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Theoretical and Computational Results of a Memory-Type Swelling Porous-Elastic System

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Abstract: In this paper, we consider a memory-type swelling porous-elastic system. First, we use the multiplier method to prove explicit and general decay results to solutions of the system with sufficient regularities. These decay results are established under a very general assumption on the relaxation function and for suitable given data. We also perform several numerical tests to illustrate our theoretical decay results. Our results generalize and improve many earlier results in the literature.

Keywords: porous problem; viscoelastic; decay; convexity; finite element; Crank–Nicolson methods

AMS Subject Classification: 93D20; 35B40

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1. Introduction

There are many different types of clay soils that swell considerably when water is added or absorbed in them and shrink with the loss of the water. This type of soil is termed expansive [1]. When a soil deposit is loaded, for example, by a structure or man-made fill, deformation will occur. Normally, deformation downward is called settlement. Deformation may also be directed upward, which is then called heave. A common cause of heaving is that a building or pavement is constructed when the topsoil layer is relatively dry. Then, the structure covering the soil increases in water content due to capillarity action, and the soil swells.

Heave will result if the pressure exerted by the pavement or building is less than the swelling pressure. The heave is usually uneven and causes structural damage [2]. Documented evidence shows that problems associated with expansive soils are worldwide and occur in areas, such as South Africa, Australia, India, United States, South America, and the Middle East [3]. Swelling soils causes serious engineering problems. Estimates indicate that about 20–25% of land area in the United State is covered with such problematic soils with an accompanied economic loss of 5.5 to 7 billions USD in 2003 [4].

Hence, it is crucial to study the ways to annihilate or at least minimize such damage. The reader is referred to [5–13] for other details concerning swelling soil. As established by Ieşan [14] and simplified by Quintanilla [15], the basic field equations for the linear theory of swelling porous elastic soils are mathematically given by

$$\begin{cases} \rho_z z_{tt} = P_{1x} - G_1 + F_1 \\ \rho_u u_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1}$$

where the constituents z and u represent the displacement of the fluid and the elastic solid material, respectively. The positive constant coefficients ρ_z and ρ_u are the densities of each constituent. The functions (P_1, G_1, F_1) represent the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similar definition holds for (P_2, G_2, F_2) but acting on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{A} \begin{bmatrix} z_x \\ u_x \end{bmatrix}, \tag{2}$$

where a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number. The matrix A is positive definite in the sense that $a_1 a_3 \geq a_2^2$. Quintanilla [15] investigated (1) by taking

$$G_1 = G_2 = \xi(z_t - u_t), \;\; F_1 = a_3 z_{xxt}, \;\; F_2 = 0,$$

where ξ is a positive coefficient, with initial and homogeneous Dirichlet boundary conditions and obtained an exponential stability result. Similarly, Wang and Guo [16] considered (1) with initial and some mixed boundary conditions, taking

$$G_1 = G_2 = 0, \;\; F_1 = -\rho_z \gamma(x) z_t, \;\; F_2 = 0,$$

where $\gamma(x)$ is an internal viscous damping function with a positive mean. They used the spectral method to establish an exponential stability result. Ramos et al. [17] looked into the following swelling porous elastic soils

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, & \text{in } (0, L) \times \mathbb{R}_+ \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \gamma(t) g(u_t) = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{3}$$

and established an exponential decay rate provided that the wave speeds of the system are equal. Recently, Apalara [18] considered the following

$$\begin{aligned} &\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \qquad \text{in } (0,1) \times (0, \infty), \\ &\rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \int_0^t g(t-s) u_{xx}(x,s) ds = 0, \text{ in } (0,1) \times (0, \infty), \\ &z(x,0) = z_0(x),\ z_t(x,0) = z_1(x),\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in [0,1], \\ &u(0,t) = u(1,t) = z(0,t) = z(1,t) = 0, \qquad t \geq 0, \end{aligned} \tag{4}$$

where the relaxation function satisfies the condition

$$g'(t) \leq -\xi(t) g(t), \;\; t \geq 0 \tag{5}$$

and established a general decay result. For more results in porous elasticity system, porous-thermo-elasticity systems, Timoshenko system and other systems, we refer the reader to see [15,16,19–33].

In this paper, we consider the problem (4), where the solution is (z, u) such that z and u represent the displacement of the fluid and the elastic solid material. The positive constant coefficients ρ_u and ρ_z are the densities of each constituent. The coefficients a_1, a_2 and a_3 are positive constants satisfying specific conditions. We establish a new asymptotic behavior of the above swelling porous elastic system under the condition

$$g'(t) \leq -\xi(t) H(g(t)), \;\; t \geq 0, \tag{6}$$

where ξ and H are two functions that satisfy some conditions to be specified later. Precisely, in our paper:

- We consider Problem (4) and establish a general decay result with a wider class of relaxation functions than the one considered in the literature such that the one by

Apalara [18] is a special case of our class. Our result is obtained by using the multiplier method and some convexity properties.

- We produce some numerical experiments to illustrate the energy decay results, for this purpose, we develop a second-order numerical scheme to solve the problem (4) based on finite element discretization and the Crank–Nicolson method in time that has the property to be unconditionally stable.
- The result is significant to engineers and architects as it might help to attenuate the harmful effects of swelling soils swiftly.

It is worth mentioning that (6) was first introduced in [34]. The rest of this paper is organized as follows. In Section 2, we present some assumptions and material needed for our work. In Section 3, we state and establish some essential lemmas needed in our proof. Finally, we state and prove our main decay result in Section 4. We also, in Section 5, present some numerical tests to illustrate our decay results as well as our conclusions.

2. Assumptions

In this section, we state some assumptions needed in the proof of our results. Throughout this paper, c is used to denote a generic positive constant. For the relaxation function g, we assume the following:
(A) $g : [0,+\infty) \to (0,+\infty)$ is a C^1 nonincreasing function satisfying

$$g(0) > 0 \qquad \textit{and} \qquad 0 < \ell := \int_0^{+\infty} g(s)ds < a_3 - \frac{a_2^2}{a_1}. \tag{7}$$

In addition, there exists a C^1 function $H : (0,\infty) \longrightarrow (0,\infty)$, which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$, $r \le g(0)$, with $H(0) = H'(0) = 0$, such that

$$g'(t) \le -\xi(t)H(g(t)), \qquad \forall t \ge 0, \tag{8}$$

where ξ is a positive nonincreasing differentiable function.

Remark 1 ([35]). *If H is a strictly increasing and strictly convex C^2 function on $(0,r]$, with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$, which is a strictly increasing and strictly convex C^2 function on $(0,\infty)$. For instance, if $H(r) = a$, $H'(r) = b$, $H''(r) = c$, we can define $\overline{H}$, for $t > r$, by*

$$\overline{H} = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \tag{9}$$

Lemma 1. *The energy functional E, is defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + \rho_u u_t^2 + \left(a_3 - \int_0^t g(s)ds\right)u_x^2 + 2a_2 z_x u_x\right]dx + \frac{1}{2}(g \circ u_x)(t), \tag{10}$$

where

$$(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)|v(t) - v(s)|^2\, dsdx \tag{11}$$

satisfies

$$E'(t) = \frac{1}{2}g' \circ u_x - \frac{1}{2}g(t)\int_0^1 u_x^2 dx \le \frac{1}{2}(g' \circ u_x)(t) \le 0. \tag{12}$$

Proof. The routine multiplication of the first two equations of (4) by z_t and u_t, respectively, and then integrating by parts over $(0,1)$, leads to

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + \rho_u u_t^2 + a_3 u_x^2 + 2a_2 z_x u_x\right]dx - \int_0^1 u_{xt}\int_0^t g(t-s)u_x(s)dsdx = 0. \tag{13}$$

The estimation of the second term in the left hand side gives

$$\begin{aligned}&\int_0^1 u_{xt}\int_0^t g(t-s)(u_x(t)-u_x(s))dsdx-\int_0^t g(s)ds\int_0^1 u_x u_{tx}dx\\=&\frac{1}{2}\frac{d}{dt}g\circ u_x-\frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\int_0^1 u_x^2dx-\frac{1}{2}g'\circ u_x+\frac{1}{2}g(t)\int_0^1 u_x^2dx.\end{aligned}\tag{14}$$

Consequently, by substituting (14) into (13), we obtain (12). □

3. Technical Lemmas

In this section, we present and establish some essential lemmas needed for our work.

Lemma 2 ([35]). *There exist positive constants β_0 and t_1 such that*

$$g'(t)\le-\beta_0 g(t),\qquad \forall t\in[0,t_1].\tag{15}$$

Lemma 3 ([35]). *For all $u\in H_0^1([0,1])$,*

$$\int_0^1\left(\int_0^t g(t-s)|u_x(s)-u_x(t)|ds\right)^2dx\le C_\alpha(h\circ u_x)(t),\tag{16}$$

for any $0<\alpha<1$,

$$C_\alpha=\left(\int_0^\infty\frac{g^2(s)}{\alpha g(s)-g'(s)}ds\right)\qquad and \qquad h(t)=\alpha g(t)-g'(t).\tag{17}$$

It is worth mentioning that the constants C_α were first introduced in [36].

Lemma 4. *The functional*

$$F_1(t):=\rho_u\int_0^1 u_t u dx-\frac{a_2}{a_1}\rho_z\int_0^1 z_t u dx$$

satisfies, for any $\varepsilon_1>0$ and some constant a_0,

$$F_1'(t)\le-\frac{a_0}{2}\int_0^1 u_x^2dx+\varepsilon_1\int_0^1 z_t^2dx+\left(\rho_u+\frac{a_2^2\rho_z^2}{4\varepsilon_1 a_1^2}\right)\int_0^1 u_t^2dx+\frac{C_\alpha}{2a_0}(h\circ u_x)(t),\tag{18}$$

where $a_0=a_3-\frac{a_2^2}{a_1}-\left(\int_0^\infty g(s)ds\right)>0$.

Proof. Direct computations using integration by parts gives

$$F_1'(t)=-\left(a_3-\frac{a_2^2}{a_1}\right)\int_0^1 u_x^2dx+\rho_u\int_0^1 u_t^2dx-\frac{a_2\rho_z}{a_1}\int_0^1 z_t u_t dx+\int_0^1 u_x\int_0^t g(t-s)u_x(s)dsdx.\tag{19}$$

Applying Young's inequalities, we obtain, for $\varepsilon_1>0$,

$$-\frac{a_2\rho_z}{a_1}\int_0^1 z_t u_t dx\le\varepsilon_1\int_0^1 z_t^2dx+\frac{a_2^2\rho_z^2}{4\varepsilon_1 a_1^2}\int_0^1 u_t^2dx,\tag{20}$$

and for $\varepsilon_2>0$,

$$\begin{aligned}\int_0^1 u_x\int_0^t g(t-s)u_x(s)dsdx&=\int_0^t g(s)ds\int_0^1 u_x^2dx-\int_0^1 u_x\int_0^t g(t-s)(u_x(t)-u_x(s))dsdx\\&\le\left(\int_0^t g(s)ds+\frac{\varepsilon_2}{2}\right)\int_0^1 u_x^2dx+\frac{C_\alpha}{2\varepsilon_2}(h\circ u_x)(t).\end{aligned}\tag{21}$$

Combining (19)–(21), we obtain

$$F_1'(t) \le -\left(a_3 - \frac{a_2^2}{a_1} - \int_0^\infty g(s)ds - \frac{\varepsilon_2}{2}\right)\int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 z_t^2 dx + \left(\rho_u + \frac{a_2^2\rho_z^2}{\varepsilon_1 a_1^2}\right)\int_0^1 u_t^2 dx + \frac{C_\alpha}{2\varepsilon_2}(h \circ u_x)(t). \tag{22}$$

Using assumption (A), taking $\varepsilon_2 = a_0$, where $a_0 = a_3 - \frac{a_2^2}{a_1} - \int_0^\infty g(s)ds$, we obtain (18). □

Lemma 5. *Assume that (A) holds. Then, for any $t_1 > 0$, the functional*

$$F_2(t) := -\rho_u \int_0^1 u_t \int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, for any $\delta_2, \delta_3 > 0$,

$$\begin{aligned} F_2'(t) \le & -\frac{\rho_u c_0}{2}\int_0^1 u_t^2 dx + \delta_1 \int_0^1 u_x^2 dx + \delta_3 a_2^2 \int_0^1 z_x^2 dx \\ & + \left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right](h \circ u_x)(t), \end{aligned} \tag{23}$$

where $c_0 = \int_0^{t_1} g(s)ds$.

Proof. Differentiating F_2, taking into account (4), and using integrating by parts, we obtain

$$\begin{aligned} F_2'(t) = & -\rho_u \int_0^t g(s)ds \int_0^1 u_t^2 dx + a_3 \int_0^1 u_x \int_0^t g(t-s)(u_x(t) - u_x(s))dsdx \\ & - \int_0^1 \int_0^t g(t-s)u_x(s)ds \int_0^t g(t-s)(u_x(t) - u_x(s))dsdx \\ & - \rho_u \int_0^1 u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx + a_2 \int_0^1 z_x \int_0^t g(t-s)(u_x(t)-u_x(s))dsdx \\ = & -\rho_u \int_0^t g(s)ds \int_0^1 u_t^2 dx + \int_0^1 \left(\int_0^t g(t-s)(u_x(t)-u_x(s))\right)^2 dx \\ & + \left(a_3 - \int_0^t g(s)ds\right)\int_0^1 u_x \int_0^t g(t-s)(u_x(t)-u_x(s))dsdx \\ & + a_2 \int_0^1 z_x \int_0^t g(t-s)(u_x(t)-u_x(s))dsdx - \rho_u \int_0^1 u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx. \end{aligned} \tag{24}$$

Using Young's inequality and Lemma 3, for any $\delta_1 > 0$, we obtain

$$\left(a_3 - \int_0^t g(s)ds\right)\int_0^1 u_x \int_0^t g(t-s)(u_x(t)-u_x(s))dsdx \le \delta_1 \int_0^1 u_x^2(t)dx + \frac{cC_\alpha}{\delta_1}(h \circ u_x)(t). \tag{25}$$

Similarly, we can find, for any $\delta_2 > 0$,

$$
\begin{aligned}
&- \rho_u \int_0^1 u_t(t) \int_0^t g'(t-s)(z(t)-z(s))dsdx \\
&= \rho_u \int_0^1 u_t(t) \int_0^t h(t-s)(u(t)-u(s))dsdx - \rho_u \int_0^t u_t(t) \int_0^t \alpha g(t-s)(u(t)-u(s))dsdx \\
&\leq \delta_2 \int_0^1 u_t^2(t)dx + \rho_u^2 \frac{\left(\int_0^t h(s)\,ds\right)}{2\delta_2}(h \circ u)(t) + \rho_u^2 \frac{\alpha^2}{2\delta_2} \int_0^1 \left(\int_0^t g(t-s)(u(t)-u(s)ds\right)^2 dx \\
&\leq \delta_2 \int_0^1 u_t^2(t)dx + \frac{c}{2\delta_2}(h \circ u_x)(t) + \frac{\alpha^2 c C_\alpha}{2\delta_2}(h \circ u_x)(t) \\
&\leq \delta_2 \int_0^1 u_t^2(t)dx + \frac{c}{\delta_2}(1 + C_\alpha)(h \circ u_x)(t).
\end{aligned}
$$

Using Young's inequality and performing similar calculations as in (25), we obtain, for any $\delta_3 > 0$

$$
a_2 \int_0^1 u_x \int_0^t g(t-s)(u_x(t) - u_x(s))dsdx \leq \delta_3 a_2^2 \int_0^1 u_x^2 dx + \frac{C_\alpha}{\delta_3}(h \circ u_x)(t). \tag{26}
$$

Employing Lemma 3, we find

$$
\int_0^1 \left(\int_0^t g(t-s)(u_x(t) - u_x(s))ds\right)^2 dx \leq C_\alpha (h \circ u_x)(t). \tag{27}
$$

Combining all the above estimates, we find, for any $t \geq t_1$

$$
\begin{aligned}
F_2'(t) \leq &-\rho_u \left(\int_0^t g(s)ds - \delta_2\right) \int_0^1 u_t^2 dx + \delta_2 \int_0^1 u_x^2 dx + \delta_3 a_2^2 \int_0^1 z_x^2 dx \\
&+ \left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right](h \circ u_x)(t).
\end{aligned} \tag{28}
$$

By taking $\delta_2 = \frac{\rho_u c_0}{2}$, we obtain the desired inequality (23). □

Lemma 6. *Assume that (A) holds. Then, the functional*

$$
F_3(t) := -a_2 \int_0^1 (u z_t - z u_t) dx
$$

satisfies

$$
\begin{aligned}
F_3'(t) \leq &-\frac{a_2^2}{2\rho_u} \int_0^1 z_x^2 dx + \left(\frac{a_2^2}{\rho_z} + \frac{3a_1}{2\rho_u} + \frac{3}{2\rho_u}\left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u}\right)^2\right) \int_0^1 u_x^2 dx \\
&+ \frac{3C_\alpha}{2\rho_u}(h \circ u_x)(t).
\end{aligned} \tag{29}
$$

Proof. Direct computation gives

$$
\begin{aligned}
F_3'(t) = &-\frac{a_2^2}{\rho_u} \int_0^1 z_x^2 dx + \frac{a_2^2}{\rho_z} \int_0^1 u_x^2 dx + a_2 \left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u}\right) \int_0^1 u_x z_x dx \\
&- \frac{a_2}{\rho_u} \int_0^1 z_x \int_0^t g(t-s) u_x(s) dsdx.
\end{aligned} \tag{30}
$$

Using Young's inequality, we find

$$a_2\left(\frac{a_1}{\rho_z}-\frac{a_3}{\rho_u}\right)\int_0^1 u_x z_x dx$$
$$\leq \frac{a_2^2\eta_2}{2}\int_0^1 z_x^2 dx + \frac{1}{2\eta_2}\left(\frac{a_1}{\rho_z}-\frac{a_3}{\rho_u}\right)^2\int_0^1 u_x^2 dx.$$

Exploiting Young's inequality again, we obtain

$$\frac{a_2}{\rho_u}\int_0^1 z_x\int_0^t g(t-s)u_x(s)dsdx$$
$$= \frac{a_2}{\rho_u}\int_0^1 z_x\int_0^t g(t-s)(u_x(s)-u_x(t))dsdx + \frac{a_2}{\rho_u}\left(\int_0^t g(s)ds\right)\int_0^1 u_x z_x dx$$
$$\leq \frac{a_2^2\eta_2}{2}\int_0^1 z_x^2 dx + \frac{1}{2\eta_2\rho_u^2}\int_0^1\left(\int_0^t g(t-s)(|u_x(s)|-|u_x(t)|)ds\right)^2 dx$$
$$+ \frac{1}{2\eta_2\rho_u^2}\left(\int_0^t g(s)ds\right)^2\int_0^1 u_x^2 dx + +\frac{a_2^2\eta_2}{2}\int_0^1 z_x^2 dx$$
$$\leq a_2^2\eta_2\int_0^1 z_x^2 dx + \frac{C_\alpha}{2\eta_2\rho_u^2}(h\circ u_x)(t) + \frac{a_1}{2\eta_2\rho_u^2}\int_0^1 u_x^2 dx.$$

By taking $\eta_2 = \frac{1}{3\rho_u}$ and using the fact that $\int_0^t g(s)ds \leq a_1$, estimate (29) is established. □

Lemma 7. *The functional*

$$F_4(t) := -\rho_z\int_0^1 z_t z dx$$

satisfies, for any $\varepsilon_3 > 0$,

$$F_4'(t) \leq -\rho_z\int_0^1 z_t^2 dx + \left(a_1 + \frac{a_2^2}{\varepsilon_3}\right)\int_0^1 z_x^2 dx + \varepsilon_3\int_0^1 u_x^2 dx. \tag{31}$$

Proof. It is straight forward to see that

$$F_4'(t) = -\rho_z\int_0^1 z_t^2 dx + a_1\int_0^1 z_x^2 dx + a_2\int_0^1 z_x u_x dx.$$

Using Young's inequality and the fact that $a_1a_3 > a_2^2$, we obtain (31). □

Lemma 8. *Under the assumption (A), the functional*

$$F_5(t) := \int_\Omega\int_0^t p(t-s)|u_x(s)|^2 dsdx \tag{32}$$

satisfies, along the solution of Equation (4), the estimate

$$F_5'(t) \leq -\frac{1}{2}(g\circ u_x)(t) + 3a_1\int_0^1 u_x^2\, dx, \tag{33}$$

where $p(t) = \int_t^{+\infty} g(s)ds$.

Proof. Taking the derivative of F_5 and using the fact $p'(t) = -g(t)$, we have

$$\begin{aligned} F_5'(t) &= p(0)\int_0^1 |u_x(t)|^2 dx + \int_0^1\int_0^t p'(t-s)|u_x(s)|^2 dsdx \\ &= -\int_0^1\int_0^t g(t-s)|u_x(s)-u_x(t)|^2 dsdx \\ &- 2\int_0^1 u_x(t)\int_0^t g(t-s)(u_x(s)-u_x(t))dsdx + p(t)\int_0^1 |u_x(t)|^2 dx. \end{aligned} \tag{34}$$

Using Young's inequality, we have the following

$$\begin{aligned} &- 2\int_0^1 u_x(t)\int_0^t g(t-s)(u_x(s)-u_x(t))dsdx \\ &\leq 2\gamma\int_0^1 |u_x(s)|^2 dx + \frac{\int_0^t g(s)}{2\gamma}\int_0^1\int_0^t g(t-s)|u_x(s)-u_x(t)|^2 dsdx \\ &\leq 2\gamma\int_0^1 |u_x(s)|^2 dx + \frac{\int_0^\infty g(s)}{2\gamma}\int_0^1\int_0^t g(t-s)|u_x(s)-u_x(t)|^2 dsdx. \end{aligned} \tag{35}$$

Choosing $\gamma = a_1 > 0$ and using the fact that $p(t) \leq p(0) < a_1$, then estimate (33) is established. □

Lemma 9. *The functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) = \mu E(t) + \sum_{i=1}^{4}\mu_i F_i(t) \tag{36}$$

satisfies, for suitable choice of $\mu, \mu_1, \mu_2, \mu_3, \mu_4$ and for all $t \geq t_1$,

$$\mathcal{L}'(t) \leq -4a_1\int_0^1 u_x^2\, dx - \int_0^1 z_x^2\, dx - \int_0^1 u_t^2\, dx - \int_0^1 z_t^2\, dx + \frac{1}{4}(g\circ u_x)(t), \tag{37}$$

where $t_1 = g^{-1}(r)$, and

$$\mathcal{L}(t) \sim E(t). \tag{38}$$

Proof. By taking the derivative of the functional $\mathcal{L}$ and using the above estimates, with choosing $\eta_1 = \frac{1}{\mu_3}$, $\varepsilon_1 = \frac{1}{\mu_1}$, $\varepsilon_3 = \frac{1}{\mu_4}$, we find

$$\begin{aligned} \mathcal{L}'(t) \leq{}& \frac{\mu\alpha}{2}(g\circ u_x)(t) - \left(\mu_3\frac{a_2^2}{2\rho_u} - \mu_2\delta_3 a_2^2 - \mu_4\left[a_1 + a_2^2\mu_4\right]\right)\int_0^1 z_x^2 dx \\ &- \left(\mu_1\frac{a_0}{2} - \mu_2\delta_1 - \mu_3\left[\frac{a_2^2}{\rho_z} + \frac{3a_1}{2\rho_u} + \frac{3}{2}\left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u}\right)\right] - 1\right)\int_0^1 u_x^2 dx \\ &- \left(\frac{\mu_2\rho_z c_0}{2} - \mu_1\left[\rho_u + \frac{a_2^2\mu_1\rho_z^2}{4a_1^2}\right]\right)\int_0^1 u_t^2 dx \\ &- (\mu_4\rho_z - 1)\int_0^1 z_t^2 dx \\ &- \left(\frac{\mu}{2} - \mu_1\frac{C_\alpha}{2a_0} - \mu_2\left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right] - \frac{3\mu_3}{2\rho_u}C_\alpha\right)(h\circ u_x)(t) \end{aligned} \tag{39}$$

First, we choose μ_4 so that $\mu_4\rho_z > 1$. Then, we select μ_3 large enough such that

$$\beta_1 := \mu_3\frac{a_2^2}{2} - \mu_4\left[a_3 + a_2^2\mu_4\right] > 1, \tag{40}$$

and then we choose μ_1 large enough that

$$\beta_2 := \mu_1 \frac{a_0}{2} - \mu_3 \left[\frac{a_2^2}{\rho_z} + \frac{3a_1}{2\rho_u} + \frac{3}{2}\left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right) \right] - 1 > 4a_1. \tag{41}$$

After fixing μ_1, we select μ_2 large enough such that

$$\beta_3 := \frac{\mu_2 \rho_z c_0}{2} - \mu_1 \left[\rho_u + \frac{a_2^2 \mu_1}{4a_1^2} \rho_z^2 \right] > 1. \tag{42}$$

Now, for any fixed $\mu_1, \mu_2, \mu_3, \mu_4$, we pick $\delta_3 < \frac{\beta_1}{2\mu_2 a_2^2}$, $\delta_1 < \frac{\beta_2}{2\mu_2}$, so that the following estimates are satisfied

$$\mu_3 \frac{a_2^2}{2} - \mu_2 \delta_3 a_2^2 - \mu_4 \left[a_3 + a_2^2 \mu_4 \right] > 1. \tag{43}$$

$$\mu_1 \frac{a_0}{2} - \mu_2 \delta_1 - \mu_3 \left[\frac{a_2^2}{\rho_z} + \frac{3a_1}{2\rho_u} + \frac{3}{2}\left(\frac{a_1}{\rho_z} - \frac{a_3}{\rho_u} \right) \right] - 1 > 4a_1. \tag{44}$$

Since $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$, using the Lebesgue dominated convergence theorem, we can find

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \, ds \longrightarrow 0, \qquad \text{as} \qquad \alpha \to 0. \tag{45}$$

Hence, there exists some $0 < \alpha^* < 1$, such that if $\alpha < \alpha^*$, then

$$\alpha C_\alpha < \frac{1}{8\left(\frac{\mu_1}{2a_0} + \mu_2 \left[\frac{c}{\delta_1} + \frac{c}{\delta_2} + \frac{1}{\delta_3} + 1 \right] + \mu_3 \frac{3}{2\rho_u} \right)}. \tag{46}$$

By putting $\alpha = \frac{1}{2\mu}$ and choosing μ sufficiently large such that

$$\frac{\mu}{4} - \frac{c\mu_2}{\delta_2} > 0, \tag{47}$$

which gives

$$\frac{\mu}{2} - \frac{c\mu_2}{\delta_2} - C_\alpha \left(\frac{\mu_1}{2a_0} + \mu_2 \left[\frac{c}{\delta_1} + \frac{c}{\delta_2} + \frac{1}{\delta_3} + 1 \right] + \frac{3\mu_3}{2\rho_u} \right) > 0. \tag{48}$$

Hence, we conclude that (37) holds. Moreover, we can choose μ even larger (if needed) so that (38) is satisfied, which means that, for some constants $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t).$$

□

4. The Main Result

In this section, we state and prove our main result and give some examples to illustrate our theoretical result.

Theorem 1. *Assume (A) holds. Then, there exist positive constants λ_1 and λ_2 such that the energy of (4) satisfies, for all $t > t_1 = g^{-1}(r)$,*

$$E(t) \le \lambda_2 H_1^{-1} \left(\lambda_1 \int_{g^{-1}(r)}^t \xi(s) ds \right), \tag{49}$$

where $H_1(t) = \int_t^r \frac{1}{sH'(s)} ds$, *is strictly decreasing and convex on* $(0, r]$, *with* $\lim_{t \to 0} H_1(t) = +\infty$.

Proof. Let the functional L defined by

$$L(t) = \mathcal{L}(t) + F_5(t). \tag{50}$$

By using Lemma 8, there exists a positive constant γ such that

$$L'(t) \le -a_1 \int_0^1 u_x^2\, dx - \int_0^1 z_x^2\, dx - \int_0^1 u_t^2\, dx - \int_0^1 z_t^2\, dx - \frac{1}{4}(g \circ u_x)(t) \le -\gamma E(t). \tag{51}$$

Then,

$$\gamma \int_0^t E(s)\, ds \le -L(t) + L(0) \le L(0). \tag{52}$$

Consequently,

$$\int_0^\infty E(s)\, ds < \infty. \tag{53}$$

Making use of (10), (12) and (37), and exploiting Lemma 2, we find, for a positive constant γ_0

$$\mathcal{F}'(t) \le -\gamma_0 E(t) + c \int_{t_1}^t g(t-s) \int_0^1 |u_x(t) - u_x(s)|^2\, dxds, \tag{54}$$

where $\mathcal{F}(t) = \mathcal{L}(t) + cE(t)$.

Now, we estimate the second term in the right-hand side of (54). For that, we distinguish two cases :
Case 1: H is linear. We multiply (54) by $\xi(t)$ and use hypothesis (A) to obtain for a positive constants γ_1,

$$\begin{aligned} \xi(t)\mathcal{F}'(t) &\le -\gamma_1 \xi(t)E(t) + c\xi(t) \int_{t_1}^t g(t-s) \int_0^1 |u_x(t) - u_x(s)|^2 dxds \\ &\le -\gamma_1 \xi(t)E(t) + c \int_{t_1}^t \xi(s) g(t-s) \int_0^1 |u_x(t) - u_x(s)|^2 dxds \\ &\le -\gamma_1 \xi(t)E(t) - c \int_{t_1}^t g'(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 dxds \\ &\le -\gamma_1 \xi(t)E(t) - cE'(t). \end{aligned} \tag{55}$$

Using the fact $\xi'(t) \le 0$, the functional $\Psi := \xi\mathcal{F} + cE$ satisfies $\Psi(t) \sim E(t)$ and

$$\Psi'(t) \le -\gamma_1 \xi(t)E(t) \le -\gamma_2 \xi(t)\Psi(t), \qquad \forall t \ge t_1. \tag{56}$$

Then, after integration over (t_1, t), we have

$$\Psi(t) \le c_1\, e^{-c_2 \int_{t_1}^t \xi(s)\, ds}, \qquad \forall t \ge t_1. \tag{57}$$

Then, using the fact that $\Psi(t) \sim E(t)$, (49) is established.
Case 2: H is nonlinear. Due to estimate (53), we can choose a constant $0 < q < 1$ so that the function I defined by

$$0 < I(t) := q \int_{t_1}^t \int_0^1 |u_x(t) - u_x(s)|^2\, dxds < 1, \qquad \forall t \ge t_1. \tag{58}$$

Let us define

$$\chi(t) := -\int_{t_1}^{t} g'(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 dxds. \tag{59}$$

By (12), we find that $\chi(t) \leq -cE'(t)$. Since H is strictly convex on $(0,r]$ and $H(0) = 0$, then

$$H(\theta z) \leq \theta H(z), \qquad 0 \leq \theta \leq 1, \qquad \text{and } z \in (0,r]. \tag{60}$$

Using (58), assumption (A) and Jensen's inequality leads to

$$\begin{aligned} \chi(t) &= \tfrac{1}{qI(t)} \int_{t_1}^{t} I(t)(-g'(s))q \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds \\ &\geq \tfrac{1}{qI(t)} \int_{t_1}^{t} I(t)\xi(s)H(g(s)) \, q \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds \\ &\geq \tfrac{\xi(t)}{qI(t)} \int_{t_1}^{t} H(I(t)g(s)) \, q \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds \\ &\geq \tfrac{\xi(t)}{q} H\Big(\tfrac{1}{I(t)} \int_{t_1}^{t} I(t)g(s) \, q \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds\Big) \\ &= \tfrac{\xi(t)}{q} H\Big(q \int_{t_1}^{t} g(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds\Big) \\ &= \tfrac{\xi(t)}{q} \overline{H}\Big(q \int_{t_1}^{t} g(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds\Big), \end{aligned} \tag{61}$$

where $\overline{H}$ is an extension of H such that $\overline{H}$ is strictly increasing and strictly convex C^2 function on $(0,\infty)$. Hence, we obtain

$$\int_{t_1}^{t} g(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 \, dx \, ds \leq \frac{1}{q} \overline{H}^{-1}\left(\frac{q\chi(t)}{\xi(t)}\right). \tag{62}$$

Therefore, (54) becomes

$$\mathcal{F}'(t) \quad \leq \quad -\sigma_1 E(t) + c\overline{H}^{-1}\left(\frac{q\chi(t)}{\xi(t)}\right), \qquad \forall t \geq t_1. \tag{63}$$

Now, for $\epsilon < r$, let

$$\mathcal{F}_1(t); = \overline{H}'\left(\epsilon \frac{E(t)}{E(0)}\right)\mathcal{F}(t) + E(t), \tag{64}$$

which is equivalent to E. Using the fact that $E' \leq 0$, $\overline{H}' > 0$, $\overline{H}'' > 0$, then (63) converts to

$$\begin{aligned} \mathcal{F}_1'(t) &= \epsilon \tfrac{E'(t)}{E(0)} \overline{H}''\Big(\epsilon \tfrac{E(t)}{E(0)}\Big)\mathcal{F}(t) + \overline{H}'\Big(\epsilon \tfrac{E(t)}{E(0)}\Big)\mathcal{F}'(t) + E'(t) \\ &\leq -\sigma_1 E(t)\overline{H}'\Big(\epsilon \tfrac{E(t)}{E(0)}\Big) + c\overline{H}'\Big(\epsilon \tfrac{E(t)}{E(0)}\Big)\overline{H}^{-1}\Big(\tfrac{q\chi(t)}{\xi(t)}\Big) + E'(t). \end{aligned} \tag{65}$$

Due to the argument given in (Arnold [37] (pp. 61–64)), we obtain

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}\Big((\overline{H}')^{-1}(s)\Big), \qquad \text{if } s \in (0, \overline{H}'(r)], \tag{66}$$

where $\overline{H}^*$ be the conjugate of $\overline{H}$ in the sense of Young, and $\overline{H}^*$ satisfies the following Young's inequality

$$AB \leq \overline{H}^*(A) + \overline{H}(B), \qquad \text{if } A \in (0, \overline{H}'(r)], \; B \in (0,r]. \tag{67}$$

Thus, with $A = \overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right)$ and $B = \overline{H}^{-1}\left(\frac{q\chi(t)}{\xi(t)}\right)$, using (12) and (65)–(67), we arrive at

$$\begin{aligned} \mathcal{F}_1'(t) &\le -\sigma_1 E(t)\overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right) + c\overline{H}^*\left(\overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right)\right) + c\frac{q\chi(t)}{\xi(t)} + E'(t) \\ &\le -\sigma_1 E(t)\overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right) + c\epsilon\frac{E(t)}{E(0)}\overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right) + c\frac{q\chi(t)}{\xi(t)} + E'(t). \end{aligned} \tag{68}$$

Using the fact that $\epsilon\frac{E(t)}{E(0)} < r$, then $\overline{H}'\left(\epsilon\frac{E(t)}{E(0)}\right) = H'\left(\epsilon\frac{E(t)}{E(0)}\right)$ and multiplying (68) by $\xi(t)$, we find

$$\begin{aligned} \xi(t)\mathcal{F}_1'(t) &\le -\sigma_1\xi(t)E(t)H'\left(\epsilon\frac{E(t)}{E(0)}\right) + c\epsilon\xi(t)\frac{E(t)}{E(0)}H'\left(\epsilon\frac{E(t)}{E(0)}\right) + cq\chi(t) + \xi(t)E'(t) \\ &\le -\sigma_1\xi(t)E(t)H'\left(\epsilon\frac{E(t)}{E(0)}\right) + c\epsilon\xi(t)\frac{E(t)}{E(0)}H'\left(\epsilon\frac{E(t)}{E(0)}\right) - cE'(t). \end{aligned} \tag{69}$$

Hence, by taking $\mathcal{F}_2(t) = \xi(t)\mathcal{F}_1(t) + cE(t)$, we find for some positives constants $b_1, b_2 > 0$,

$$b_1\mathcal{F}_2(t) \le E(t) \le b_2\mathcal{F}_2(t). \tag{70}$$

Consequently, with a suitable choice of ϵ, we obtain, for some constant $\sigma > 0$,

$$\mathcal{F}_2'(t) \le -\sigma\xi(t)\left(\frac{E(t)}{E(0)}\right)H'\left(\epsilon\frac{E(t)}{E(0)}\right) = -\sigma\xi(t)H_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \ge t_1. \tag{71}$$

where $H_2(t) = tH'(\epsilon t)$. Since $H_2'(t) = H'(\epsilon t) + \epsilon tH''(\epsilon t)$, then, making use of the strict convexity of H on $(0, r]$, we find that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \frac{b_1\mathcal{F}_2(t)}{E(0)} \sim E(t). \tag{72}$$

By (71), we find for some positive constant $\vartheta_1 > 0$,

$$R'(t) \le -\vartheta_1\xi(t)H_2(R(t)), \qquad \forall t \ge t_1. \tag{73}$$

Then, by integration over (t_1, t) yields

$$\int_{t_1}^{t} \frac{-R'(t)}{H_2(R(t))}\, ds \ge \vartheta_1 \int_{t_1}^{t} \xi(s)\, ds, \tag{74}$$

which gives

$$\int_{\epsilon R(t)}^{\epsilon R(t_1)} \frac{1}{sH'(s)}\, ds \ge \vartheta_1 \int_{t_1}^{t} \xi(s)\, ds. \tag{75}$$

Hence, let us define a function $H_1(t) = \int_t^r \frac{1}{sH'(s)}\, ds$, using the properties of H, the fact H_1 is strictly decreasing function on $(0, r]$ and $\lim_{t\to 0} H_1(t) = +\infty$, we obtain

$$R(t) \le \frac{1}{\epsilon}H_1^{-1}\left(\vartheta_1 \int_{t_1}^{t} \xi(s)\, ds\right). \tag{76}$$

Consequently, from (72) and (76), we obtain the desired result (49). □

Example 1. *(1) Let $g(t) = ae^{-bt}$, $t \ge 0$, a, $b > 0$ are constants, and a is chosen so that (A) is satisfied; then*

$$g'(t) = -bH(g(t)) \qquad \text{with} \qquad \xi(t) = b \qquad \text{and} \qquad H(s) = s.$$

Thus, under the assumptions of Theorem 1, we conclude that the solution of (4) satisfies, for two constants $d_1, d_2 > 0$, the energy estimate

$$E(t) \leq d_1 e^{-d_2 t}, \qquad \forall\, t > t_1.$$

(2) *For $g(t) = ae^{-(1+t)^\nu}$, for $t \geq 0$, $0 < \nu < 1$, and a is chosen so that condition (A) is satisfied, then*

$$g'(t) = -\xi(t)H(g(t)) \qquad \text{with} \qquad \xi(t) = \nu(1+t)^{\nu-1} \qquad \text{and} \qquad H(s) = s.$$

Thus, under the assumptions of Theorem 1, we conclude that the solution of (4) satisfies, for some constant $C > 0$, the energy estimate

$$E(t) \leq Ce^{-c(1+t)^\nu}, \qquad \text{when} \quad t \ \text{is large enough.}$$

(3) *Consider the following relaxation function,*

$$g(t) = \frac{a}{(1+t)^\nu}, \qquad t \geq 0,$$

for $\nu > 1$, and a is chosen so that hypothesis (A) remains valid. Then

$$g'(t) = -bH(g(t)) \qquad \text{with} \qquad \xi(t) = b \qquad \text{and} \qquad H(s) = s^p,$$

where b is a fixed constant, $p = \frac{1+\nu}{\nu}$, which satisfies $1 < p < 2$. Thus, under the assumptions of Theorem 1, we conclude that the solution of (4) satisfies, for some constant $C > 0$ and $t_1 > 0$, the energy estimate

$$E(t) \leq \frac{C}{(1+t)^\nu}, \qquad \forall\, t > t_1.$$

5. Numerical Tests

In this section, numerical experiments are performed to illustrate the energy decay results in Theorem 1. For this purpose, we developed a second-order numerical scheme to solve the problem (4) based on finite element discretization and the Crank–Nicolson method in time that has the property of being unconditionally stable.

The spatial interval $(0, L) = (0, 1)$ is subdivided into 100 sub-intervals, where the temporal interval $(0, T) = (0, 1000)$ is subdivided into N with a time step $\Delta t = T/N$. We ran our code for N time steps $(N = T/\Delta t)$ using the following initial conditions:

$$z_0(x) = \sin(\pi x), \quad u_0(x) = 2\sin(\pi x),$$

$$z_1(x) = u_1(x) = \sin(\frac{\pi}{2}x)$$

The numerical tests are done in the light of the Example 1 as follows

- **Test 1:** For the first numerical test, we choose the following entries:

$$\rho_z = \rho_u = a_1 = a_2 = 1, \quad a_3 = 2,$$

and the relaxation function

$$g(t) = e^{-2t}, \quad t \geq 0.$$

Thus, under the assumptions of Theorem 1, the solution of (4) satisfies the energy estimate

$$E(t) \leq E_h(t) = Ce^{-\frac{1}{5}t}, \quad \forall t > t_1$$

where C is a constant depends on the energy at $t = 0$.

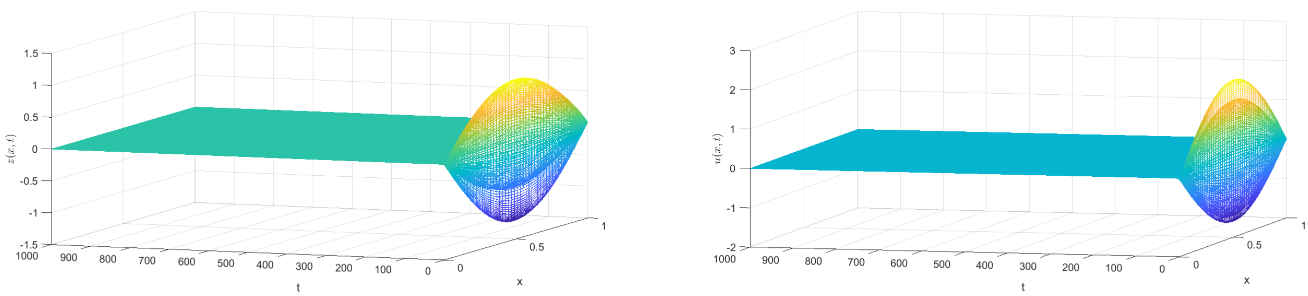
- **Test 2:** In the second numerical test, we consider the following entries so that condition (A) is satisfied
$$\rho_z = \rho_u = a_1 = a_2 = 1, \quad a_3 = 3$$
and the following relaxation function
$$g(t) = \frac{1}{3}e^{-\sqrt{1+t}}, \quad t \geq 0$$
Then, under the assumptions of Theorem 1, the solution of (4) satisfies the energy estimate
$$E(t) \leq E_h(t) = Ce^{-\sqrt{1+t}}, \quad \text{for a } t \text{ large enough}$$
where C is a constant depends on the energy at $t = 0$.
- **Test 3:** For last test, we consider the third case of Example 1 with the same entries of Test 1 and with an polynomial relaxation function
$$g(t) = \frac{1}{(1+t)^3}, \quad t \geq 0.$$
Under the assumptions of Theorem 1, the solution of (4) satisfies the energy estimate
$$E(t) \leq E_h(t) = C\frac{1}{(1+t)^3}, \quad \forall t > t_1,$$
where C is a constant depends on the energy at $t = 0$.

For Test 1, we examine the exponential decay case. Under the initial and boundary conditions above, we plot in Figure 1 the decay behavior of the solution (z, u) in time and space. In Figure 2, we plot the energy decay and we made a zoom on the time interval $[950, 1000]$ to show the difference between the curves of $E(t)$ and $E_h(t)$ and according to the parameters chosen in this test the time t_1 is too small.

Next, by following the same process, we present the numerical results of Test 2 in Figures 3 and 4. We observed that the energy decay satisfies the energy inequality in Test 2, when the time is large enough that $t \geq 840$ (as we can observe in the zoomed in part of Figure 4).

Finally, Figures 5 and 6 show the results of Test 3, which are demonstrated the damping behavior. Furthermore, the energy inequality is satisfied for $t_1 = 120$ (as we can observe on the zoom part of Figure 6).

As a conclusion, we observed that the energy decay uniformly for all tests and satisfies the results of Theorem 1.



(a) **(b)**

Figure 1. Test 1: The approximate solution (z, u) in the x-t plane. (**a**) $z(x, t)$. (**b**) $u(x, t)$.

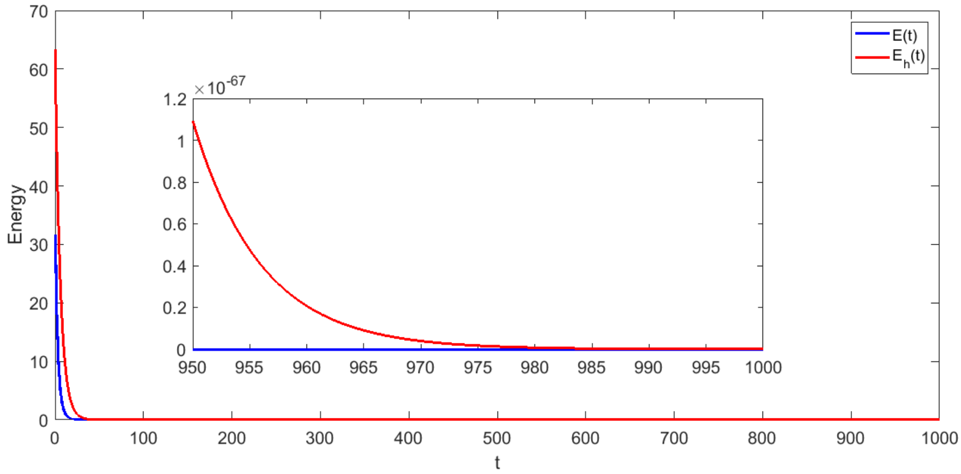


Figure 2. Test 1: energy decay.

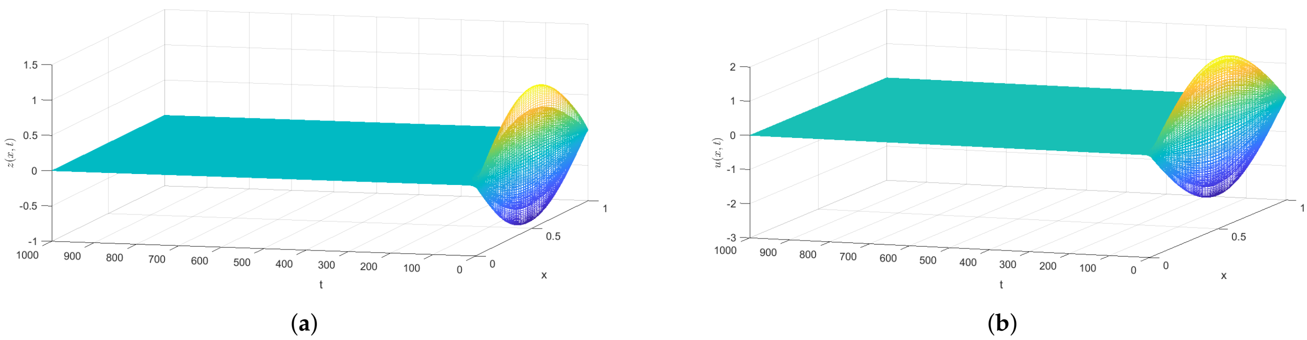


Figure 3. Test 2: The approximate solution (z, u) in the x-t plane. (**a**) $z(x, t)$. (**b**) $u(x, t)$.

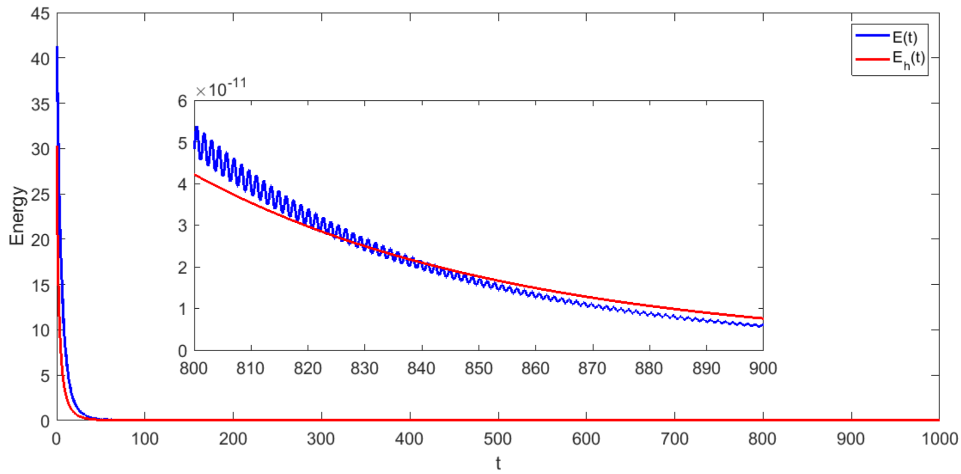
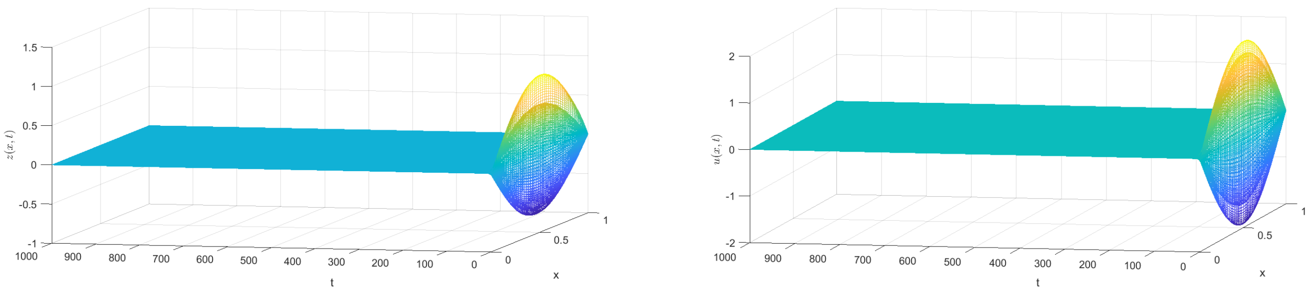


Figure 4. Test 2: energy decay.



(**a**) (**b**)

Figure 5. Test 3: The approximate solution (z, u) in the x-t plane. (**a**) $z(x, t)$. (**b**) $u(x, t)$.

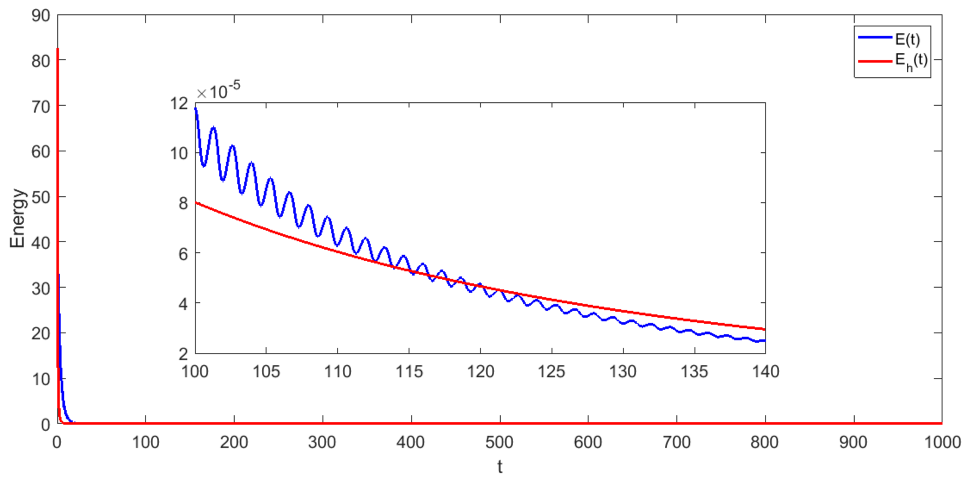


Figure 6. Test 3: energy decay.

6. Conclusions

In this work, we considered a swelling porous elastic soil system with a viscoelastic damping term. We obtained a general decay result with a large class of the relaxation functions associated with the memory term. We also preformed numerical tests to justify our theoretical decay result. Our result generalizes the one of [18] and gives a sharper decay rate in specific cases.

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