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On the Modeling of Five-Layer Thin Prismatic Bodies

Mikhail U. Nikabadze [1], Armine R. Ulukhanyan [2,*], Tamar Moseshvili [3], Ketevan Tskhakaia [3], Nodar Mardaleishvili [3] and Zurab Arkania [3]

1 Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, 119991 Moscow, Russia
2 Department of Computational Mathematics and Mathematical Physics, Bauman Moscow State Technical University, 105005 Moscow, Russia
3 Mechanics and Mechanical Engineering Department, Akaki Tsereteli State University, 4600 Kutaisi, Georgia
* Correspondence: armine_msu@mail.ru

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Abstract: Proceeding from three-dimensional formulations of initial boundary value problems of the three-dimensional linear micropolar theory of thermoelasticity, similar formulations of initial boundary value problems for the theory of multilayer thermoelastic thin bodies are obtained. The initial boundary value problems for thin bodies are also obtained in the moments with respect to systems of orthogonal polynomials. We consider some particular cases of formulations of initial boundary value problems. In particular, the statements of the initial-boundary value problems of the micropolar theory of K-layer thin prismatic bodies are considered. From here, we can easily get the statements of the initial-boundary value problems for the five-layer thin prismatic bodies.

Keywords: new parameterization; multilayer body; thin prismatic body; micropolar theory; orthogonal polynomials

1. Introduction

Any problem of the theory of a thin body can be considered in a three-dimensional formulation, which is more accurate than a two-dimensional one. However, it is not always possible to realize this treatment in practice due to the high complexity of solving three-dimensional problems and a wide variety of formulations of problems that are practically necessary. Thus, it is necessary to create new exact theories of thin bodies within the framework of the classical as well as the micropolar theories of thin bodies and improved methods for their calculation. Therefore, the construction of refined theories of thin bodies and the development of effective methods for their calculation are important and urgent problems.

In this work, the new parameterization of a multilayer thin domain is applied [1–3]. In contrast with classic approaches, several base surfaces and an analytic method with application of orthogonal polynomial systems are used. Geometric characteristics typical for the proposed parameterizations are introduced into consideration. The new parameterization in the case of a one-layered thin body is described in detail in [3,4]. Various representations of the equations of motion and the constitutive relations of physical content are given under the new parameterization of the body domain. Various variants of the equations of motion in moments with respect to orthogonal polynomial systems are obtained. The equations in moments of displacement and rotation vectors with respect to any system of orthogonal polynomials for the micropolar theory of prismatic multilayer elastic bodies of constant thickness are given. The interlayer conditions are written down under various connections of adjacent layers of a multilayer body. The formulations of initial-boundary value problems in the theory of multilayer elastic thin bodies in moments are discussed.

It should be noted that the analytic method with the use of the Legendre polynomial system in constructing the one-layer thin body theory [5–14] and multilayer thin body theory [15,16] was also applied by other authors. In this direction, the authors published many papers (e.g., [1–3,17–21]) with the application of Legendre and Chebyshev polynomial systems. These expansions can be successfully used in constructing any thin body theory. Despite this, classical theories, especially micropolar theories and theories of other rheologies, constructed by this method, are far from perfection.

2. Parameterization of a Multilayer Thin Domain with Several Base Surfaces

Consider a multilayer thin domain of the Euclidean space consisting of not more than countably many layers. We perform the parameterization of this domain in the same way as in [1]. Let the layers be enumerated in the ascending order, i.e., for example, if $\alpha \geq 2$ is the serial number of a certain layer, then the serial number of the previous layer is $\alpha - 1$ and the serial number of the next layer is $\alpha + 1$. Each layer has two frontal surfaces. The frontal surface of the layer α, which lies to the side of the previous layer $\alpha - 1$, is called the interior base surface and is denoted by $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$, whereas the frontal surface of the layer α, which lies to the side of the next layer $\alpha + 1$, is called the exterior base surface and is denoted by $\underset{\alpha}{\overset{(+)}{\mathrm{S}}}$.

If the multilayer structure consists of K layers, then $\underset{1}{\overset{(-)}{\mathrm{S}}}$ ($\underset{1}{\overset{(+)}{\mathrm{S}}}$) and $\underset{K}{\overset{(-)}{\mathrm{S}}}$ ($\underset{K}{\overset{(+)}{\mathrm{S}}}$) are the interior (exterior) surfaces of the first and last layers, respectively. In this case, $\underset{1}{\overset{(-)}{\mathrm{S}}}$ and $\underset{K}{\overset{(+)}{\mathrm{S}}}$ are also called the interior and exterior surfaces of the multilayer structure.

We assume that the frontal surfaces of each layer are regular surfaces and its lateral surface is a ruled surface in the case where the layer is bounded and unclosed.

2.1. *Vector Parametric Equation of the Layer α and the System of Vector Parametric Equations of a Multilayer Thin Domain*

The radius-vector of an arbitrary point $\underset{\alpha}{\mathrm{M}}$ of the layer α is represented in the form

$$\underset{\alpha}{\mathbf{r}}(x^1,x^2,x^3) = \underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2) + x^3\underset{\alpha}{\mathbf{h}}(x^1,x^2) = (1-x^3)\underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2) + x^3\underset{\alpha}{\overset{(+)}{\mathbf{r}}}(x^1,x^2) \tag{1}$$

for all $\alpha \in \mathbb{N}$ and $\forall\, x^3 \in [0,1]$, where the vector relations

$$\underset{\alpha}{\overset{(-)}{\mathbf{r}}} = \underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2), \quad \underset{\alpha}{\overset{(+)}{\mathbf{r}}} = \underset{\alpha}{\overset{(+)}{\mathbf{r}}}(x^1,x^2), \quad \alpha \in \mathbb{N}, \tag{2}$$

are the vector equations of the base surfaces $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$ and $\underset{\alpha}{\overset{(+)}{\mathrm{S}}}$, respectively, x^1 and x^2 are curvilinear (Gaussian) coordinates on the interior base surface $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$, and $\mathbb{N}$ is the set of natural numbers. The vector $\underset{\alpha}{\mathbf{h}}(x^1,x^2) = \underset{\alpha}{\overset{(+)}{\mathbf{r}}}(x^1,x^2) - \underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2)$, which topologically maps the interior base surface $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$ onto the exterior base surface $\underset{\alpha}{\overset{(+)}{\mathrm{S}}}$, is, in general, not orthogonal to the base surfaces, and, moreover, the endpoint of each $\underset{\alpha}{\mathbf{h}}(x^1,x^2)$ is the initial point of $\underset{\alpha+1}{\mathbf{h}}(x^1,x^2)$, $\forall\alpha$, i.e., the following relation holds:

$$\underset{\alpha+\delta}{\overset{(+)}{\mathbf{r}}}(x^1,x^2) = \underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2) + \sum_{\nu=\alpha}^{\alpha+\delta}\underset{\nu}{\mathbf{h}} = \underset{\alpha}{\overset{(-)}{\mathbf{r}}}(x^1,x^2) + \sum_{\nu=\alpha}^{\alpha+\delta}\left[\underset{\nu}{\overset{(+)}{\mathbf{r}}}(x^1,x^2) - \underset{\nu}{\overset{(-)}{\mathbf{r}}}(x^1,x^2)\right]. \tag{3}$$

Let a multilayer domain consist of K layers (we use the usual rules of tensor calculus [4,22–25]. We mainly preserve the notation and conventions of the previous works. Under symbols, we write indices denoting the serial numbers of layers. The Greek indices under symbols assume their values according to circumstances, and capital and small Latin indices assume the values 1, 2 and 1, 2, 3, respectively). Then, introducing the notation

$$\mathbf{h} = \sum_{\nu=1}^{K} \underset{\nu}{\mathbf{h}} = \sum_{\nu=1}^{K} \left[\underset{\nu}{\overset{(+)}{\mathbf{r}}}(x^1, x^2) - \underset{\nu}{\overset{(-)}{\mathbf{r}}}(x^1, x^2) \right], \tag{4}$$

we have

$$\underset{K}{\overset{(+)}{\mathbf{r}}}(x^1, x^2) = \underset{1}{\overset{(-)}{\mathbf{r}}}(x^1, x^2) + \mathbf{h}(x^1, x^2) = \underset{1}{\overset{(-)}{\mathbf{r}}}(x^1, x^2) + \sum_{\nu=1}^{K} \left[\underset{\nu}{\overset{(+)}{\mathbf{r}}}(x^1, x^2) - \underset{\nu}{\overset{(-)}{\mathbf{r}}}(x^1, x^2) \right]. \tag{5}$$

It should be noted that Equation (1) for a fixed α is the vector parametric equation of the layer α, and, when α varies in the corresponding range and the conditions in Equation (3) hold, it is the system of vector parametric equations of the multilayer thin domain considered. It is easy to see that Equation (1) for any x^1, x^2, and $x^3 = 0$ defines the interior base surface $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$, and for any x^1, x^2, and $x^3 = 1$, it defines the exterior lateral surface $\underset{\alpha}{\overset{(+)}{\mathrm{S}}}$, whereas for any x^1, x^2 and $x^3 = \text{const}$,where $x^3 \in (0, 1)$, it defines the equidistance surface for the base surfaces $\underset{\alpha}{\overset{(-)}{\mathrm{S}}}$ and $\underset{\alpha}{\overset{(+)}{\mathrm{S}}}$, which is denoted by $\underset{\alpha}{\mathrm{S}}$.

2.2. Three-Dimensional Families of Bases and the Families of Parameterization of the Domain of the Layer α Generated by Them

Taking into account the expression of the radius-vector $\underset{\alpha}{\mathbf{r}}$ in Equation (1) and introducing the notation $\underset{\alpha}{\mathbf{h}}_P \equiv \partial \underset{\alpha}{\mathbf{h}} / \partial x^P \equiv \partial_P \underset{\alpha}{\mathbf{h}}$, we obtain

$$\underset{\alpha}{\mathbf{r}}_P = \underset{\alpha P}{\mathbf{r}_{-}} + x^3 \underset{\alpha}{\mathbf{h}}_P = (1 - x^3) \underset{\alpha P}{\mathbf{r}_{-}} + x^3 \underset{\alpha P}{\mathbf{r}_{+}}, \quad \underset{\alpha}{\mathbf{r}}_P \equiv \frac{\partial_P \underset{\alpha}{\mathbf{r}}}{\partial x^P}, \quad \underset{\alpha P}{\mathbf{r}_{\star}} \equiv \frac{\partial \underset{\alpha}{\overset{(\star)}{\mathbf{r}}}}{\partial x^P}, \ \star \in \{-, +\}, \ \forall \alpha. \tag{6}$$

Now, differentiating Equation (1) in x^3, we have

$$\underset{\alpha}{\mathbf{r}}_3 \equiv \partial_3 \underset{\alpha}{\mathbf{r}} \equiv \frac{\partial \underset{\alpha}{\mathbf{r}}}{\partial x^3} = \underset{\alpha}{\mathbf{h}}(x^1, x^2), \quad \forall x^3 \in [0, 1], \quad \forall \alpha. \tag{7}$$

According to Equation (7), we can assume that

$$\underset{\alpha 3}{\mathbf{r}_{-}} \equiv \underset{\alpha}{\mathbf{r}}_3 \equiv \underset{\alpha 3}{\mathbf{r}_{+}} \equiv \partial_3 \underset{\alpha}{\mathbf{r}} = \underset{\alpha}{\mathbf{h}}(x^1, x^2), \quad \forall x^3 \in [0, 1], \quad \forall \alpha. \tag{8}$$

The relation in Equation (8) allows us to define the spatial covariant bases $\underset{\alpha p}{\mathbf{r}_{\star}}$, $\star \in \{-, +\}$, $\forall \alpha$ at the points $\underset{\alpha}{\overset{(\star)}{\mathrm{M}}} \in \underset{\alpha}{\overset{(\star)}{\mathrm{S}}}$ and $\star \in \{-, +\}$, $\forall \alpha$, respectively. Therefore, the third basis vector of the spatial covariant bases at the points $\underset{\alpha}{\overset{(\star)}{\mathrm{M}}} \in \underset{\alpha}{\overset{(\star)}{\mathrm{S}}}$, $\star \in \{-, \varnothing, +\}$, for each layer α is the same vector $\underset{\alpha}{\mathbf{h}}(x^1, x^2)$. In view of Equation (8), we can join the relations in Equations (6)and (7) and represent them as

$$\underset{\alpha}{\mathbf{r}}_p = \underset{\alpha p}{\mathbf{r}_{-}} + x^3 \underset{\alpha}{\mathbf{h}}_p = (1 - x^3) \underset{\alpha}{\overset{(-)}{\mathbf{r}}}_{-p} + x^3 \underset{\alpha}{\overset{(+)}{\mathbf{r}}}_{+p}, \quad \forall \alpha. \tag{9}$$

The triples of vectors $\underset{\alpha 1}{\mathbf{r}_{\star}} \underset{\alpha 2}{\mathbf{r}_{\star}}, \underset{\alpha 3}{\mathbf{r}_{\star}}$, $\star \in \{-, \varnothing, +\}$, $\forall \alpha$ defined at the points $\underset{\alpha}{\overset{(\star)}{\mathrm{M}}} \in \underset{\alpha}{\overset{(\star)}{\mathrm{S}}}$, $\star \in \{-, \varnothing, +\}$, $\forall \alpha$ obviously compose three-dimensional covariant spatial bases. As is known [4,22–24], according to these frames (bases), we can construct the corresponding contravariant bases $\underset{\alpha}{\overset{\star}{\mathbf{r}}}{}^{1}\underset{\alpha}{\overset{\star}{\mathbf{r}}}{}^{2}\underset{\alpha}{\overset{\star}{\mathbf{r}}}{}^{3}$, $\star \in \{-, \varnothing, +\}$, $\forall\, \alpha$. Indeed, by their definition [4,22–24], we have

$$\underset{\alpha}{\mathbf{r}}{}^{\tilde{k}} = \frac{1}{2} \underset{\alpha}{\overset{(\sim)}{\mathrm{C}}}{}^{\tilde{k}\tilde{p}\tilde{q}} \underset{\alpha\tilde{p}}{\mathbf{r}} \times \underset{\alpha\tilde{q}}{\mathbf{r}}, \quad \sim \in \{-, \varnothing, +\},\ \forall\, \alpha, \tag{10}$$

where $\underset{\alpha}{\overset{(\sim)}{\mathrm{C}}}{}^{\tilde{k}\tilde{p}\tilde{q}} = (\underset{\alpha}{\mathbf{r}}{}^{\tilde{k}} \times \underset{\alpha}{\mathbf{r}}{}^{\tilde{p}}) \cdot \underset{\alpha}{\mathbf{r}}{}^{\tilde{q}}, \sim \in \{-, \varnothing, +\},\ \ \forall\, \alpha$, are contravariant components of the discriminant tensors [22] of the layer α at the points $\underset{\alpha}{\overset{(\star)}{\mathrm{M}}} \in \underset{\alpha}{\overset{(\star)}{\mathrm{S}}}, \star \in \{-, \varnothing, +\},\ \forall\, \alpha$. It is easy to see that Equation (9) is shortly represented in the form

$$\underset{\alpha}{\mathbf{r}}{}_{p} = \underset{\alpha}{g}{}_{p}^{\overset{\star}{q}} \underset{\alpha\overset{\star}{q}}{\mathbf{r}} = \underset{\alpha}{g}{}_{p\overset{\star}{q}} \underset{\alpha}{\mathbf{r}}{}^{\overset{\star}{q}}, \quad \underset{\alpha}{g}{}_{\breve{p}\tilde{q}} = \underset{\alpha\breve{p}}{\mathbf{r}} \cdot \underset{\alpha\tilde{q}}{\mathbf{r}}, \quad \underset{\alpha}{g}{}_{\breve{p}}^{\tilde{q}} = \underset{\alpha\breve{p}}{\mathbf{r}} \cdot \underset{\alpha}{\mathbf{r}}{}^{\tilde{q}}, \quad \smile \in \{-, \varnothing, +\},\ \sim, \star \in \{-, +\},\ \forall\alpha. \tag{11}$$

It is easy to express $\underset{\alpha}{\mathbf{r}}{}^{k}$, $\forall\alpha$, through the vectors $\underset{\alpha\tilde{m}}{\mathbf{r}}$ or $\underset{\alpha}{\mathbf{r}}{}^{\tilde{m}}$, $\sim \in \{-, +\}$, $\forall\, \alpha$. Indeed, taking into account the first relation in Equation (11) in (10) for $\sim = \varnothing$, we obtain

$$\underset{\alpha}{\mathbf{r}}{}^{k} = \frac{1}{2} \underset{\alpha}{\overset{(\sim)}{\vartheta}}{}^{-1} \epsilon^{kpq} \epsilon_{lmn} \underset{\alpha}{g}{}_{p}^{\tilde{m}} \underset{\alpha}{g}{}_{q}^{\tilde{n}} \underset{\alpha}{\mathbf{r}}{}^{\tilde{l}}, \quad \underset{\alpha}{\overset{(\sim)}{\vartheta}} \equiv= \det(\underset{\alpha}{g}{}_{P}^{\tilde{Q}}), \quad \sim \in \{-, +\}, \quad \forall\alpha, \tag{12}$$

where ϵ^{kpq} and ϵ_{lmn} are the Levi–Civita symbols. By Equation (12), we can introduce the notation

$$\underset{\alpha}{g}{}_{\tilde{l}}^{k} = \frac{1}{2} \underset{\alpha}{\overset{(\sim)}{\vartheta}}{}^{-1} \epsilon^{kpq} \epsilon_{lmn} \underset{\alpha}{g}{}_{p}^{\tilde{m}} \underset{\alpha}{g}{}_{q}^{\tilde{n}}, \quad \underset{\alpha}{g}{}^{k\tilde{l}} = \frac{1}{2} \underset{\alpha}{\overset{(\sim)}{\vartheta}}{}^{-1} \epsilon^{kpq} \epsilon_{smn} \underset{\alpha}{g}{}_{p}^{\tilde{m}} \underset{\alpha}{g}{}_{q}^{\tilde{n}} \underset{\alpha}{g}{}^{\tilde{s}\tilde{l}}, \quad \sim \in \{-, +\}, \quad \forall\alpha. \tag{13}$$

Using this notation, we represent the relation in Equation (12) in the desired form

$$\underset{\alpha}{\mathbf{r}}{}^{p} = \underset{\alpha}{g}{}_{\tilde{q}}^{p} \underset{\alpha}{\mathbf{r}}{}^{\tilde{q}} = \underset{\alpha}{g}{}^{p\tilde{q}} \underset{\alpha\tilde{q}}{\mathbf{r}}, \quad \sim \in \{-, +\}, \quad \forall\alpha. \tag{14}$$

Let us introduce into consideration the following objects (matrices):

$$\underset{\alpha\beta}{g}{}_{\breve{p}\,\cdot}^{\,\cdot\,\tilde{q}} = \underset{\alpha\breve{p}}{\mathbf{r}} \cdot \underset{\beta}{\mathbf{r}}{}^{\tilde{q}}, \quad \smile, \sim \in \{-, \varnothing, +\}, \quad \forall\alpha, \beta, \tag{15}$$

and the objects are obtained from Equation (15) by alternating the indices. It is easy to calculate that the number of such objects is equal to 36. It is easy to see that, for $\alpha = \beta$, Equation (15) contains Equations (11) and (13). Indeed, from Equation (15), we have

$$\underset{\alpha}{g}{}_{\breve{p}}^{\tilde{q}} = \underset{\alpha\alpha}{g}{}_{\breve{p}\,\cdot}^{\,\cdot\,\tilde{q}} = \underset{\alpha\breve{p}}{\mathbf{r}} \cdot \underset{\alpha}{\mathbf{r}}{}^{\tilde{q}}, \quad \smile, \sim \in \{-, \varnothing, +\}, \quad \forall\alpha, \tag{16}$$

and alternating the indices, we obviously obtain the objects considered above and also $\underset{\alpha}{g}{}^{pq} = \underset{\alpha}{\mathbf{r}}{}^{p} \cdot \underset{\alpha}{\mathbf{r}}{}^{q}$, $\forall\alpha$, i.e., in this case, the number of the introduced quantities is equal to 36. It is easy to see that by Equations (15) and (16), the connections between the families of bases are represented in the form

$$\underset{\alpha\tilde{p}}{\mathbf{r}} = \underset{\alpha}{g}{}_{\tilde{p}}^{\breve{n}} \underset{\alpha\breve{n}}{\mathbf{r}} = \underset{\alpha\beta}{g}{}_{\tilde{p}\,\cdot}^{\,\cdot\,\breve{n}} \underset{\beta\breve{n}}{\mathbf{r}}, \quad \smile, \sim \in \{-, \varnothing, +\}, \quad \forall\alpha, \beta, \tag{17}$$

which remains valid under index alternation. By Equation (17), it is easy to show that the following relation holds:

$$\underset{\alpha\beta}{g}{}_{\tilde{p}\,\cdot}^{\,\cdot\,\breve{q}} = \underset{\alpha\delta}{g}{}_{\tilde{p}\,\cdot}^{\,\cdot\,\overset{\star}{n}} \underset{\beta\beta}{g}{}_{\overset{\star}{n}\,\cdot}^{\,\cdot\,\breve{q}}, \quad \smile, \sim, \star \in \{-, \varnothing, +\}, \quad \forall\alpha, \beta, \delta. \tag{18}$$

Differentiating Equations (3)–(5) in x^I and taking into account Equation (17), we obtain

$$
\begin{aligned}
&\underset{\alpha+\beta}{\mathbf{r}}{}_{\overset{+}{I}} = \underset{\alpha}{\mathbf{r}}{}_{\overset{-}{I}} + \sum_{\nu=\alpha}^{\alpha+\beta}\Big[\underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{+}{I}} - \underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{-}{I}}\Big]\underset{\nu}{\mathbf{r}}{}_{\overset{-}{k}} = \underset{\alpha}{\mathbf{r}}{}_{\overset{+}{I}} + \sum_{\nu=\alpha+1}^{\alpha+\beta}\Big[\underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{+}{I}} - \underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{-}{I}}\Big]\underset{\nu}{\mathbf{r}}{}_{\overset{-}{k}},\\
&\partial_I\mathbf{h}(x^1,x^2) = \sum_{\nu=1}^{N}\partial_I\underset{\nu}{\mathbf{h}}(x^1,x^2) = \sum_{\nu=1}^{N}\Big[\underset{\nu}{\mathbf{r}}{}_{\overset{+}{I}}(x^1,x^2) - \underset{\nu}{\mathbf{r}}{}_{\overset{-}{I}}(x^1,x^2)\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \sum_{\nu=1}^{N}\Big[\underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{+}{I}}(x^1,x^2) - \underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{-}{I}}(x^1,x^2)\Big]\underset{\nu}{\mathbf{r}}{}_{\overset{-}{k}}(x^1,x^2),\\
&\underset{N}{\mathbf{r}}{}_{\overset{+}{I}}(x^1,x^2) = \underset{1}{\mathbf{r}}{}_{\overset{-}{I}}(x^1,x^2) + \sum_{\nu=1}^{N}\Big[\underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{+}{I}}(x^1,x^2) - \underset{\nu}{g}{}^{\overset{-}{k}}_{\overset{-}{I}}(x^1,x^2)\Big]\underset{\nu}{\mathbf{r}}{}_{\overset{-}{k}}(x^1,x^2).
\end{aligned}
\tag{19}
$$

Next, using Equations (17) and (18), we can obtain the relation [2–4,23,24]

$$
\underset{\sim}{\mathbf{E}} = \underset{\underset{\alpha}{\sim}}{\mathbf{E}} = \underset{\alpha}{g}{}^{\breve{n}}_{\tilde{p}}\,\underset{\alpha}{\mathbf{r}}{}^{\tilde{p}}\,\underset{\alpha}{\mathbf{r}}{}_{\breve{n}} = \underset{\underset{\beta}{\sim}}{\mathbf{E}} = \underset{\beta}{g}{}^{\breve{n}}_{\tilde{p}}\,\underset{\beta}{\mathbf{r}}{}^{\tilde{p}}\,\underset{\beta}{\mathbf{r}}{}_{\breve{n}} = \underset{\alpha\beta}{g}{}_{\tilde{p}}{}^{\cdot\,\breve{n}}_{\cdot}\,\underset{\alpha}{\mathbf{r}}{}^{\tilde{p}}\,\underset{\beta}{\mathbf{r}}{}_{\breve{n}}, \quad \sim, \smile \in \{-,\varnothing,+\},\ \ \forall\alpha,\beta,
\tag{20}
$$

which remains valid under index alternating. As is seen from Equation (20), the quantities in Equations (15) and (16) introduced above represent the components of the second rank unit tensor for a multilayer thin domain of the three-dimensional Euclidean space.

It is seen from the material presented above that, in the parameterization of the multilayer domain considered, for each layer, all the corresponding relations for a one-layered thin body under a new parameterization in [3,4], as well as for other parameterizations considered in [3,5,22], hold under the condition that the root letters of quantities entering these relations must be equipped with the bottom index, which denotes the number of the layer considered. In this connection, we do not consider the problems on the parameterization of a multilayer domain in detail.

3. Equations of Motion and Constitutive Relations of Micropolar Theory

In what follows, for brevity, we present certain representations of equations of motion and constitutive relations (CR) in the case of a one-layer thin body, and then we show how one can obtain the desired relations using the rule presented above and write certain relations.

3.1. Motion Equations and CR of Physical Contents of the Micropolar Theory of One-Layer Thin Bodies with One Small Size

The new parameterization of a one-layer thin domain [1–4] is performed by the relation, which is obtained from Equation (1) under the absence of index α under the symbols. To obtain the representations of equations of motion and CR, we need the representations of the gradient and the divergence under the parameterization considered. Let us obtain the representations of these operators. Omitting the index, from Equations (11) and (14), we find that

$$
\mathbf{r}_p = g_p^{\overset{-}{m}}\mathbf{r}_{\overset{-}{m}} = g_{p\overset{+}{m}}\mathbf{r}^{\overset{+}{m}}, \quad \mathbf{r}^p = g^p_{\overset{-}{m}}\mathbf{r}^{\overset{-}{m}} = g^p_{\overset{+}{m}}\mathbf{r}^{\overset{+}{m}},
\tag{21}
$$

and also from Equation (13), we have

$$
\begin{aligned}
&g^P_{\overset{-}{M}} = \overset{(-)}{\vartheta}{}^{-1}A^P_{\overset{-}{M}}, \quad \overset{(-)}{\vartheta} = \det(g^{\overset{-}{J}}_I), \quad g^3_{\overset{-}{M}} = -g^{\overset{-}{3}}_P g^P_{\overset{-}{M}}, \quad g^{\overset{-}{3}}_P = x^3 g^{\overset{-}{3}}_{\overset{+}{P}},\\
&g^{\overset{-}{3}}_{\overset{+}{P}} = h^{-1}\partial_P h, \quad h = |\mathbf{h}|, \quad A^P_{\overset{-}{M}} \equiv g^{\overset{-}{P}}_{\overset{-}{M}} + x^3 a^P_{\overset{+}{M}}, \quad a^P_{\overset{+}{M}} \equiv (g^{\overset{-}{I}}_{\overset{+}{I}} - 1)g^{\overset{-}{P}}_{\overset{-}{M}} - g^{\overset{-}{P}}_{\overset{+}{M}}.
\end{aligned}
\tag{22}
$$

Moreover, it is easy to note that the following relations hold [1–4]:

$$g_{\underset{-}{M}}^{P} = \sum_{s=0}^{\infty} \underset{(s)\bar{M}}{A}^{\underset{+}{\bar{P}}} (x^3)^s, \quad \underset{(s)\bar{M}}{A}^{\underset{+}{\bar{P}}} = (g_{\underset{-}{N_1}}^{\bar{P}} - g_{\underset{+}{N_1}}^{\bar{P}}) \cdot (g_{\underset{-}{N_2}}^{\bar{N}_1} - g_{\underset{+}{N_2}}^{\bar{N}_1}) \cdot \ldots \cdot (g_{\underset{-}{M}}^{\bar{N}_{s-1}} - g_{\underset{+}{M}}^{\bar{N}_{s-1}}), \quad \underset{(0)\bar{M}}{A}^{\underset{+}{\bar{P}}} = g_{\underset{-}{\bar{M}}}^{\bar{P}}. \tag{23}$$

By the first and third relations in Equation (22) and the second relation in Equation (21), we find that

$$\mathbf{r}^P = g_{\underset{-}{M}}^{P} \mathbf{r}^{\bar{M}}, \quad \mathbf{r}^3 = g_{\underset{-}{M}}^{3} \mathbf{r}^{\bar{M}} + \mathbf{r}^{\bar{3}} = \mathbf{r}^{\bar{3}} - g_{P}^{\bar{3}} \mathbf{r}^P = \mathbf{r}^{\bar{3}} - g_{P}^{\bar{3}} g_{\underset{-}{M}}^{P} \mathbf{r}^{\bar{M}}. \tag{24}$$

The gradient operator can be applied to any tensor. Therefore, denoting a certain tensor quantity by $\mathbb{F}(x', x^3)$, by the definition of the gradient [22–24] and by Equation (24), we have [3]

$$\mathrm{grad}\mathbb{F} = \nabla\mathbb{F} = \mathbf{r}^p \partial_p \mathbb{F} = \mathbf{r}^P \partial_P \mathbb{F} + \mathbf{r}^3 \partial_3 \mathbb{F} = \mathbf{r}^{\bar{M}} g_{\underset{-}{M}}^{P} (\partial_P - g_{P}^{\bar{3}} \partial_3)\mathbb{F} + \mathbf{r}^{\bar{3}} \partial_3 \mathbb{F}.$$

whence, introducing the differential operator

$$N_p = \partial_p - g_p^{\bar{3}} \partial_3, \quad \mathbf{N} = \mathbf{r}^p N_p = \mathbf{r}^P N_P = \mathbf{r}^{\bar{M}} g_{\underset{-}{M}}^{P} N_P, \quad N_3 = 0, \tag{25}$$

we obtain the desired representation of the gradient in the form

$$\mathrm{grad}\mathbb{F} = \nabla\mathbb{F} = \mathbf{N}\mathbb{F} + \mathbf{r}^{\bar{3}} \partial_3 \mathbb{F} = \mathbf{r}^P N_p \mathbb{F} + \mathbf{r}^{\bar{3}} \partial_3 \mathbb{F} = \mathbf{r}^{\bar{M}} g_{\underset{-}{M}}^{P} N_p \mathbb{F} + \mathbf{r}^{\bar{3}} \partial_3 \mathbb{F}. \tag{26}$$

The divergence operator is applied to a tensor whose rank is no less than 1. Applying this operator, e.g., to a second-rank tensor $\underset{\sim}{\mathbf{P}}$, by the definition, the third relation in Equations (22), and (25), we obtain

$$\mathrm{div}\, \underset{\sim}{\mathbf{P}} = \nabla \cdot \underset{\sim}{\mathbf{P}} = g_{\underset{-}{M}}^{P} N_p \mathbf{P}^{\bar{M}} + \partial_3 \mathbf{P}^{\bar{3}} \quad (\mathbf{P}^{\bar{m}} = \mathbf{r}^{\bar{m}} \cdot \underset{\sim}{\mathbf{P}}). \tag{27}$$

Note that Equation (27) can also be easily obtained from Equation (26) if in this relation, we replace the sign of tensor product, which is omitted, with the sign of inner product.

3.1.1. Representations of Motion Equations

As is known [26–28], the three-dimensional equations of motion of micropolar deformable rigid bodies are represented in the form

$$\nabla \cdot \underset{\sim}{\mathbf{P}} + \rho \mathbf{F} = \rho \partial_t^2 \mathbf{u}, \quad \nabla \cdot \underset{\sim}{\boldsymbol{\mu}} + \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{P}} + \rho \mathbf{m} = \underset{\sim}{\mathbf{J}} \cdot \partial_t^2 \boldsymbol{\varphi}. \tag{28}$$

Here, $\underset{\sim}{\mathbf{P}}$ and $\underset{\sim}{\boldsymbol{\mu}}$ are tensors of stresses and couple stresses, $\underset{\approx}{\mathbf{C}}$ is the discriminant tensor (third-rank tensor) [22], $\mathbf{u}$ is the vector of displacements, $\boldsymbol{\varphi}$ is the vector of (inner) rotation, ρ is the material density, $\mathbf{F}$ is the mass force density, $\mathbf{m}$ is the mass moment density, and $\overset{2}{\otimes}$ is the inner 2-product (for example, $\underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{P}} = \mathbf{r}^i C_{ijk} P^{jk}$). The definition of inner r-product and the problems related to it are considered in [1–3,22,29]. It is easy to see that by Equation (27), Equation (28) can be rewritten in the form

$$g_{\underset{-}{M}}^{P} N_p \mathbf{P}^{\bar{M}} + \partial_3 \mathbf{P}^{\bar{3}} + \rho \mathbf{F} = \rho \partial_t^2 \mathbf{u}, \quad g_{\underset{-}{M}}^{P} N_p \boldsymbol{\mu}^{\bar{M}} + \partial_3 \boldsymbol{\mu}^{\bar{3}} + \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{P}} + \rho \mathbf{m} = \underset{\sim}{\mathbf{J}} \cdot \partial_t^2 \boldsymbol{\varphi}. \tag{29}$$

Note that Equation (29) is called the motion equation of the micropolar deformable rigid thin body under the new parameterization of thin body domain. Taking into account the first relation in Equation (23), we can rewrite Equation (29) in the form

$$\begin{aligned}\sum_{s=0}^{\infty} \underset{(s)M}{A}{}^{\bar{P}}_{+} (x^3)^s N_P \mathbf{P}^{\bar{M}} + \partial_3 \mathbf{P}^{\bar{3}} + \rho \mathbf{F} = \rho \partial_t^2 \mathbf{u},\\ \sum_{s=0}^{\infty} \underset{(s)M}{A}{}^{\bar{P}}_{+} (x^3)^s N_P \boldsymbol{\mu}^{\bar{M}} + \partial_3 \boldsymbol{\mu}^{\bar{3}} + \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{P}} + \rho \mathbf{m} = \underset{\sim}{\mathbf{J}} \cdot \partial_t^2 \boldsymbol{\varphi}.\end{aligned} \tag{30}$$

It is seen that Equation (30) contains infinitely many summands. Therefore, they cannot be used in practice. Naturally, we need to consider approximate equations with finitely many summands. In this connection, let us introduce the following definition.

Definition 1. *The equations, which are obtained from Equation (29) if in the expansion of* $g^{P}_{\underset{M}{-}}$ *(see the first formula in Equation (23)) we preserve the first* $s+1$ *terms, are called the equations of approximation of order* s*.*

Obviously, the equation of approximation of order s are represented in the form

$$\underset{(s)M}{g}{}^{P}_{-} N_P \mathbf{P}^{\bar{M}} + \partial_3 \mathbf{P}^{\bar{3}} + \rho \mathbf{F} = \rho \partial_t^2 \mathbf{u}, \quad \underset{(s)M}{g}{}^{P}_{-} N_P \boldsymbol{\mu}^{\bar{M}} + \partial_3 \boldsymbol{\mu}^{\bar{3}} + \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{P}} + \rho \mathbf{m} = \underset{\sim}{\mathbf{J}} \cdot \partial_t^2 \boldsymbol{\varphi}, \tag{31}$$

where

$$\underset{(s)M}{g}{}^{P}_{-} = \sum_{m=0}^{s} \underset{M}{A}{}^{\bar{P}}_{+} (x^3)^m.$$

From Equation (31), for $s = 0$, we obtain the equations of zero approximation, for $s = 1$, the equations of first approximation, etc.

3.1.2. Representations of Constitutive Relations of Physical and Heat Contents

Let us represent the constitutive relations of physical content under non-isothermal processes in the linear micropolar elasticity theory and the corresponding representation for the Fourier heat conduction law under the new parameterization of the thin body domain in the approximation of order s [3]

$$\begin{aligned}\underset{\sim}{\mathbf{P}}_{(s)} &= \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} (\underset{(s)M}{g}{}^{P}_{-} \mathbf{r}^{\bar{M}} N_p \mathbf{u} + \mathbf{r}^{\bar{3}} \partial_3 \mathbf{u}) + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} (\underset{(s)M}{g}{}^{P}_{-} \mathbf{r}^{\bar{M}} N_p \boldsymbol{\varphi} + \mathbf{r}^{\bar{3}} \partial_3 \boldsymbol{\varphi}) - \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\simeq}{\mathbf{C}} \cdot \boldsymbol{\varphi} - \underset{\sim}{\mathbf{b}} \vartheta,\\ \underset{\sim}{\boldsymbol{\mu}}_{(s)} &= \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} (\underset{(s)M}{g}{}^{P}_{-} \mathbf{r}^{\bar{M}} N_p \mathbf{u} + \mathbf{r}^{\bar{3}} \partial_3 \mathbf{u}) + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} (\underset{(s)M}{g}{}^{P}_{-} \mathbf{r}^{\bar{M}} N_p \boldsymbol{\varphi} + \mathbf{r}^{\bar{3}} \partial_3 \boldsymbol{\varphi}) - \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\simeq}{\mathbf{C}} \cdot \boldsymbol{\varphi} - \underset{\sim}{\boldsymbol{\beta}} \vartheta\\ \mathbf{q}_{(s)} &= -\boldsymbol{\Lambda}^{\bar{M}} \underset{(s)M}{g}{}^{P}_{-} N_p T - \boldsymbol{\Lambda}^{\bar{3}} \partial_3 T, \quad \boldsymbol{\Lambda}^{\bar{m}} = \underset{\sim}{\boldsymbol{\Lambda}} \cdot \mathbf{r}^{\bar{m}}, \quad \underset{(s)M}{g}{}^{P}_{-} = \sum_{n=0}^{s} \underset{(n)M}{A}{}^{\bar{P}}_{+} (x^3)^n,\end{aligned} \tag{32}$$

where $\underset{\approx}{\mathbf{A}}$, $\underset{\approx}{\mathbf{B}}$, $\underset{\approx}{\mathbf{D}}$ ($\underset{\approx}{\mathbf{C}} = \underset{\approx}{\mathbf{B}}^T$) are material tensors of the fourth rank, and ϑ is the temperature drop, $\underset{\sim}{\mathbf{b}} = \underset{\approx}{\mathbf{A}} \overset{2}{\otimes} \underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{B}} \overset{2}{\otimes} \underset{\sim}{\mathbf{d}}$, $\underset{\sim}{\boldsymbol{\beta}} = \underset{\approx}{\mathbf{C}} \overset{2}{\otimes} \underset{\sim}{\mathbf{a}} + \underset{\approx}{\mathbf{D}} \overset{2}{\otimes} \underset{\sim}{\mathbf{d}}$, which are called the tensors of thermomechanical properties, $\underset{\sim}{\mathbf{a}}$ and $\underset{\sim}{\mathbf{d}}$ are the thermal expansion tensors. It should be noted that, if we consider a body with center of symmetry [27,28,30], then $\underset{\approx}{\mathbf{B}} = 0$, $\underset{\approx}{\mathbf{C}} = 0$, and, in this case, the constitutive relations presented above are simplified. The Fourier heat conduction law [25,28] has the form $\mathbf{q} = -\underset{\sim}{\Lambda} \cdot \nabla T$, where the second-rank positive-definite tensor $\underset{\sim}{\boldsymbol{\Lambda}}$ is called the heat conduction tensor.

The relations obtained from Equation (32) for $s = 0$ are called CR of zero approximation, and for $s = 1$, they are called CR of the first approximation.

4. Equations of Motion in Moments

4.1. Motion Equations in Moments with Respect to Chebyshev Polynomial Systems for Multilayer Thin Bodies with One Small Size

We restrict ourselves to obtaining the systems of equations of motion of approximations $(0, N)$ in moments. Using the rule presented above, by analogous systems of equations from [3], we represent the desired systems of equations in the form

$$
\begin{aligned}
&\left\{ \underset{\alpha}{\nabla}_I \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} - \underset{\alpha}{g}{}^{\bar 3}_{+I}\left[k\underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} + 2(k+1)\left(\sum_{p=k}^{N} \underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar I} - \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I}\right)\right]\right.\\
&\left. +2(k+1)\sum_{p=k}^{N}\left[1-(-1)^{k+p}\right]\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar 3}\right\} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{F}}} = \underset{\alpha}{\rho}\,\partial_t^2 \underset{\alpha}{\overset{(k)}{\mathbf{u}}},\\
&\{\mathbf{P} \Rightarrow \boldsymbol{\mu}\} + \underset{\alpha}{\underset{\approx}{\mathbf{C}}} \overset{2}{\otimes} \underset{\alpha}{\underset{\sim}{\overset{(k)}{\mathbf{P}}}} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{m}}} = \underset{\alpha}{\underset{\sim}{\mathbf{J}}} \cdot \partial_t^2 \underset{\alpha}{\overset{(k)}{\boldsymbol{\varphi}}}, \quad k = \overline{0, N}, \quad \alpha = \overline{1, K}.
\end{aligned}
\tag{33}
$$

Here, the notation $\{\mathbf{P} \Rightarrow \boldsymbol{\mu}\}$ means that the expression in brackets is obtained from the expression in brackets of the previous relation if the letter $\mathbf{P}$ is replaced with $\boldsymbol{\mu}$. Similar designations are used in below.

4.2. Motion Equations in Moments with Respect to Legendre Polynomials for Multilayer thin Bodies with One Small Size

Let us write systems of equations of motion of approximations $(0, N)$ and $(1, N)$ in moments taking into account only the boundary conditions of physical content on the frontal surfaces, which can be obtained by using the corresponding systems of equations from [3,31–35]

$$
\begin{aligned}
&\left\{ \underset{\alpha}{\nabla}_I \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} - \underset{\alpha}{g}{}^{\bar 3}_{+I}\left[k\underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} - (2k+1)\sum_{p=0}^{k}\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar I}\right] - (2k+1)\sum_{p=0}^{k}\left[1-(-1)^{k+p}\right]\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar 3}\right.\\
&\left. +(2k+1)\left[\sqrt{\underset{\alpha}{g}{}^{\overset{++}{33}}}\,\underset{\alpha}{\overset{(+)}{\mathbf{P}}} + (-1)^k\sqrt{\underset{\alpha}{g}{}^{\overset{--}{33}}}\,\underset{\alpha}{\overset{(-)}{\mathbf{P}}}\right]\right\} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{F}}} = \underset{\alpha}{\rho}\,\partial_t^2 \underset{\alpha}{\overset{(k)}{\mathbf{u}}},\\
&\{\mathbf{P} \Rightarrow \boldsymbol{\mu}\} + \underset{\alpha}{\underset{\approx}{\mathbf{C}}} \overset{2}{\otimes} \underset{\alpha}{\underset{\sim}{\overset{(k)}{\mathbf{P}}}} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{m}}} = \underset{\alpha}{\underset{\sim}{\mathbf{J}}} \cdot \partial_t^2 \underset{\alpha}{\overset{(k)}{\boldsymbol{\varphi}}}, \quad k = \overline{0, N}, \quad \alpha = \overline{1, K};
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&\left\{ \underset{\alpha}{\nabla}_I \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} + \frac{1}{2}\left(\underset{\alpha}{g}{}^{\bar P}_{-M} - \underset{\alpha}{g}{}^{\bar P}_{+M}\right)\left(\frac{k}{2k-1}\underset{\alpha}{\nabla}_P \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar M} + \underset{\alpha}{\nabla}_P \underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar M} + \frac{k+1}{2k+3}\underset{\alpha}{\nabla}_P \underset{\alpha}{\overset{(k+1)}{\mathbf{P}}}{}^{\bar M}\right)\right.\\
&-(2k+1)\sum_{p=0}^{k}\left[1-(-1)^{k+p}\right]\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar 3} - \underset{\alpha}{g}{}^{\bar 3}_{+P}\left[\underset{\alpha}{g}{}^{\bar P}_{-M}\left[k\underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar M} - (2k+1)\sum_{p=0}^{k}\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar M}\right]\right.\\
&\left.+\left(\underset{\alpha}{g}{}^{\bar P}_{-I} - \underset{\alpha}{g}{}^{\bar P}_{+I}\right)\left[\frac{(k-1)k}{2(2k-1)}\underset{\alpha}{\overset{(k-1)}{\mathbf{P}}}{}^{\bar I} + k\underset{\alpha}{\overset{(k)}{\mathbf{P}}}{}^{\bar I} - \frac{(k+1)(k+2)}{2(2k+3)}\underset{\alpha}{\overset{(k+1)}{\mathbf{P}}}{}^{\bar I} - (2k+1)\sum_{p=0}^{k}\underset{\alpha}{\overset{(p)}{\mathbf{P}}}{}^{\bar I}\right]\right]\\
&\left. +(2k+1)\left[\sqrt{\underset{\alpha}{g}{}^{\overset{++}{33}}}\,\underset{\alpha}{\overset{(+)}{\mathbf{P}}} + (-1)^k\sqrt{\underset{\alpha}{g}{}^{\overset{--}{33}}}\,\underset{\alpha}{\overset{(-)}{\mathbf{P}}}\right]\right\} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{F}}} = \underset{\alpha}{\rho}\,\partial_t^2 \underset{\alpha}{\overset{(k)}{\mathbf{u}}},\\
&\{\mathbf{P} \Rightarrow \boldsymbol{\mu}\} + \underset{\alpha}{\underset{\approx}{\mathbf{C}}} \overset{2}{\otimes} \underset{\alpha}{\underset{\sim}{\overset{(k)}{\mathbf{P}}}} + \underset{\alpha}{\rho}\,\underset{\alpha}{\overset{(k)}{\mathbf{m}}} = \underset{\alpha}{\underset{\sim}{\mathbf{J}}} \cdot \partial_t^2 \underset{\alpha}{\overset{(k)}{\boldsymbol{\varphi}}}, \quad k = \overline{0, N}, \quad \alpha = \overline{1, K}.
\end{aligned}
\tag{35}
$$

Note that Equations (34) and (35) are deduced by using the recurrence relations for Legendre polynomials [3]. Note also that $\underset{\alpha}{\overset{(+)}{\mathbf{P}}}$ ($\underset{\alpha}{\overset{(+)}{\boldsymbol{\mu}}}$) and $\underset{\alpha+1}{\overset{(-)}{\mathbf{P}}}$ ($\underset{\alpha+1}{\overset{(-)}{\boldsymbol{\mu}}}$) ($\alpha = \overline{1, K-1}$) are stress vectors (couple stresses) of interaction between the layers α and $\alpha + 1$, which act on the surfaces $\underset{\alpha}{\overset{(+)}{S}}$ and $\underset{\alpha+1}{\overset{(-)}{S}}$, respectively, and $\underset{1}{\overset{(+)}{\mathbf{P}}}$

$(\overset{(+)}{\underset{1}{\boldsymbol{\mu}}})$ and $\overset{(-)}{\underset{K}{\mathbf{P}}}$ $(\overset{(-)}{\underset{K}{\boldsymbol{\mu}}})$ are given stress vectors (couple stresses) on the frontal surfaces $\overset{(+)}{\underset{1}{S}}$ and $\overset{(-)}{\underset{K}{S}}$, respectively. The equations of heat influx of approximations $(0, N)$ and $(1, N)$, and also constitutive relations of physical and heat contents in moments for multilayer thin bodies are obtained in full analogy with Equations (34) and (35). To help the readers understand this work, we refer them to [3,21,33–36], where for the theories of one-layer thin body with one small size and two small sizes, and also for theory of multilayer constructions with the use of the Legendre and Chebyshev polynomial systems, and many analogous problems, are presented in detail.

4.3. Equations in Moments of the Displacement Vector for Multilayer Thin Bodies with One Small Size

Let us write down the systems of equations of zero approximation in moments of the displacement vector for the classical theory [3]

$$\begin{aligned} &\underset{\widetilde{\alpha}}{\mathbf{A}}^{\bar{I}\cdot\bar{J}\cdot} \cdot \underset{\alpha}{\nabla}_I \underset{\alpha}{\nabla}_J \overset{(k)}{\underset{\alpha}{\mathbf{u}}} + \Big(\underset{\widetilde{\alpha}}{\mathbf{A}}^{\bar{3}\cdot\bar{I}\cdot} + \underset{\widetilde{\alpha}}{\mathbf{A}}^{\bar{I}\cdot\bar{3}\cdot}\Big) \cdot \underset{\alpha}{\nabla}_I \overset{(k)}{\underset{\alpha}{\mathbf{u}}}{}' + \underset{\widetilde{\alpha}}{\mathbf{A}}^{\bar{3}\cdot\bar{3}\cdot} \cdot \overset{(k)}{\underset{\alpha}{\mathbf{u}}}{}'' \\ &-\Big(\underset{\alpha}{\mathbf{b}}^{\bar{I}} \underset{\alpha}{\nabla}_I \overset{(k)}{\underset{\alpha}{\vartheta}} + \underset{\alpha}{\mathbf{b}}^{\bar{3}} \overset{(k)}{\underset{\alpha}{\vartheta}}{}'\Big) + \underset{\alpha}{\rho} \overset{(k)}{\underset{\alpha}{\mathbf{F}}} = \underset{\alpha}{\rho}\, \partial_t^2 \overset{(k)}{\underset{\alpha}{\mathbf{u}}}, \quad k \geq 0, \quad \alpha = \overline{1, K}, \end{aligned} \tag{36}$$

where $\underset{\widetilde{\alpha}}{\mathbf{A}}^{\bar{I}\cdot\bar{J}\cdot} = \underset{\alpha}{\mathbf{r}}^{\bar{I}} \cdot \underset{\widetilde{\widetilde{\alpha}}}{\mathbf{A}} \overset{2}{\otimes} \underset{\alpha}{\mathbf{r}}^{\bar{J}} \underset{\sim}{\mathbf{E}}$, $\underset{\alpha}{\mathbf{b}}^{\bar{m}} = \underset{\alpha}{\mathbf{r}}^{\bar{m}} \cdot \underset{\widetilde{\alpha}}{\mathbf{b}}$ and, in the case of Legendre polynomials, we have

$$\begin{aligned} &\overset{(k)}{\mathbf{u}}{}' = (2k+1) \sum_{p=0}^{\infty} \overset{(k+2p+1)}{\mathbf{u}} = \frac{2k+1}{2}\big[\overset{(-)}{\mathbf{u}} - (-1)^k \overset{(-)}{\mathbf{u}}\big] - \frac{2k+1}{2} \sum_{p=0}^{k-1} \big[1 - (-1)^{k+p}\big] \overset{(k)}{\mathbf{u}}, \\ &\overset{(k)}{\mathbf{u}}{}'' = 4(2k+1)\big[(2k+3)\overset{(k+2)}{\mathbf{u}} + 2(2k+5)\overset{(k+4)}{\mathbf{u}} + 3(2k+7)\overset{(k+6)}{\mathbf{u}} + \ldots\big] \\ &= (2k+1)\big[\overset{(+)}{\partial_3 \mathbf{u}} - (-1)^k \overset{(-)}{\partial_3 \mathbf{u}} - k(k+1)\big(\overset{(+)}{\mathbf{u}} + (-1)^k \overset{(-)}{\mathbf{u}}\big)\big] + \overset{(k)}{\underline{\mathbf{u}}}{}'', \\ &\overset{(k)}{\underline{\mathbf{u}}}{}'' = 4(2k+1)\big[(2k-1)\overset{(k-2)}{\mathbf{u}} + 2(2k-3)\overset{(k-4)}{\mathbf{u}} + 3(2k-5)\overset{(k-6)}{\mathbf{u}} + \ldots\big]. \end{aligned} \tag{37}$$

Here, $\overset{(-)}{\partial_3 \mathbf{u}} = (\partial_3 \mathbf{u})|_{x^3=0}$, $\overset{(+)}{\partial_3 \mathbf{u}} = (\partial_3 \mathbf{u})|_{x^3=1}$, $\overset{(-)}{\mathbf{u}} = \mathbf{u}|_{x^3=0}$ and $\overset{(+)}{\mathbf{u}} = \mathbf{u}|_{x^3=1}$.

Taking into account the expressions for $\overset{(k)}{\underset{\alpha}{\mathbf{u}}}{}'$ and $\overset{(k)}{\underset{\alpha}{\mathbf{u}}}{}''$, which are obtained from Equation (37) (see also Equation (40)) when $\mathbf{u}$ is replaced by $\underset{\alpha}{\mathbf{u}}$ from Equation (36), we obtain various representations of the equations of different approximations in moments of the displacement vector with respect to Legendre polynomials.

It should be noted that to close the systems of Equations (34) and (35), we need to add to them the system of equations of heat influx, constitutive relations, boundary and initial conditions of physical and heat contents in moments of the corresponding approximations, as well as the inter-layer contact conditions depending on the connections of neighboring surfaces.

Similar to the above, for the closure of Equation (36), we need to add to them the system of equations of heat influx and boundary and initial conditions of physical and heat contents in moments of the corresponding approximations, as well asa the inter-layer contact conditions depending on the connections of neighboring surfaces.

4.4. Quasi-Static Problems of the Micropolar Theory of Multilayer Prismatic Bodies in Displacements and Rotations and in Moments of Displacement and Rotation Vectors

Using the rule, set out in [3,34–36], to obtain the desired relation of the multilayer thin body from the corresponding relation of monolayer thin body under the new parameterization, the system of equations of the micropolar theory of multilayer prismatic bodies with constant thickness (each layer has a constant thickness) in displacements and rotations can be written as [3,37]

$$\begin{aligned}
&[\underset{s}{\bar{\Delta}}^3+\underset{s}{A}\underset{s}{\bar{\Delta}}^2+\underset{s}{h}^{-2}(3\underset{s}{\bar{\Delta}}+2\underset{s}{A})\underset{s}{\bar{\Delta}}\partial_3^2+\underset{s}{h}^{-4}(3\underset{s}{\bar{\Delta}}+\underset{s}{A})\partial_3^4+\underset{s}{h}^{-6}\partial_3^6]\underset{s}{\mathbf{u}}+\underset{s}{\mathbf{S}}^{**}=0,\ s=\overline{1,K},\\
&[\underset{s}{\bar{\Delta}}^3+(\underset{s}{B}\underset{s}{\bar{\Delta}}+\underset{s}{A})\underset{s}{\bar{\Delta}}+\underset{s}{h}^{-2}[(3\underset{s}{\bar{\Delta}}+2\underset{s}{B})\underset{s}{\bar{\Delta}}+\underset{s}{C}]\partial_3^2+\underset{s}{h}^{-4}(3\underset{s}{\bar{\Delta}}+\underset{s}{B})\partial_3^4+\underset{s}{h}^{-6}\partial_3^6]\underset{s}{\boldsymbol{\varphi}}+\underset{s}{\mathbf{H}}^{**}=0,
\end{aligned}\tag{38}$$

where $\underset{s}{h}=const$ is the sth layer thickness, $\underset{s}{\bar{\Delta}}=\underset{s}{g}^{\bar{I}\bar{J}}\partial_I\partial_J$. Here, we have introduced the notations

$$\underset{s}{\mathbf{S}}^{**}=\frac{\underset{s}{\mathbf{S}}^{*}}{(\underset{s}{\lambda}+2\underset{s}{\mu})(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})},\quad \underset{s}{\mathbf{H}}^{**}=\frac{\underset{s}{\mathbf{H}}^{*}}{(\underset{s}{\gamma}+2\underset{s}{\delta})(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})},\quad \underset{s}{A}=-\frac{4\underset{s}{\alpha}\underset{s}{\mu}}{(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})},$$

$$\underset{s}{B}=-\frac{4\underset{s}{\alpha}[\underset{s}{\mu}(\underset{s}{\gamma}+2\underset{s}{\delta})+(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})]}{(\underset{s}{\gamma}+2\underset{s}{\delta})(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})},\quad \underset{s}{C}=\frac{16\underset{s}{\alpha}^2\underset{s}{\mu}}{(\underset{s}{\gamma}+2\underset{s}{\delta})(\underset{s}{\mu}+\underset{s}{\alpha})(\underset{s}{\delta}+\underset{s}{\beta})},\quad s=\overline{1,K}.$$

Applying to Equation (38), the kth moment operator of any system of orthogonal polynomials (Legendre and Chebyshev), we obtain for the micropolar theory of prismatic bodies of constant thickness the following equations in moments of displacement and rotation vectors:

$$\begin{aligned}
&[\underset{s}{\bar{\Delta}}^3+\underset{s}{A}\underset{s}{\bar{\Delta}}^2]\overset{(k)}{\underset{s}{\mathbf{u}}}+\underset{s}{h}^{-2}(3\underset{s}{\bar{\Delta}}+2\underset{s}{A})\underset{s}{\bar{\Delta}}\overset{(k)}{\underset{s}{\mathbf{u}}}''+\underset{s}{h}^{-4}(3\underset{s}{\bar{\Delta}}+\underset{s}{A})\overset{(k)}{\underset{s}{\mathbf{u}}}^{IV}+\underset{s}{h}^{-6}\overset{(k)}{\underset{s}{\mathbf{u}}}^{VI}+\overset{(k)}{\underset{s}{\mathbf{S}}}^{**}=0,\\
&[\underset{s}{\bar{\Delta}}^3+(\underset{s}{B}\underset{s}{\bar{\Delta}}+\underset{s}{A})\underset{s}{\bar{\Delta}}]\overset{(k)}{\underset{s}{\boldsymbol{\varphi}}}+\underset{s}{h}^{-2}[(3\underset{s}{\bar{\Delta}}+2\underset{s}{B})\underset{s}{\bar{\Delta}}+\underset{s}{C}]\overset{(k)}{\underset{s}{\boldsymbol{\varphi}}}''+\underset{s}{h}^{-4}(3\underset{s}{\bar{\Delta}}+\underset{s}{B})\overset{(k)}{\underset{s}{\boldsymbol{\varphi}}}^{IV}+\underset{s}{h}^{-6}\overset{(k)}{\underset{s}{\boldsymbol{\varphi}}}^{VI}\\
&+\overset{(k)}{\underset{s}{\mathbf{H}}}^{**}=0,\quad k\geq 0,\ s=\overline{1,K}.
\end{aligned}\tag{39}$$

Note that, in the case of Legendre polynomials, expressions for $\overset{(k)}{\underset{\alpha}{\mathbf{u}}}''$, $\overset{(k)}{\underset{\alpha}{\mathbf{u}}}^{IV}$, $\overset{(k)}{\underset{\alpha}{\mathbf{u}}}^{VI}$, $\overset{(k)}{\underset{\alpha}{\boldsymbol{\varphi}}}''$, $\overset{(k)}{\underset{\alpha}{\boldsymbol{\varphi}}}^{IV}$ and $\overset{(k)}{\underset{\alpha}{\boldsymbol{\varphi}}}^{VI}$ in Equation (39) are found using [3,35] (see also Equation (37))

$$\begin{aligned}
&\overset{(n)}{\mathbf{u}}{}^{(2m)}(x^1,x^2)=(2n+1)\sum_{k=1}^{\infty}C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n+2k+2s-1)\overset{(n+2k+2m-2)}{\mathbf{u}}\\
&=\frac{2n+1}{2}\sum_{k=1}^{2m}(-1)^{k+1}[(\partial_3^{2m-k}\mathbf{u})^{+}+(-1)^{n+k}(\partial_3^{2m-k}\mathbf{u})^{-}]P_n^{k-1}(1)+\overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)},\\
&\overset{(k)}{\underline{\mathbf{u}}}{}^{(2m)}=(2n+1)\sum_{k=1}^{[n/2-m+1]}C_{k+2m-2}^{2m-1}\prod_{s=1}^{2m-1}(2n-2k-2s+3)\overset{(n-2k-2m+2)}{\mathbf{u}},\quad n\geq 0,\ m>0.
\end{aligned}\tag{40}$$

In the relations in Equation (39), s is the index of layers and K is the number of layers. It should be noted that, as in the case of a single layer prismatic body, in the case of multilayer prismatic body for each of the equations, obtained after splitting Equation (39), using the method of Vekua [38], we can write an analytic solution. Consequently, for correct statement of problems the boundary conditions in moments and the interlayer contact conditions must be added to Equation (39) (see in [3,34–37]). To satisfy the boundary conditions on the frontal surfaces, as well as to describe the interlayer contact conditions, under the simplified method of reducing the three-dimensional problem to two dimensions, it is necessary to construct the correcting terms [3,34]. At the same time, the analytical solution of each layer (except the first and last) with the corrective terms can be written so that it satisfies the interlayer contact conditions. For the first (last) layer, the analytical solution by means of corrective terms can be represented in such a way that it satisfies the boundary conditions on the inner (outer) surface and the interlayer contact conditions on the outer (inner) surface. Therefore, we suppose that the interlayer contact conditions would be taken into account better if the order of approximation is higher. This is very important in the theory of multilayer structures. We note that the questions considered above are described in some detail in [37] (see also [3,33–37]).

Note that, if $K = 5$, then we obtain relations for the five-layer theory of prismatic bodies.

5. Inter-Layer Contact Conditions

In studying strained-deformed states of multilayer constructions and composite media, as a rule, one assumes that component layers (elements and phases) work jointly, without sliding. Obviously, such a model does not cover the variety of connection methods used in technology and does not take into account the existence of interphase defects, which manifest themselves in non-perfect connection of phases in contact. Defects of such a type often are undoubted because of peculiarities of technological character (see [39]). Therefore, the deformation of multilayer thin bodies can be without violation or with violation of complete layer contact owing to their separation in normal or tangential direction. Between the layers, there can arise contact domain and contact-free domain. Moreover, the boundaries of these domains can vary in the deformation process, the layer can slide with respect to each other, the sliding can be with friction, etc. All these phenomena can essentially influence the mechanical behavior of a thin body and its strained-deformed state. Of course, accounting for these phenomena is necessary in studying strained-deformed state of multilayer bodies. In contrast to other parameterizations, the use of frontal surfaces as base surfaces in parameterization of multilayer thin body domain allows one to easily take into account these phenomena. In consideration of the phenomena occurring on frontal surfaces, the main problem is the problem of modelling the interface. In this direction, there exist two approaches. The first approach is physical, which takes into account thin adhesion layers via generalized weld condition of elements being in contact. For the first time, such an approach was proposed for heat conduction problems in [40]. Later on, it was generalized to mechanical problems [41]. The second approach is phenomenological; it is based on the assumption that, a priori, the exist discontinuity zones of displacements. To study these problems, we assume that a multilayer thin construction consists of K layers. Denote by $\underset{\alpha}{\overset{(+)}{S}}$ and $\underset{\alpha}{\overset{(-)}{S}}$ $(\alpha = \overline{1,K})$ the exterior and inner surfaces of the layer α $(\alpha = \overline{1,K})$, respectively, and consider some cases of mutual relation of neighboring surfaces $\underset{\alpha}{\overset{(+)}{S}}$ and $\underset{\alpha+1}{\overset{(-)}{S}}$ $(\alpha = \overline{1,K-1})$, which are important in practice.

5.1. Weld Conditions (Complete Ideal Contact Conditions)

In this case, the forces and moments of interaction between the layers α and $\alpha+1$ $(\alpha = \overline{1,K-1})$ are unknown. These forces and moments certainly are equal and have opposite directions. Therefore, there additionally arise six unknown functions. However, in the case considered, we have six additional conditions, which express the continuity of displacement vectors and the rotation of welded surface points. In other words, displacement vectors and rotation vectors of contacted surfaces are equal. Denoting the forces and moments of interaction of the contacted surfaces $\underset{\alpha}{\overset{(+)}{S}}$ and $\underset{\alpha+1}{\overset{(-)}{S}}$ $(\alpha = \overline{1,K-1})$ by $\underset{\alpha}{\overset{(+)}{\mathbf{P}}}$, $\underset{\alpha}{\overset{(+)}{\boldsymbol{\mu}}}$ and $\underset{\alpha+1}{\overset{(-)}{\mathbf{P}}}$, $\underset{\alpha+1}{\overset{(-)}{\boldsymbol{\mu}}}$ $(\alpha = \overline{1,K-1})$, respectively, and the displacement and rotation vectors of points of these surfaces by $\underset{\alpha}{\overset{(+)}{\mathbf{u}}}$, $\underset{\alpha}{\overset{(+)}{\boldsymbol{\varphi}}}$ and $\underset{\alpha+1}{\overset{(-)}{\mathbf{u}}}$, $\underset{\alpha+1}{\overset{(-)}{\boldsymbol{\varphi}}}$ $(\alpha = \overline{1,K-1})$, we can represent the complete contact conditions in micropolar theory of multilayer thin bodies in the form

$$\underset{\alpha}{\overset{(+)}{\mathbf{P}}} = -\underset{\alpha+1}{\overset{(-)}{\mathbf{P}}}, \quad \underset{\alpha}{\overset{(+)}{\boldsymbol{\mu}}} = -\underset{\alpha+1}{\overset{(-)}{\boldsymbol{\mu}}}, \quad \underset{\alpha}{\overset{(+)}{\mathbf{u}}} = \underset{\alpha+1}{\overset{(-)}{\mathbf{u}}}, \quad \underset{\alpha}{\overset{(+)}{\boldsymbol{\varphi}}} = \underset{\alpha+1}{\overset{(-)}{\boldsymbol{\varphi}}}, \quad \alpha = \overline{1,K-1}. \tag{41}$$

Neglecting the characteristics of micropolar theory (the second and fourth relations) in Equation (41), we obtain the ideal contact conditions for the classical theory (the first and third relations).

5.2. Conditions under Relative Displacement of Points of Uneven Contacted Surfaces of Layers

In the case considered here, the slipping with the friction of layers with respect to each other can take place in the process of deformation of the multilayer thin body. The relative slipping does not occur until the magnitude of the tangent component of the interaction force $\overset{(+)}{\underset{\alpha}{\mathbf{P}}}{}_{(s)}$ ($\overset{(-)}{\underset{\alpha+1}{\mathbf{P}}}{}_{(s)}$) (force of friction) between the contacted surfaces reaches its limit (maximal possible) value $|\overset{(+)}{\underset{\alpha}{\mathbf{P}}}{}^{*}|$ ($|\overset{(-)}{\underset{\alpha+1}{\mathbf{P}}}{}^{*}|$), therefore,

$$\underset{\alpha}{\mathbf{v}}(x^1,x^2)=0,\quad \alpha=\overline{1,K-1}. \tag{42}$$

In the case of the classic theory of multilayer thin bodies, we have [3]

$$\begin{gathered}
\overset{(-)}{\underset{\alpha+1}{u}}{}_{(n)}=\overset{(+)}{\underset{\alpha}{u}}{}_{(n)},\quad \overset{(-)}{\underset{\alpha+1}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha+1}{\mathbf{u}},\underset{\alpha+1}{\vartheta})\cdot\overset{(-)}{\underset{\alpha+1}{\mathbf{s}}}=\overset{(-)}{\underset{\alpha+1}{P}}{}^{*}_{(s)},\quad \overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{s}}}=\overset{(+)}{\underset{\alpha}{P}}{}^{*}_{(s)},\\
\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}}=\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha+1}{\mathbf{u}},\underset{\alpha+1}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}},\quad \alpha=\overline{1,K-1},\quad x'\in\overset{(+)}{\underset{\alpha}{S}}{}^{0}\subset\overset{(+)}{\underset{\alpha}{S}},
\end{gathered} \tag{43}$$

and in the case of the micropolar theory of multilayer thin bodies whose layers do not have a center of symmetry, we assume the following relations [3]

$$\begin{gathered}
\overset{(-)}{\underset{\alpha+1}{u}}{}_{(n)}=\overset{(+)}{\underset{\alpha}{u}}{}_{(n)},\quad \overset{(-)}{\underset{\alpha+1}{\varphi}}{}_{(n)}=\overset{(+)}{\underset{\alpha}{\varphi}}{}_{(n)},\quad \overset{(-)}{\underset{\alpha+1}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha+1}{\mathbf{u}},\underset{\alpha+1}{\boldsymbol{\varphi}},\underset{\alpha+1}{\vartheta})\cdot\overset{(-)}{\underset{\alpha+1}{\mathbf{s}}}=\overset{(-)}{\underset{\alpha+1}{P}}{}^{*}_{(s)},\\
\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\boldsymbol{\varphi}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{s}}}=\overset{(+)}{\underset{\alpha}{P}}{}^{*}_{(s)},\quad \overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\boldsymbol{\varphi}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}}=\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\mathbf{P}}}}(\underset{\alpha+1}{\mathbf{u}},\underset{\alpha+1}{\boldsymbol{\varphi}},\underset{\alpha+1}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}},\\
\overset{(-)}{\underset{\alpha+1}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\boldsymbol{\mu}}}}(\underset{\alpha+1}{\boldsymbol{\varphi}},\underset{\alpha+1}{\mathbf{u}},\underset{\alpha+1}{\vartheta})\cdot\overset{(-)}{\underset{\alpha+1}{\mathbf{n}}}=\overset{(-)}{\underset{\alpha+1}{\mu}}{}^{*}_{(n)},\quad \overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\boldsymbol{\mu}}}}(\underset{\alpha}{\boldsymbol{\varphi}},\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}}=\overset{(+)}{\underset{\alpha}{\mu}}{}^{*}_{(n)},\\
\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(+)}{\underset{\alpha}{\underset{\sim}{\boldsymbol{\mu}}}}(\underset{\alpha}{\boldsymbol{\varphi}},\underset{\alpha}{\mathbf{u}},\underset{\alpha}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{s}}}=\overset{(+)}{\underset{\alpha}{\mathbf{n}}}\cdot\overset{(-)}{\underset{\alpha+1}{\underset{\sim}{\boldsymbol{\mu}}}}(\underset{\alpha+1}{\boldsymbol{\varphi}},\underset{\alpha}{\mathbf{u}},\underset{\alpha+1}{\vartheta})\cdot\overset{(+)}{\underset{\alpha}{\mathbf{s}}},\quad \alpha=\overline{1,K-1},\quad x'\in\overset{(+)}{\underset{\alpha}{S}}{}^{0}\subset\overset{(+)}{\underset{\alpha}{S}}.
\end{gathered} \tag{44}$$

Here, naturally, $\overset{(+)}{\underset{\alpha}{\mu}}{}^{*}_{(n)}=\overset{(+)}{\underset{\alpha}{\boldsymbol{\mu}}}{}^{*}\cdot\overset{(+)}{\underset{\alpha}{\mathbf{n}}}$, $\overset{(-)}{\underset{\alpha+1}{\mu}}{}^{*}_{(n)}=\overset{(-)}{\underset{\alpha+1}{\boldsymbol{\mu}}}{}^{*}\cdot\overset{(+)}{\underset{\alpha+1}{\mathbf{n}}}$, where $\overset{(+)}{\underset{\alpha}{\boldsymbol{\mu}}}{}^{*}$ ($\overset{(-)}{\underset{\alpha+1}{\boldsymbol{\mu}}}{}^{*}$) is the intensity of the limit momentum. Therefore, $\overset{(+)}{\underset{\alpha}{P}}{}^{*}_{(s)}$, $\overset{(-)}{\underset{\alpha+1}{P}}{}^{*}_{(s)}$, $\overset{(+)}{\underset{\alpha}{\mu}}{}^{*}_{(n)}$ and $\overset{(-)}{\underset{\alpha+1}{\mu}}{}^{*}_{(n)}$ are unknown values in the relations in Equations (43) and (44) determined from some a priori dependencies, conditions of slipping with friction, which, generally speaking, must depend on geometric and physical-mechanical properties of contacted bodies. In the classic case, we may suppose the following relations hold:

$$\mathbf{L}(x^1,x^2,\mathbf{v}_s,\dot{\mathbf{v}}_s,[T],\mathbf{P}^{(l)*},\dots)=0, \tag{45}$$

where $\mathbf{v}_s$ and $\dot{\mathbf{v}}_s$ are the tangent components of the vectors of the relative displacement and relative velocity, $[T]$ is the temperature jump, $\mathbf{P}^{(l)*}$ is the limit stress vector on a plane element with the normal $\mathbf{l}$, and the ellipsis denotes the dependence on some other parameters. Based on Equation (45), we can accept that the generalized model of Coulomb friction is valid:

$$\mathbf{P}^{*}_{(s)}=\underset{\sim}{\mathbf{f}}(x^1,x^2,[T],\mathbf{P}^{*}_{(n)})\cdot\dot{\mathbf{v}}_s, \tag{46}$$

which takes into account the anisotropy of the friction. Here, $\mathbf{P}^*_{(s)}$ and $\mathbf{P}^*_{(n)}$ are the limit tangent and normal components of the stress vector $\mathbf{P}^{(l)*}$. The second-rank tensor $\underset{\sim}{\mathbf{f}}(x^1, x^2, [T], \mathbf{P}^*_{(n)})$ is called the tensor of friction coefficients. Obviously, in the isotropic case, we have $\underset{\sim}{\mathbf{f}} = f\underset{\sim}{\mathbf{E}}$, where $\underset{\sim}{\mathbf{E}}$ is the unit second rank tensor. Representing Equation (46) for contacted surfaces of a multilayer thin body, we obtain the missing required relations. Based on similar arguments in the case of the micropolar theory, we can assert that the following a priori relations are valid:

$$\begin{aligned} &\mathbf{L}(x^1, x^2, \mathbf{v}_s, \dot{\mathbf{v}}_s, \boldsymbol{\psi}_n, \dot{\boldsymbol{\psi}}_n, [T], \mathbf{P}^{(l)*}, \ldots) = 0, \\ &\mathbf{M}(x^1, x^2, \mathbf{v}_s, \dot{\mathbf{v}}_s, \boldsymbol{\psi}_n, \dot{\boldsymbol{\psi}}_n, [T], \boldsymbol{\mu}^{(l)*}, \ldots) = 0, \end{aligned} \tag{47}$$

where $\boldsymbol{\psi}_n$ and $\dot{\boldsymbol{\psi}}_n$ are the normal components of the vectors of the relative internal rotation and relative internal rotation velocity of adjacent layers, $\boldsymbol{\mu}^{(l)*}$ is the limit vector of the couple stress on a plane element with the normal $\mathbf{l}$, and the other parameters are the same as in Equation (45). Based on Equation (47) and similar to Equation (46), for the micropolar theory, we can assume that the following relations are valid:

$$\begin{aligned} &\mathbf{P}^*_{(s)} = \underset{\sim}{\mathbf{f}}(x^1, x^2, [T], \mathbf{P}^*_{(n)}) \cdot \dot{\mathbf{v}}_s + \underset{\sim}{\mathbf{h}}(x^1, x^2, [T], \mathbf{P}^*_{(n)}) \cdot \dot{\boldsymbol{\psi}}_n, \\ &\boldsymbol{\mu}^*_{(n)} = \underset{\sim}{\mathbf{g}}(x^1, x^2, [T], \boldsymbol{\mu}^*_{(s)}) \cdot \dot{\boldsymbol{\psi}}_n + \underset{\sim}{\mathbf{l}}(x^1, x^2, [T], \boldsymbol{\mu}^*_{(s)}) \cdot \dot{\mathbf{v}}_s, \end{aligned} \tag{48}$$

that take into account the anisotropy of the friction. Here, $\underset{\sim}{\mathbf{f}}$, $\underset{\sim}{\mathbf{h}}$, $\underset{\sim}{\mathbf{g}}$ and $\underset{\sim}{\mathbf{l}}$ are the second-rank tensors called the tensors of friction coefficients. Therefore, in the case of an isotropic friction, we have $\underset{\sim}{\mathbf{f}} = f\underset{\sim}{\mathbf{E}}$, $\underset{\sim}{\mathbf{h}} = h\underset{\sim}{\mathbf{E}}$, $\underset{\sim}{\mathbf{g}} = g\underset{\sim}{\mathbf{E}}$ and $\underset{\sim}{\mathbf{l}} = l\underset{\sim}{\mathbf{E}}$, where $\underset{\sim}{\mathbf{E}}$ is the unit second-rank tensor. It should be noted here that the coefficients of friction are determined by experiments and are given in tables. The authors know little in this direction for the micropolar theory, but for the classic theory these coefficients can be obtained, e.g., from the work in [42–44]. Representing Equation (48) for the contacted surfaces of a multilayer thin body, we get the missing required relations in the case of the micropolar theory.

The very important direction is the study of eigenvalue problems for the tensor and tensor-block matrix of any even rank, since the constitutive relations for most classical and micropolar media of different rheology (which here, of course, also include porous and multilayer textile media) are written using a tensor and tensor-block matrix of even rank, respectively. Some questions concerning these problems, as well as tensor calculus, have been studied in some detail in [4,29,45–49]. A very important direction is also the investigation of internal structures of differential tensors-operators and tensor-block matrix operators of even rank. This is due to the fact that such operators are operators of systems of equations of motion and static boundary conditions with respect to kinematic characteristics (displacement, rotation, velocity, angular velocity) for most classical and micropolar mediums. The study of these problems promotes the decomposition of initial-boundary value problems in the case of linear theories. Some questions about the decomposition of initial-boundary value problems can be found in [3,37]. We note that the boundary and initial conditions are also given there in the moments which, for the formulation of initial-boundary value problems in moments for multilayer thin bodies, must be added to the equations and CR in moments and interlayer contact conditions. We do not write out them to shorten the letter, but we refer the interested reader to the works mentioned above.

6. Conclusions

In this paper, some questions regarding the new parameterization of a multilayer thin domain are considered. In contrast with classic approaches, several base surfaces and an analytic method with application of orthogonal polynomial systems are used. Geometric characteristics typical for the new parameterizations are introduced into consideration. Various presentations of the equations of motion and the constitutive relations of physical and heat contents are given under the new parameterization of the body domain. Various variants of the equations of motion in moments with respect to orthogonal polynomial systems are obtained. The equations in moments of (displacement vector) displacement and rotation vectors with respect to any system of orthogonal polynomials (Legendre or Chebyshev) for the (classical) micropolar theory of prismatic multilayer elastic bodies of constant thickness are given. The complete ideal contact conditions and the conditions under relative displacement of points of uneven contacted surfaces of layers are given. The formulations of initial-boundary value problems in the theory of multilayer elastic thin bodies in moments are discussed.

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