{\rho}$, c_0 : speed of the longitudinal wave and

$$p_r=0, \quad \frac{\rho}{\lambda+2\mu}p_z=\frac{F}{\lambda+2\mu}\frac{\delta(r)}{2\pi r}\delta(z)$$

The infinitesimal generator X for the Lie point symmetries is

$$X=\xi\frac{\partial}{\partial r}+\eta\frac{\partial}{\partial z}+\zeta\frac{\partial}{\partial u}+\tau\frac{\partial}{\partial w} \tag{2.3}$$

The prolonged infinitesimal generator of second order is in the form

$$\underset{2}{X} = \xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial z} + \zeta\frac{\partial}{\partial u} + \tau\frac{\partial}{\partial w} + \zeta_r\frac{\partial}{\partial u_r} + \zeta_z\frac{\partial}{\partial u_z} + \tau_r\frac{\partial}{\partial w_r} + \tau_z\frac{\partial}{\partial w_z} + \zeta_{rr}\frac{\partial}{\partial u_{rr}} + \zeta_{rz}\frac{\partial}{\partial u_{rz}} +$$

$$\zeta_{zz}\frac{\partial}{\partial u_{zz}} + \tau_{rr}\frac{\partial}{\partial w_{rr}} + \tau_{rz}\frac{\partial}{\partial w_{rz}} + \tau_{zz}\frac{\partial}{\partial w_{zz}} \tag{2.4}$$

where

$$\zeta_r = D_r\zeta - u_r D_r\xi - u_z D_r\eta, \quad \zeta_z = D_z\zeta - u_r D_z\xi - u_z D_z\eta, \quad \zeta_{rr} = D_r\zeta_r - u_{rr}D_r\xi - u_{rz}D_r\eta$$

$$\zeta_{rz} = D_z\zeta_r - u_{rr}D_z\xi - u_{rz}D_z\eta, \quad \zeta_{zz} = D_z\zeta_r - u_{zr}D_z\xi - u_{zz}D_z\eta, \quad \tau_r = D_r\tau - w_r D_r\xi - w_z D_r\eta$$

$$\tau_z = D_z\tau - w_r D_z\xi - w_z D_z\eta, \quad \tau_{rr} = D_r\tau_r - w_{rr}D_r\xi - w_{rz}D_r\eta, \quad \tau_{rz} = D_z\tau_r - w_{rr}D_z\xi - w_{rz}D_z\eta$$

$$\tau_{zz} = D_z\tau_r - w_{zr}D_z\xi - w_{zz}D_z\eta \tag{2.5}$$

D represents the total derivative with respect to the independent variable of the subscript.
If the second prolongation of X is applied to the homogeneous part of (2.2), we reach the equations:

$$\zeta_{rr} + \frac{2u}{r^3}\xi - \frac{1}{r^2}\zeta - \frac{u_r}{r^2}\xi + \frac{1}{r}\zeta_r + \kappa\left(\zeta_{zz} - \tau_{zr}\right) + \tau_{rz} = 0$$

$$\zeta_{rz} + \tau_{zz} - \frac{u_z}{r^2}\xi + \frac{1}{r}\zeta_z - \kappa\left[\zeta_{rz} - \tau_{rr} + \frac{-u_z + w_r}{r^2}\xi + \frac{1}{r}\left(\zeta_z - \tau_r\right)\right] = 0 \tag{2.6}$$

Using $(2.2)_1$ and $(2.2)_2$ u_{rr} and w_{zz} can be written as follows

$$u_{rr} = -\kappa u_{zz} + (\kappa - 1)w_{rz} + \frac{u}{r^2} - \frac{u_r}{r}$$

$$w_{zz} = -\kappa w_{rr} + (\kappa - 1)\left(u_{rz} + \frac{u_z}{r}\right) - \kappa\frac{w_r}{r} \tag{2.7}$$

where

$$\kappa = \frac{\mu}{\lambda + 2\mu}$$

The structures of the functions of ξ, η, ζ and τ have the form (see [17])

$$\xi=\xi(r,z),\quad \eta=\eta(r,z)$$

$$\zeta=f(r,z)u+g(r,z)w \tag{2.8}$$

$$\tau=h(r,z)u+k(r,z)w$$

We get the following determining equations substituting (2.8) to (2.5) and (2.6)(see Appendix):

1) $r^3(f_{rr}+\kappa f_{zz})+(1-\kappa)r^3h_{rz}+r^2f_r+2(\xi-r\xi_r)=0$ (2.9)

2) $r^2(g_{rr}+\kappa g_{zz})+(1-\kappa)r^2k_{rz}+rg_r-g=0$ (2.10)

3) $2r^2f_r+(1-\kappa)r^2h_z-r^2\xi_{rr}+r\xi_r-\xi-r^2\kappa\xi_{zz}=0$ (2.11)

4) $2rg_r+(1-\kappa^2)g+(1-\kappa)rk_z-(1-\kappa)r\xi_{rz}+\kappa(1-\kappa)\eta_r=0$ (2.12)

5) $-\kappa(1-\kappa)g+(1-\kappa)rh_r+2\kappa rf_z-r\eta_{rr}-\kappa(2-\kappa)\eta_r-\kappa r\eta_{zz}=0$ (2.13)

6) $(1-\kappa)k_r-(1-\kappa)\eta_{rz}+2\kappa g_z=0$ (2.14)

7) $-\kappa(1-\kappa)g+(\kappa^2-2\kappa-1)\eta_r-2\kappa\xi_z+(1-\kappa)h=0$ (2.15)

8) $\xi_r=\eta_z$ (2.16)

9) $(1+\kappa)g+\kappa\eta_r-\xi_z=0$ (2.17)

10) $-f+\xi_r+k-\eta_z=0$ (2.18)

11) $(1-\kappa)r^2f_{rz}-(1-\kappa)\xi_z+r^2h_{zz}+(1-\kappa)rf_z+\kappa r^2h_{rr}+\kappa h+\kappa rh_r=0$ (2.19)

12) $(1-\kappa)rg_{rz}+rk_{zz}+(1-\kappa)g_z+\kappa rk_{rr}+\kappa k_r=0$ (2.20)

13) $(1-\kappa)f_z-(1-\kappa)\xi_{zr}+2\kappa h_r=0$ (2.21)

14) $-r^2\xi_{zz}+2\kappa r\eta_z+(1-\kappa)r^2g_z+2\kappa r^2k_r-\kappa r^2\xi_{rr}-\kappa\xi-\kappa r\xi_r=0$ (2.22)

15) $(1-\kappa)(r^2f_r-r^2\eta_{rz}-rk+r\eta_z-\xi+rf)+2r^2h_z=0$ (2.23)

16) $(1-\kappa)rg_r+g(1-\kappa)+2rk_z-r\eta_{zz}-\kappa r\eta_{rr}-\kappa\eta_r=0$ (2.24)

17) $-\eta_r + \kappa\xi_z + (1+\kappa)h = 0$ (2.25)

18) $f - k - \xi_r + \eta_z = 0$ (2.26)

19) $(1-\kappa)\left(g + (1-\kappa)\xi_z - \kappa h\right) - 2\kappa\eta_r = 0$ (2.27)

Integrating the determining equations, we get the following set of infinitesimal generators:

$$X_1 = r\partial_r + z\partial_z, \quad X_2 = u\partial_u + w\partial_w \tag{2.28}$$

3. SIMILARITY SOLUTION

The infinitesimal generator X can be written as follows

$$X = \alpha X_1 + \beta X_2 \tag{3.1}$$

where α, β are constants. If we apply (3.1) to $u = \theta(r,z)$ and $w = \psi(r,z)$ which are assumed to be solution of the homogeneous part of PDE (2.2) we reach the following PDE' s

$$X(u - \theta(r,z))\big|_{u=\theta} = 0, \quad X(w - \psi(r,z))\big|_{w=\psi} = 0 \tag{3.2}$$

which give the general solution as follows

$$\theta = r^{\gamma} f\left(\frac{z}{r}\right), \quad \psi = r^{\gamma} g\left(\frac{z}{r}\right) \tag{3.3}$$

where $\gamma = \dfrac{\alpha}{\beta}$

Now we define a new independent variable

$$s = \frac{z}{r} \tag{3.4}$$

If we rearrange $(2.2)_2$ taking

$$\delta(r) = \frac{1}{z}\delta\left(\frac{1}{s}\right), \quad \delta(z) = \frac{1}{r}\delta(s) \tag{3.5}$$

and use (3.3) we reach the following ordinary differential equation:

$$r^{\gamma-2}\{(\gamma-1)f'(s)-sf''(s)+g''(s)+f'(s)-\kappa[(\gamma-1)f'(s)-sf''(s)-\gamma(\gamma-1)g(s)+2(1-\gamma)sg'(s)-$$

$$s^2g''(s)+f'(s)-\gamma g(s)+sg'(s)]\}=-\frac{F}{\rho c_0^2}\frac{\delta\left(\frac{1}{s}\right)}{2\pi rz}\frac{1}{r}\delta(s) \tag{3.6}$$

(3.1) is a symmetry of homogeneous part of (2.4). From (3.6) it is seen that if

$$\gamma=-1 \tag{3.7}$$

the symmetry (3.1) is also a symmetry of non-homogeneous system (2.2). So

$$X=X_1-X_2 \tag{3.8}$$

is a symmetry of the non-homogeneous system. Hence we get the following similarity solutions:

$$u=\frac{1}{r}f\left(\frac{z}{r}\right),\quad w=\frac{1}{r}g\left(\frac{z}{r}\right) \tag{3.9}$$

Substituting (3.9) to (2.2) we reach the following system of ordinary differential equations

$$\begin{aligned}&\left(\kappa+s^2\right)f''(s)+s(\kappa-1)g''(s)+3sf'(s)+2(\kappa-1)g'(s)=0\\&s(\kappa-1)f''(s)+\left(1+\kappa s^2\right)g''(s)+(\kappa-1)f'(s)+3s\kappa g'(s)+\kappa g(s)=0\end{aligned} \tag{3.10}$$

Solving $f(s), g(s)$ from (3.10) and substituting into (3.9) we get the general form of the displacement components as follows:

$$u(r,z)=\frac{1}{r\left(r^2+z^2\right)^{3/2}}\Big(3Ar^2z-3A\kappa r^2z+M_2r^2z+2Az^3-2A\kappa z^3+M_2z^3-Br^2\sqrt{r^2+z^2}+$$

$$B\kappa r^2\sqrt{r^2+z^2}+M_1r^2\sqrt{r^2+z^2}+M_1z^2\sqrt{r^2+z^2}+$$

$$2B(\kappa-1)z\left(r^2+z^2\right)ArcSinh\left(\frac{z}{r}\right)-B(\kappa-1)z\left(3r^2+2z^2\right)Log\left(\frac{z}{r}+\sqrt{1+\frac{z^2}{r^2}}\right)\Big) \tag{3.11}$$

and

$$w(r,z)=\frac{1}{z\left(r^2+z^2\right)^{3/2}}\left(-Ar^2z+3A\kappa r^2z-M_2r^2z+2A\kappa z^3-M_2z^3+\right.$$

$$Br^2\sqrt{r^2+z^2}-B\kappa r^2\sqrt{r^2+z^2}+M_3r^2\sqrt{r^2+z^2}+M_3z^2\sqrt{r^2+z^2}-$$

$$\left.-2B(\kappa-1)z\left(r^2+z^2\right)ArcSinh\left(\frac{z}{r}\right)+Bz\left(-r^2+3\kappa r^2+2\kappa z^2\right)Log\left(\frac{z}{r}+\sqrt{1+\frac{z^2}{r^2}}\right)\right) \tag{3.12}$$

The stress components expressed in terms of displacement components are :

$$\sigma_z=\lambda\left(u_r+\frac{u}{r}\right)+(\lambda+2\mu)w_z$$

$$\tau_{zr}=2\mu\left(u_z+w_r\right) \tag{3.13}$$

If we rearrange (3.13) using (3.11) and (3.12) we find the stress components as follows:

$$\sigma_z=\frac{1}{z^2\left(r^2+z^2\right)^{3/2}}\left(-2A\kappa r^2z^3\lambda-2A\kappa z^5\lambda-Br^4\lambda\sqrt{r^2+z^2}+B\kappa r^4\lambda\sqrt{r^2+z^2}-M_3r^4\lambda\sqrt{r^2+z^2}-\right.$$

$$2Br^2z^2\lambda\sqrt{r^2+z^2}+4Br^2z^2\lambda\kappa\sqrt{r^2+z^2}-2M_3r^2z^2\lambda\sqrt{r^2+z^2}-Bz^3\lambda\sqrt{r^2+z^2}+$$

$$3B\kappa z^3\lambda\sqrt{r^2+z^2}-M_3z^4\lambda\sqrt{r^2+z^2}+6Ar^2z^3\mu-10A\kappa r^2z^3\mu+2M_2r^2z^3\mu-$$

$$4A\kappa z^5\mu+2M_2z^5\mu-2Br^4\mu\sqrt{r^2+z^2}+2B\kappa r^4\mu\sqrt{r^2+z^2}-2M_3r^4\mu\sqrt{r^2+z^2}-$$

$$4Br^2z^2\mu\sqrt{r^2+z^2}+8B\kappa r^2z^2\mu\sqrt{r^2+z^2}-4M_3r^2z^2\mu\sqrt{r^2+z^2}+4Bz^4\mu\sqrt{r^2+z^2}-$$

$$2M_3z^4\mu\sqrt{r^2+z^2}+4B(\kappa-1)z^3\left(r^2+z^2\right)\mu ArcSinh\left(\frac{z}{r}\right)-$$

$$\left.2Bz^3\left(\kappa r^2\lambda+\kappa z^2\lambda-3r^2\mu+5\kappa r^2\mu+2\kappa z^2\mu\right)Log\left(\frac{z}{r}+\sqrt{1+\frac{z^2}{r^2}}\right)\right) \tag{3.14}$$

and

$$\tau_{rz} = -\frac{4\mu}{\left(r^2+z^2\right)^{3/2}}\left(-2Ar^3 + 3A\kappa r^3 - M_2 r^3 + Arz^2 - M_2 rz^2 - 2Brz\sqrt{r^2+z^2} + 3B\kappa rz\sqrt{r^2+z^2} + \right.$$

$$B\frac{z^3}{r}\sqrt{r^2+z^2} - 2B(\kappa-1)r\left(r^2+z^2\right)ArcSinh\left(\frac{z}{r}\right) +$$

$$\left. Br\left(-2r^2+3\kappa r^2+z^2\right)Log\left(\frac{z}{r}+\sqrt{1+\frac{z^2}{r^2}}\right)\right) \tag{3.15}$$

The boundary conditions are

$$\lim_{z\to 0}\sigma_z = 0, \quad r \neq 0$$

$$\lim_{z\to 0}\tau_{rz} = 0, \quad r \neq 0 \tag{3.16}$$

which give

$$B = M_3 = 0, \quad M_2 = A(3\kappa - 2) \tag{3.17}$$

By the substitution of (3.17) into (3.11) and (3.12), the exact solution of the Boussinesq problem is obtained.

$$u(r,z) = \frac{M_1\left(r^2+z^2\right)^2 + A\,r\,z\left(r^2+\kappa z^2\right)\sqrt{1+\frac{z^2}{r^2}}}{r\left(r^2+z^2\right)^2}$$

$$w(r,z) = \frac{A\left(r^2-(\kappa-2)z^2\right)}{\left(r^2+z^2\right)^{3/2}} \tag{3.18}$$

where

$$A = \frac{(1-2\nu)(1+\nu)F}{2\pi E\kappa}$$

$$M_1 = \frac{(1-2\nu)(1+\nu)F}{2\pi E} \tag{3.19}$$

that is exactly identical with the classical result in literature (see [12], [15], [16]).

4.CONCLUSION

The solution of Boussinesq problem has been reached by using Lie groups. It is clear that the solutions of the problems we face in many fields can be calculated by this method. Lie groups method gives some advantages for solving problems. As it is seen here the system of partial differential equations have been converted to a system of ordinary differential equations by Lie groups and the solution has been found by using boundary conditions.

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Appendix

For $(2.6)_1$:

$$
\begin{aligned}
& f_{rr}u+2f_r u_r+f\left(-\kappa u_{zz}+(\kappa-1)w_{rz}+\frac{u}{r^2}-\frac{u_r}{r}\right)+g_{rr}w+2g_r w_r+gw_{rr} \\
& -2\xi_r\left(-\kappa u_{zz}+(\kappa-1)w_{rz}+\frac{u}{r}-\frac{u_r}{r}-\right)-u_r\xi_{rr}-2u_{zr}\eta_r-u_z\eta_{rr}+\frac{2u}{r^3}\xi-\frac{f_u}{r^2}-\frac{gw}{r^2}-\frac{u_r}{r^2}\xi+ \\
& +\frac{1}{r}\left(f_r u+fu_r+g_r w+gw_r-u_r\xi_r-u_z\eta_r\right)+h_{rz}u+h_r u_z+h_z u_r+hu_{rz}+k_{rz}w+ \\
& k_r w_z+k_z w_r+kw_{rz}-w_{rz}\xi_r-w_r\xi_{rz}-\eta_r\left(-\kappa w_{rr}+(\kappa-1)\left(u_{rz}+\frac{u_z}{r}\right)-\kappa\frac{w_r}{r}\right)- \\
& w_z\eta_{rz}-w_{rr}\xi_z-w_{rz}\eta_z+\kappa f_{zz}u+2\kappa f_z u_z+f\kappa u_{zz}+\kappa g_{zz}w+2\kappa g_z w_z+\kappa g+ \\
& \left[-\kappa w_{rr}(\kappa-1)\left(u_{rz}+\frac{u_z}{r}\right)-\kappa\frac{w_r}{r}\right]-2\kappa u_{zr}\xi_z-\kappa u_r\xi_{zz}-2\kappa u_{zz}\eta_z-\kappa u_z\eta_{zz}-\kappa h_{rz}- \\
& -\kappa h_r u_z-\kappa h_z u_r-\kappa hu_{rz}-\kappa k_{rz}w-\kappa k_r w_z-\kappa k_z w_r-\kappa kw_{rz}+\kappa w_{zr}\xi_r+\kappa w_r\xi_{rz}+ \\
& \kappa\eta_r\left[-\kappa w_{rr}(\kappa-1)\left(u_{rz}+\frac{u_z}{r}\right)-\kappa\frac{w_r}{r}\right]+\kappa w_z\eta_{rz}+\kappa w_{rr}\xi_z+\kappa w_{rz}\eta_z=0
\end{aligned}
$$

For $(2.6)_2$:

$$
\begin{aligned}
& f_{rz}u+f_r u_z+f_z u_r+fu_{zr}+g_{rz}w+g_r w_z+g_z w_r+gw_{zr}-u_{zr}\xi_{zr}-u_r\xi_{zr}-u_{zz}\eta_r- \\
& u_z\eta_{rz}-\xi_z\left(-\kappa u_{zz}+(\kappa-1)w_{rz}+\frac{u}{r^2}-\frac{u_r}{r}\right)-u_{rz}\eta_z+h_{zz}u+2h_z u_z+hu_{zz}+k_{zz}w+ \\
& +2k_z w_z+k\left[-\kappa w_{rr}+(\kappa-1)\left(u_{rz}+\frac{u_z}{r}\right)-\kappa\frac{w_r}{r}\right]-2w_{zr}\xi_r-w_r\xi_{zz}- \\
& -2\eta_z\left[-\kappa w_{rr}+(\kappa-1)\left(u_{rz}+\frac{u_z}{r}\right)-\kappa\frac{w_r}{r}\right]-w_z\eta_{zz}-\frac{u_z}{r^2}\xi+\frac{1}{r}\left(f_z u+fu_z+g_z w+gw_z-u_r\xi_z-u_z\eta_z\right)
\end{aligned}
$$

$$-\kappa\Big\{f_{rz}u + f_r u_z + f_z u_r + fu_{zr} + g_{rz}w + g_r w_z + g_z w_r + gw_{zr} - u_{zr}\xi_r - u_r\xi_{zr} - u_{zz}\eta_r - u_z\eta_{rz}$$

$$-\xi_z\left[-\kappa u_{zz} + (\kappa-1)w_{rz} + \frac{u}{r^2} - \frac{u_r}{r}\right] - u_{rz}\eta_z - h_{rr}u - 2h_r u_r - h\left(-\kappa u_{zz} + (\kappa-1)w_{rz} + \frac{u}{r^2} - \frac{u_r}{r}\right)$$

$$-k_{rr}w - 2k_r w_r - kw_{rr} + 2w_{rr}\xi_r + w_r\xi_{rr} + 2w_{zr}\eta_r + w_z\eta_{rr} + \frac{-u_z + w_r}{r^2}\xi +$$

$$\frac{1}{r}\left(f_z u + fu_z + g_z w + gw_z - u_r\xi_z - u_z\eta_z - h_r u - hu_r - k_r w - kw_r + w_r\xi_r + w_z\eta_r\right)\Big\} = 0$$