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ON THE THEORY OF GENERALIZED BALLISTICAL PROBLEM FOR NONLINEAR ABSTRACT DIFFERENTIAL EQUATIONS

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Abstract In the present work, we have obtained principally new results for solvability and non-solvability of one problem for second order abstract differential equations with the main linear part.

The method, based on solving corresponding Dirichlet problem, lies on the basis of present investigations.

In section 1, we present the formulation of the basic problem and reduce the short review of same investigations related to given problems.

In section 2, we show the roll of the corresponding subsidiary boundary value problem for linear equation in the study of the basic problem.

In section 3, we present the theorems of existence and nonexistence of the solution of the ballistical problem for nonlinear abstract differential equations with the main linear part, in the case, when the initial and the latter points of the desired trajectory are not coinciding.

Also we can prove the theorems of solvability and non-solvability of considering problem in the case, when the initial and the latter points of desired trajectory are coinciding.

1. INTRODUCTION

Let H be a Hilbert space with norm $\|\cdot\|_H$; let $C^{(K)}([0,T];H)$ be the space of all functions u(t) with values in H, these are continuous together with their all derivatives up to K-th order inclusive on $[0,T]$.

Let us consider the second-order differential equation in the space H

$$\ddot{u}(t) - A u(t) = P u(t), \quad t \in [0,T] \tag{1}$$

where A is a self-adjoint positive-definite operator in H, P is a nonlinear operator.

We want to find pair $(u(t), \tau)$ such that $\tau \in [0,T]$ and $u \in C^{(2)}([0,\tau];H)$, satisfies (1) for $t \in [0,\tau]$ with the boundary conditions

$$u(0) = a, \tag{2}$$

$$u(\tau) = b, \tag{3}$$

and the special condition

$$\left\| \dot{u}(0) \right\|_H = v, \tag{4}$$

where $v > 0$ is a given number; a and b are given elements of a linear manifold H' from the space H.

Problems of this kind occur in ballistics, optimum control theory and dynamics of charged particles.

The similar problem on the minimum of given functional on solutions of Dirichlet problem for partial differential equations of elliptic type is considered in [1] and other papers.

Problem (1)-(4) has been studied in the case of ordinary differential equations and functional-differential equations in [3]-[4], in the case of linear partial differential equations of elliptic type and in the case of linear abstract differential equations in [6]-[11].

Certain general questions of the theory of functional-differential equations and methods of investigations of problem (1)-(4) (method of dependent variables, method of independent variables) for separate classes of differential equations are given in [5].

We have obtained principally new results for solvability and non-solvability of problem (1)-(4).

The study of the properties of the solution of Dirichlet problem for linear non-homogeneous elliptic equation, depending on the boundary values of the variable "t", side by side with other propositions, lies on the basis of present investigations.

This allows us to establish the existence and nonexistence of the solutions of the special boundary value problem for linear abstract differential equations by the natural hypothesis on initial data of the problem without the application of the traditional fixed point theorems requiring a map of the concrete set into itself. Such questions also can be established for the nonlinear abstract differential equations with the main linear part, if one uses fixed point theorems [2].

2. REDUCTION OF PROBLEM TO THE SYSTEM OF OPERATOR EQUATIONS

The solution of the subsidiary linear equation

$$u(t) = Au(t) + \varphi(t) \tag{5}$$

satisfying the boundary conditions (2), (3) can be expressed by the formula

$$u(t) = P_0\varphi = \sum_{n=1}^{\infty} C_n(t;\tau)u_n, \tag{6}$$

where

$$C_n(t;\tau) = a_n \frac{\operatorname{sh}\sqrt{\lambda_n}(\tau - t)}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} + b_n \frac{\operatorname{sh}\sqrt{\lambda_n}\,t}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} - \int_0^\tau G_n(t,s;\tau)\varphi_n(s)\,ds,$$

$$G_n(t,s;\tau) = \frac{1}{\sqrt{\lambda_n}\operatorname{sh}\sqrt{\lambda_n}\,\tau}\begin{cases}\operatorname{sh}\sqrt{\lambda_n}\,t\operatorname{sh}\sqrt{\lambda_n}(\tau - s), & 0 \le t \le s\\ \operatorname{sh}\sqrt{\lambda_n}(\tau - t)\operatorname{sh}\sqrt{\lambda_n}\,s, & s \le t \le \tau\end{cases}$$

Here $a_n, b_n, \varphi_n(t)$ are associated Fourier coefficients of elements a, b, $\varphi(t)$ for system $\{u_n\}$ of eigen elements corresponding to eigen elements λ_n of operator A ([8]-[9]).

The transformation $u = P_0\varphi$ reduces problem (1)-(4) to the system of equations

$$\left.\begin{aligned}\varphi(t) &= (P_\tau\varphi)(t)\\ g_\upsilon(\varphi) &= \upsilon\end{aligned}\right\}, \tag{7}$$

where

$$(P_\tau\,\varphi)(t) = (P\,P_0\,\varphi)(t),$$

$$g_\tau(\varphi) = \left\{\sum_{n=1}^{\infty}\left[-a_n\sqrt{\lambda_n}\frac{\operatorname{ch}\sqrt{\lambda_n}\,\tau}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} + b_n\sqrt{\lambda_n}\frac{1}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} - \frac{1}{\operatorname{sh}\sqrt{\lambda_n}\,\tau}\int_0^\tau \operatorname{sh}\sqrt{\lambda_n}(\tau - s)\varphi_n(s)\,ds\right]^2\right\}^{\frac{1}{2}}.$$

The problem (1)-(4) is equivalent to the system (7) in the following mean.

If $(u(t),\tau)$ is the solution of problem (1)-(4), then $(\varphi(t),\tau)$ is the solution of (7), where $\varphi(t) = \ddot{u}(t) - Au(t)$; and if $(\varphi(t),\tau)$ is the solution of (7), then $(u(t),\tau)$ is the solution of problem (1)-(4), where $u(t) = (P_0\varphi)(t)$.

Consequently, the conditions, guaranteed the existence or nonexistence of solution of problem (5), (2)-(4) may be constituted the part of the sufficient conditions to solvability or non-solvability of problem (1)-(4).

3. THE THEOREMS IN THE CASE $a \neq b$

Theorem 1. Assume that the following conditions are satisfied:

1) A is a self adjoint positive-definite operator in Hilbert space and has a discrete spectrum;

2) $P: C^{(i)}([0,T];H) \to C^{(i)}([0,T];H)$ i=0,1 $(C^0 = C)$ and satisfies the Lipshitz condition: $\forall \bar{u}, u \in C([0,T];H)$

$$\left\| P\bar{u} - Pu \right\|_C \leq \theta \left\| \bar{u} - u \right\|_C , \theta > 0;$$

3) $a \neq b$; $a, b \in D(A)$;

4) For the number ξ, where $\xi \in (0,T]$, then the inequality

$$\upsilon - \|b\|_{H_A} - \|b-a\|_{H_A} \operatorname{cth}\sqrt{\lambda_1}\xi > 0,$$

holds,

Here λ_1 is the smallest eigenvalue of operator A, H_A is the energy space generated by operator A;

5) The Inequalities

$$\theta_0 = \frac{\theta T}{2\sqrt{\lambda_1}} < 1; \quad \frac{\|PP_0 0\|_C}{1-\theta_0} \leq \frac{\upsilon - \|b\|_{H_A} - \|b-a\|_{H_A} \operatorname{cth}\sqrt{\lambda_1}\,\xi}{\xi}$$

are satisfied for all $\tau \in (0,\xi]$ (or $\forall \tau \in (0,T]$),

where

$$P_0 0 = \left\{ \sum_{n=1}^{\infty} a_n \frac{\operatorname{sh}\sqrt{\lambda_n}(\tau - t)}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} + b_n \frac{\operatorname{sh}\sqrt{\lambda_n}\, t}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} \right\} u_n .$$

Then problem (1)-(4), where conditions (2) and (3) hold even in the sense of the norm in H_A, has at least one solution $(u(t);\tau)$, for which $u(t) \in C^{(2)}([0,\tau];H)$, $\tau \in [\tau_0, \xi]$,

where

$$\tau_0 = \frac{1}{2\sqrt{\lambda_k}} \ln \frac{k_0+1}{k_0-1}; \quad k_0 = \frac{2\upsilon - \|b-a\|_{H_A} \operatorname{cth}\sqrt{\lambda_1}\,\xi}{\sqrt{\lambda_k}\,\|b_k - a_k\|}.$$

$b_k - a_k$ is the first nonzero Fourier coefficient of b-a, for which according to the condition 3) is $b - a \neq 0$;

$$\xi \geq \frac{1}{2\sqrt{\lambda_1}} \ln \frac{k+1}{k-1} = T_0; \quad k = \frac{\upsilon - \|b\|_{H_A}}{\|b-a\|_{H_A}}.$$

Besides, there exists no solution $(u(t);\tau)$ of problem (1)-(4), for which $u(t) \in C^{(2)}([0,\tau];H)$, $\tau \in (0,\tau_0)$.

Theorem 2. The problem (1)-(4), where the conditions (2) and (3) hold even in the sense of the norm in H_A can not have a solution $(u(t);\tau)$: $u(t) \in C^{(2)}([0,\tau];u)$; $\tau \in (0,T]$ if:

1) The conditions 1)-3) of Theorem 1 are satisfied;

2) The Inequality

$$\|b-a\|_{H_A} - \|b\|_{H_A} - \upsilon \rangle 0$$

holds;

3) The Inequalities

$$\theta_0 = \frac{\theta T}{2\sqrt{\lambda_1}} \langle 1; \quad \frac{\|P P_0 O\|_C}{1-\theta_0} \langle \frac{\|b-a\|_{H_A} - \|b\|_{H_A} - \upsilon}{T}, \forall \tau \in (0,T]$$

hold, where

$$P_0 0 = \sum_{n=1}^{\infty} \left\{ a_n \frac{\operatorname{sh}\sqrt{\lambda_n}(\tau - t)}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} + b_n \frac{\operatorname{sh}\sqrt{\lambda_n}\,t}{\operatorname{sh}\sqrt{\lambda_n}\,\tau} \right\} u_n.$$

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