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A Perturbed Algorithm for Generalized Nonlinear Quasi-Variational Inclusions

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Abstract: In this paper, a perturbed iterative method for solving a generalized nonlinear quasi-variational inclusions, is presented and a convergence result which generalizes some known results in this field, is given.

1. INTRODUCTION

In 1994, Hassouni and Moudafi [4], have introduced a perturbed method for solving a new class of variational inclusions and presented a convergence result. In 1996, Samir Adly [2], has studied a perturbed iterative method in order to approximate a solution for a general class of variational inclusions and proved the convergence of the iterative algorithm by using some fixed point theorems.

The aim of this paper is, firstly to present a new iterative algorithm for solving a generalized nonlinear quasi-variational inclusions. Then we prove the convergence of this algorithm, by using the definition of multi-valued relaxed Lipschitz operators. Our result is more general than the one considered in [3,4,5,6,8,9,10] which motivated us for the present work.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$. Let $\phi : H \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\delta\phi$ be the subdifferential of ϕ. Given a multivalued map $T : H \to 2^H$, where 2^H denotes the family of nonempty subsets of H, and $f, g, m : H \to H$ be single-valued maps, then we consider the following *generalized nonlinear quasi-variational inclusions problem (GNQVIP)*:

(GNQVIP): Find $x \in H$, $w \in T(x)$ such that $g(x) \in \ dom\ (\delta\phi) + m(x)$, and

$$< g(x) - f(w),\ y + m(x) - g(x) > \geq \ \phi(g(x) - m(x)) - \phi(y), \quad \forall y \in H. \quad (2.1)$$

Inequality (2.1) is called *generalized nonlinear quasi-variational inclusion.*

It is clear that the *generalized nonlinear quasi-variational inclusion*(2.1), for the appropriate suitable choice of operators T, f, g and m, includes many kinds of variational inequalities and quasi-variational inequalities of [4,6,8,9,10], as special cases.

3. ITERATIVE ALGORITHM

To begin with, let us show the equivalence of the generalized nonlinear quasi-variational inclusion (2.1) to a nonlinear equation.

LEMMA 3.1: Elements $x \in H$ and $w \in T(x)$ are the solutions of (GN-QVIP) if and only if x and w satisfy the following relation

$$g(x) \ = \ m(x) + J_\alpha^\phi \left(g(x) - m(x) - \alpha\left(g(x) - f(w)\right)\right). \qquad (3.1)$$

where $\alpha > 0$ is a constant and $J_\alpha^\phi := (I + \alpha\delta\phi)^{-1}$ is the so-called proximal mapping on H, I stands for the identity operator on H.

PROOF: From the definition of J_α^ϕ , we have

$$g(x) - m(x) - \alpha(g(x) - f(w)) \in g(x) - m(x) + \alpha\delta\phi(g(x) - m(x)),$$

and hence

$$f(w) - g(x) \in \delta\phi(g(x) - m(x)).$$

This implies that $g(x) \in$ *dom* $(\delta\phi) + m(x)$ and by the definition of $\delta\phi$, we have

$$\phi(y) \geq \phi(g(x) - m(x)) + < f(w) - g(x),\ y + m(x) - g(x) >, \quad \forall y \in H.$$

Thus x and w are solutions of (GNQVIP).

To obtain an approximate solution of (2.1), we can apply a successive approximation method to the problem of solving

$$x = F(x) \tag{3.2}$$

where

$$F(x) = x - g(x) + m(x) + J_\alpha^\phi(g(x) - \alpha(g(x) - f(w)) - m(x)).$$

Based on (3.1) and (3.2), we suggest the following iterative algorithm.

ALGORITHM 3.1: Given $x_0 \in H$, compute x_{n+1} by the rule

$$x_{n+1} = x_n - g(x_n) + m(x_n) + J_\alpha^\phi(g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)). \tag{3.3}$$

for each $x \in N$, where $\alpha > 0$ is a constant.

To perturb scheme (3.3), first, we add in the righthand side of (3.3), an error e_n to take into account a possible in exact computation of the proximal point and we consider an other perturbation by replacing in (3.3) ϕ by ϕ_n, where the sequence $\{\phi_n\}$ approximates ϕ. Finally, we obtain the perturbed algorithm which generates from any starting point x_0 in H a sequence $\{x_n\}$ by the rule

$$x_{n+1} = x_n - g(x_n) + m(x_n) + J_\alpha^{\phi_n}(g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)) + e_n \tag{3.4}$$

our algorithm (3.4) is more general than the algorithms considered by Hassouni and Moudafi [4], Noor [6] and Siddiqi and Ansari [8].

4. CONVERGENCE THEORY

We need the following concepts and result to prove the main result of this paper.

DEFINITION 4.1: A mapping $g : H \to H$ is said to be

(i) *Strongly monotone* if there exists $r > 0$ such that

$$< g(x_1) - g(x_2),\ x_1 - x_2 > \geq r \parallel x_1 - x_2 \parallel^2, \quad \forall\, x_1, x_2 \in H,$$

(ii) *Lipschitz continuous* if there exists $s > 0$ such that

$$\parallel g(x_1) - g(x_2) \parallel \leq s \parallel x_1 - x_2 \parallel, \quad \forall\, x_1, x_2 \in H.$$

DEFINITION 4.2: Let $f : H \to H$ be a map. Then a multivalued map $T : H \to 2^H$ is said to be *relaxed Lipschitz with respect to* f if for given $k \leq 0$,

$$< f(w_1) - f(w_2),\ x_1 - x_2 > \leq k \parallel x_1 - x_2 \parallel^2, \quad \forall\, w_1 \in T(x_1) \text{ and } w_2 \in T(x_2),$$

$$\text{and} \quad \forall\, x_1, x_2 \in H.$$

The multivalued map T is called *Lipschitz continuous* if for $m \geq 1$,

$$\parallel w_1 - w_2 \parallel \leq m \parallel x_1 - x_2 \parallel, \quad \forall\, w_1 \in T(x_1) \text{ and } w_2 \in T(x_2), \text{ and } \forall\, x_1, x_2 \in H.$$

Lemma 4.1 [1]: Let ϕ be a proper convex lower semicontinuous function. Then $J_\alpha^\phi = (I + \alpha\delta\phi)^{-1}$ is nonexpansive, that is

$$\parallel J_\alpha^\phi(x) - J_\alpha^\phi(y) \parallel \leq \parallel x - y \parallel, \quad \forall\, x, y \in H.$$

Now we prove the following main result of this paper.

THEOREM 4.1: Let $g : H \to H$ be strongly monotone and Lipschitz continuous with corresponding constants $r > 0$ and $s > 0$; $f : H \to H$ be Lipschitz continuous with constant $t > 0$, and $m : H \to H$ be Lipschitz continuous with constant $\mu > 0$. Let $T : H \to 2^H$ be relaxed Lipschitz with respect to f and Lipschitz continuous with corresponding constants $k \leq 0$ and $m \geq 1$. Assume

$$\lim_{n \to +\infty} \parallel J_\alpha^{\phi_n}(y) - J_\alpha^\phi(y) \parallel = 0, \quad \text{for all } y \in H \text{ and } \lim_{n \to +\infty} \parallel e_n \parallel = 0\ ,$$

then the sequences $\{x_n\}$ and $\{w_n\}$, generated by (3.4) with $x_0 \in H$ and $w_0 \in T(x_0)$, and

$$\left| \alpha - \frac{1-k+p[1-2(p+\mu)]}{1-2k+t^2m^2-p^2} \right|$$
$$< \frac{\sqrt{[1-k+p(1-2(p+\mu))]^2 - 4(p+\mu)(1-(p+\mu))(1-2k+t^2m^2-p^2)}}{1-2k+t^2m^2-p^2}, \tag{4.1}$$

where $1-k > p(2(p+\mu)-1) + \sqrt{4(p+\mu)(1-(p+\mu))(1-2k+t^2m^2-p^2)}$, for $p = \sqrt{(1-2r+s^2)}$, converges strongly to x and w, respectively, the solution of (2.1).

PROOF: Using (3.2), we can write

$$x = x - g(x) + m(x) + J_\alpha^\phi(g(x)) - \alpha(g(x) - f(w)) - m(x)) \tag{4.2}$$

Denoting $h(x) = g(x) - \alpha(g(x) - f(w)) - m(x)$
and $h(x_n) = g(x_n) - \alpha(g(x_n) - f(w_n)) - m(x_n)$,

then we have

$$\begin{aligned} \| x_{n+1} - x \| \le \| x_n - x - (g(x_n) - g(x)) \| + \| m(x_n) - m(x) \| \\ \| J_\alpha^{\phi_n}(h(x_n)) - J_\alpha^\phi(h(x)) \| + \| e_n \| \end{aligned} \tag{4.3}$$

On the other hand, by introducing the term $J_\alpha^{\phi_n}(h(x))$, we get

$$\| J_\alpha^{\phi_n}(h(x_n)) - J_\alpha^\phi(h(x)) \| \le \| h(x_n) - h(x) \| + \| J_\alpha^{\phi_n}(h(x)) - J_\alpha^\phi(h(x)) \|$$

Since J_α^ϕ is nonexpansive.

Hence,

$$\begin{aligned} \| J_\alpha^{\phi_n}(h(x_n)) - J_\alpha^\phi(h(x)) \| \le (1-\alpha) \| x_n - x - (g(x_n) - g(x)) \| + \\ \| (1-\alpha)(x_n - x) + \alpha(f(w_n) - f(w)) \| + \| m(x_n) - m(x) \| + \\ \| J_\alpha^{\phi_n}(h(x)) - J_\alpha^\phi(h(x)) \| \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we get

$$\| x_{n+1} - x \| \leq (2-\alpha) \| x_n - x - (g(x_n) - g(x)) \| + 2 \| m(x_n) - m(x) \| +$$
$$\| (1-\alpha)(x_n - x) + \alpha(f(w_n) - f(w)) \| + \| J_\alpha^{\phi_n}(h(x) -$$
$$J_\alpha^{\phi}(h(x)) \| + \| e_n \| \tag{4.5}$$

By Lipschitz continuity and strong monotonocity of g, we obtain

$$\| x_n - x - (g(x_n) - g(x)) \|^2 \leq (1 - 2r + s^2) \| x_n - x \|^2 \tag{4.6}$$

Since T is Lipschitz continuous and relaxed Lipschitz with respect to f, and f is Lipschitz continuous, we have

$$\| (1-\alpha)(x_n - x) + \alpha(f(w_n) - f(w_{n-1})) \|^2 = (1-\alpha)^2 \| x_n - x \|^2 +$$
$$2\alpha(1-\alpha) < f(w_n) - f(w) , x_n - x > + \alpha^2 \| f(w_n) - f(w) \|^2 \leq$$
$$(1-\alpha)^2 \| x_n - x \|^2 + 2\alpha(1-\alpha)k \| x_n - x \|^2 + \alpha^2 t^2 m^2 \| x_n - x \|^2$$
$$((1-\alpha)^2 + 2\alpha(1-\alpha)k + \alpha^2 t^2 m^2) \| x_n - x \|^2 \tag{4.7}$$

Again, since m is Lipschitz continuous, we have

$$\| m(x_n) - m(x) \| \leq \mu \| x_n - x \| \tag{4.8}$$

By combining (4.5) to (4.8), we finally obtain

$$\| x_{n+1} - x \| \leq [(2-\alpha)p + 2\mu + \{(1-\alpha)^2 + 2\alpha(1-\alpha)k + \alpha^2 t^2 m^2\}^{1/2}] \| x_n - x \|,$$

where $p = (1 - 2r + s^2)^{1/2}$. Therefore

$$\| x_{n+1} - x \| \leq \theta \| x_n - x \| + \| J_\alpha^{\phi_n}(h(x) - J_\alpha^{\phi}(h(x)) \| + \| e_n \|,$$

where $\theta = (2-\alpha)p + 2\mu + \{(1-\alpha)^2 + 2\alpha(1-\alpha)k + \alpha^2 t^2 m^2\}^{1/2}$. It follows from (4.1) that $\theta < 1$.

By setting $\epsilon_n = \| J_\alpha^{\phi_n}(h(x)) - J_\alpha^{\phi}(h(x)) \| + \| e_n \|$, we can write

$$\| x_{n+1} - x \| \leq \theta \| x_n - x \| + \epsilon_n$$

Hence

$$\| x_{n+1} - x \| \leq \theta^{n+1} \| x_0 - x \| + \sum_{j=1}^{n} \theta^j \epsilon_{n+1-j}$$

By the assumption of Theorem, $\lim_{n\to\infty} \epsilon_n = 0$. Hence the sequence $\{x_n\}$ strongly converges to x (e.g.; see, Ortega and Rheinboldt [7]). Now the Lipschitz continuity of T implies that the sequence $\{w_n\}$ strongly converges to w. This completes the proof of the Theorem.

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