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A Unified Generalization of the Catalan, Fuss, and Fuss—Catalan Numbers

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Abstract: In the paper, the authors introduce a unified generalization of the Catalan numbers, the Fuss numbers, the Fuss–Catalan numbers, and the Catalan–Qi function, and discover some properties of the unified generalization, including a product-ratio expression of the unified generalization in terms of the Catalan–Qi functions, three integral representations of the unified generalization, and the logarithmically complete monotonicity of the second order for a special case of the unified generalization.

Keywords: unified generalization; Catalan number; Fuss number; Fuss–Catalan number; Catalan–Qi function; product-ratio expression; logarithmically complete monotonicity of the second order; integral representation

MSC: Primary 11B83; Secondary 11B75, 11R33, 11S23, 11Y35, 11Y55, 11Y60, 26A48, 26A51, 30E20, 33B15

1. Introduction

As well known from [1,2], Catalan numbers C_n are used in the study of set partitions in different areas of mathematics. In particular, in combinatorial mathematics, the Catalan numbers C_n form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. There are many counting problems in combinatorics whose solution is given by the Catalan numbers C_n. The book [3] contains a set of exercises which describe 66 different interpretations of the Catalan numbers.

The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}}=\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}C_nx^n=1+x+2x^2+5x^3+14x^4+42x^5+132x^6+\cdots.$$

One of the explicit formulas of C_n for $n\ge 0$ reads that

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

is the classical Euler gamma function. In [2,4,5] and ([1] pp. 110–111), it was mentioned that there exists an asymptotic expansion

$$C_x \triangleq \frac{4^x\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(x+2)} \sim \frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac{9}{8}\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right) \tag{1}$$

for the Catalan function C_x. For new developments on (1), see [6] and the review paper in [7], in which there are plenty of closely related references.

A generalization of the Catalan numbers C_n was defined in [8,9] by

$${}_pd_n = \frac{1}{n}\binom{pn}{n-1} = \frac{1}{(p-1)n+1}\binom{pn}{n}$$

for $n,p\ge 1$. It is obvious that $C_n = {}_2d_n$. In ([1] pp. 375–376), the generalization ${}_{p+1}d_n$ of the Catalan numbers C_n is denoted by $C(n,p)$ for $p\ge 0$ and is called as the generalized Catalan numbers. In ([1] pp. 377–378), the Fuss numbers

$$F(m,n) = \frac{1}{mn+1}\binom{mn+1}{n}$$

were given and discussed. It is apparent that $F(2,n)=C_n$.

In combinatorial mathematics and statistics, the Fuss–Catalan numbers $A_n(p,r)$ are defined [10] as numbers of the form

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n} = \frac{r\Gamma(np+r)}{\Gamma(n+1)\Gamma(n(p-1)+r+1)}. \tag{2}$$

It is easy to see that

$$A_n(p,1)=F(p,n), \quad A_n(2,1)=C_n, \quad n\ge 0$$

and

$$A_{n-1}(p,p) = {}_pd_n = C(n,p-1), \quad n,p\ge 1.$$

There have existed some literature, such as [1,2,11–21], on the investigation of the Fuss–Catalan numbers $A_n(p,r)$.

In [22], an alternative and analytical generalization of the Catalan numbers C_n and the Catalan function C_x was introduced by

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(z+a)}{\Gamma(z+b)}, \quad \Re(a),\Re(b)>0, \quad \Re(z)\ge 0.$$

For uniqueness and convenience of referring to the quantity $C(a,b;z)$, we call the quantity $C(a,b;z)$ the Catalan–Qi function and, when taking $z=n\ge 0$, call $C(a,b;n)$ the Catalan–Qi numbers. It is not difficult to verify that $C\left(\frac{1}{2},2;n\right)=C_n$ and

$$C(n+1,2;(m-1)n) = \left(\frac{2}{n+1}\right)^{(m-1)n}{}_md_n = \left(\frac{2}{n+1}\right)^{(m-1)n}C(n,m-1)$$

for $m,n\ge 1$. In [22], it was obtained that

$$C(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{(z+a)^z}{(z+b)^{z+b-a}}\exp\left[b-a+\int_0^\infty\frac{1}{t}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-a\right)\left(e^{-at}-e^{-bt}\right)e^{-zt}\,\mathrm{d}\,t\right] \tag{3}$$

for $\Re(a), \Re(b) > 0$ and $\Re(z) \ge 0$. Recently, we discovered in ([23] Theorem 1.1) relations between the Fuss–Catalan numbers $A_n(p,r)$ and the Catalan–Qi numbers $C(a,b;n)$, one of which reads that

$$A_n(p,r) = r^n \frac{\prod_{k=0}^{p-1} C\left(\frac{k+r}{p}, 1; n\right)}{\prod_{k=0}^{p-2} C\left(\frac{k+r+1}{p-1}, 1; n\right)} \tag{4}$$

for integers $n \ge 0$, $p > 1$, and $r > 0$. In recent papers [6,22–32], among other things, some properties, including the general expression and a generalization of the asymptotic expansion (1), the monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, minimality, Schur-convexity, product and determinantal inequalities, exponential representations, integral representations, a generating function, connections with the Bessel polynomials and the Bell polynomials of the second kind, and identities, of the Catalan numbers C_n, the Catalan function C_x, the Catalan–Qi numbers $C(a,b;n)$, the Catalan–Qi function $C(a,b;z)$, and the Fuss–Catalan numbers $A_n(p,r)$ were established.

In this paper, we will introduce a unified generalization of the Catalan numbers C_n, the generalized Catalan numbers $C(n,k)$, the Fuss numbers $F(k,n)$, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;z)$. Hereafter, we will find a product-ratio expression, similar to the product-ratio expression (4), of the unified generalization in terms of the Catalan–Qi function $C(a,b;z)$. Furthermore, based on the integral representation (3), on the Gauss multiplication formula for the gamma function, and on an integral representation for the logarithm of the gamma function $\Gamma(z)$, we will derive three integral representations of the unified generalization. Finally, we will establish the logarithmically complete monotonicity of the second order for the unified generalization.

2. A Unified Generalization of the Catalan and Other Numbers

Does there exist a unified and analytic generalization of the Catalan numbers C_n, the Fuss numbers $F(m,n) = C(n,m)$, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;z)$? What is it concretely? In the early morning of September 15th 2015, a unified generalization was framed out eventually and successfully, and which can be described by a five-variable function

$$Q(a,b;p,q;z) = \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a}\right)^{(q-p+1)z} [\Gamma(z+1)]^{q-p} \frac{\Gamma(pz+a)}{\Gamma(qz+b)}, \tag{5}$$

where $\Re(a), \Re(b) > 0$, $\Re(p), \Re(q) > 0$, and $\Re(z) \ge 0$. For uniqueness and convenience of referring to the quantity $Q(a,b;p,q;z)$, we call $Q(a,b;p,q;z)$ the Fuss–Catalan–Qi function and, when taking $z = n \ge 0$, call $Q(a,b;p,q;n)$ the Fuss–Catalan–Qi numbers.

It is easy to see that

$$\begin{gathered} Q\left(\frac{1}{2}, 2; 1, 1; n\right) = Q(1,2;2,1;n) = C_n, \quad Q(r, r+1; p, p-1; n) = A_n(p,r), \\ Q(p, p+1; p, p-1; n-1) = {}_pd_n = C(n, p-1), \quad Q(a,b;1,1;z) = C(a,b;z). \end{gathered} \tag{6}$$

Accordingly, the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$ is a unified generalization of the Catalan numbers C_n, the generalized Catalan numbers $C(n,m)$, the Fuss numbers $F(m,n)$, the Fuss–Catalan numbers $A_n(p,r)$, and the Catalan–Qi function $C(a,b;z)$.

It is easy to see that the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$ meets

$$Q(b,a;p,q;z) = \left(\frac{a}{b}\right)^{2(q-p)z} \frac{1}{Q(a,b;q,p;z)}$$

and, when $p = q$ or $a = b$,

$$Q(a,b;q,p;z)Q(b,a;p,q;z) = 1.$$

If only interchanging the role of a and b, then

$$Q(a,b;p,q;z)Q(b,a;p,q;z) = \begin{cases} \dfrac{R(a,b;2p;z)}{R(b,a;2q;z)}, \\ S(a;p,q;z)S(b;p,q;z), \end{cases}$$

where

$$R(a,b;r;z) = \frac{\Gamma(rz+a)\Gamma(rz+b)}{[\Gamma(z+1)]^r}$$

and

$$S(c;p,q;z) = [\Gamma(z+1)]^{q-p}\frac{\Gamma(pz+c)}{\Gamma(qz+c)}.$$

If only swapping p and q, then

$$Q(a,b;p,q;z)Q(a,b;q,p;z) = \begin{cases} \dfrac{F(b;p,q;z)}{F(a;p,q;z)}, \\ G(a,b;p;z)G(a,b;q;z), \end{cases}$$

where

$$F(c;p,q;z) = \frac{[c^z\Gamma(c)]^2}{\Gamma(pz+c)\Gamma(qz+c)}$$

and

$$G(a,b;r;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z\frac{\Gamma(rz+a)}{\Gamma(rz+b)}.$$

3. A Product-Ratio Expression of the Fuss–Catalan–Qi Function

Motivated by the product-ratio expression (4), we now find out a product-ratio expression of the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$.

Theorem 1. *For $\Re(a),\Re(b)>0$ and $\Re(z)\ge 0$, when $p,q\in\mathbb{N}$, we have*

$$Q(a,b;p,q;z) = \left[\left(\frac{b}{a}\right)^{q-p+1}\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(q+b)}\right]^z\frac{\prod_{k=0}^{p-1}C\left(\frac{k+a}{p},1;z\right)}{\prod_{k=0}^{q-1}C\left(\frac{k+b}{q},1;z\right)}. \tag{7}$$

Proof. By the Gauss multiplication formula

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right), \quad n\in\mathbb{N} \tag{8}$$

in ([33] p. 256, 6.1.20), the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$ can be written as

$$\begin{aligned} Q(a,b;p,q;z) &= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z}[\Gamma(z+1)]^{q-p}\frac{\Gamma\left(p\left(z+\frac{a}{p}\right)\right)}{\Gamma\left(q\left(z+\frac{b}{q}\right)\right)} \\ &= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z}[\Gamma(z+1)]^{q-p}\frac{\dfrac{p^{pz+a-1/2}}{(2\pi)^{(p-1)/2}}\prod_{k=0}^{p-1}\Gamma\left(z+\frac{k+a}{p}\right)}{\dfrac{q^{qz+b-1/2}}{(2\pi)^{(q-1)/2}}\prod_{k=0}^{q-1}\Gamma\left(z+\frac{k+b}{q}\right)} \end{aligned}$$

$$= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z} \frac{\dfrac{p^{pz+a-1/2}}{(2\pi)^{(p-1)/2}} \displaystyle\prod_{k=0}^{p-1} \frac{\Gamma\left(z+\frac{k+a}{p}\right)}{\Gamma(z+1)}}{\dfrac{q^{qz+b-1/2}}{(2\pi)^{(q-1)/2}} \displaystyle\prod_{k=0}^{q-1} \frac{\Gamma\left(z+\frac{k+b}{q}\right)}{\Gamma(z+1)}}$$

$$= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z} \frac{\displaystyle\prod_{k=0}^{q-1} \frac{\Gamma(1)}{\Gamma\left(\frac{k+b}{q}\right)}\left(\frac{q}{k+b}\right)^{z}}{\displaystyle\prod_{k=0}^{p-1} \frac{\Gamma(1)}{\Gamma\left(\frac{k+a}{p}\right)}\left(\frac{p}{k+a}\right)^{z}} \frac{\dfrac{p^{pz+a-1/2}}{(2\pi)^{(p-1)/2}} \displaystyle\prod_{k=0}^{p-1} \frac{\Gamma(1)}{\Gamma\left(\frac{k+a}{p}\right)}\left(\frac{p}{k+a}\right)^{z} \frac{\Gamma\left(z+\frac{k+a}{p}\right)}{\Gamma(z+1)}}{\dfrac{q^{qz+b-1/2}}{(2\pi)^{(q-1)/2}} \displaystyle\prod_{k=0}^{q-1} \frac{\Gamma(1)}{\Gamma\left(\frac{k+b}{q}\right)}\left(\frac{q}{k+b}\right)^{z} \frac{\Gamma\left(z+\frac{k+b}{q}\right)}{\Gamma(z+1)}}$$

$$= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z} \frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{a}{p}+\frac{k}{p}\right)}{\prod_{k=0}^{q-1}\Gamma\left(\frac{b}{q}+\frac{k}{q}\right)} \frac{\prod_{k=0}^{p-1}(k+a)^{z}}{\prod_{k=0}^{q-1}(k+b)^{z}} \frac{q^{zq}}{p^{zp}} \frac{\dfrac{p^{pz+a-1/2}}{(2\pi)^{(p-1)/2}} \displaystyle\prod_{k=0}^{p-1} C\left(\tfrac{k+a}{p},1;z\right)}{\dfrac{q^{qz+b-1/2}}{(2\pi)^{(q-1)/2}} \displaystyle\prod_{k=0}^{q-1} C\left(\tfrac{k+b}{q},1;z\right)}$$

$$= \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z} \frac{\Gamma(a)\dfrac{(2\pi)^{(p-1)/2}}{p^{a-1/2}}}{\Gamma(b)\dfrac{(2\pi)^{(q-1)/2}}{q^{b-1/2}}} \left[\frac{\prod_{k=0}^{p-1}(k+a)}{\prod_{k=0}^{q-1}(k+b)}\right]^{z} \frac{p^{a-1/2}}{(2\pi)^{(p-1)/2}} \frac{(2\pi)^{(q-1)/2}}{q^{b-1/2}} \frac{\prod_{k=0}^{p-1} C\left(\frac{k+a}{p},1;z\right)}{\prod_{k=0}^{q-1} C\left(\frac{k+b}{q},1;z\right)}$$

$$= \left(\frac{b}{a}\right)^{(q-p+1)z} \left[\frac{\prod_{k=0}^{p-1}(k+a)}{\prod_{k=0}^{q-1}(k+b)}\right]^{z} \frac{\prod_{k=0}^{p-1} C\left(\frac{k+a}{p},1;z\right)}{\prod_{k=0}^{q-1} C\left(\frac{k+b}{q},1;z\right)}$$

$$= \left(\frac{b}{a}\right)^{(q-p+1)z} \left[\frac{\Gamma(b)\Gamma(p+a)}{\Gamma(a)\Gamma(q+b)}\right]^{z} \frac{\prod_{k=0}^{p-1} C\left(\frac{k+a}{p},1;z\right)}{\prod_{k=0}^{q-1} C\left(\frac{k+b}{q},1;z\right)}.$$

The identity (7) is thus proved. The proof of Theorem 1 is complete. □

Remark 1. *Before getting (7), we did not appreciate the analytic meanings of the form of the product-ratio expression (4) because before catching sight of the unified generalization (5), we did not appreciate the analytic meanings of the form of the Fuss–Catalan numbers $A_n(p,r)$ in (2).*

Remark 2. *From (6) and (7), we derive the identity (4) and*

$${}_{p}d_n = C(n,p-1) = p^{n-1} \frac{\prod_{k=0}^{p-1} C\left(1+\frac{k}{p},1;n-1\right)}{\prod_{k=0}^{p-2} C\left(1+\frac{k+2}{p-1},1;n-1\right)}.$$

Remark 3. *When $p=q$, the product-ratio expression (7) can be reformulated as*

$$Q(a,b;q,q;z) = \frac{K(a,q,z)}{K(b,q,z)},$$

where

$$K(c,q,z) = \left[\frac{\Gamma(q+c)}{\Gamma(1+c)}\right]^{z} \prod_{k=0}^{q-1} C\left(\frac{k+c}{q},1;z\right).$$

If taking $a=b$, then

$$Q(a,a;p,q;z) = \frac{\dfrac{\Gamma(pz+a)}{[\Gamma(z+1)]^{p}}}{\dfrac{\Gamma(qz+a)}{[\Gamma(z+1)]^{q}}} = \frac{[\Gamma(p+a)]^{z}\prod_{k=0}^{p-1} C\left(\frac{k+a}{p},1;z\right)}{[\Gamma(q+a)]^{z}\prod_{k=0}^{q-1} C\left(\frac{k+a}{q},1;z\right)}.$$

4. Integral Representations of the Fuss–Catalan–Qi Function

Making use of the integral representation (3) and the product-ratio expression (7), we now derive the first integral representation of the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$.

Theorem 2. *For $\Re(a), \Re(b) > 0$, and $\Re(z) \ge 0$, when $p, q \in \mathbb{N}$, we have*

$$
\begin{aligned}
Q(a,b;p,q;z) = {} & (2\pi)^{(q-p)/2}\frac{\Gamma(b)}{\Gamma(a)}\frac{p^{a-1/2}}{q^{b-1/2}}(z+1)^{(q-p)(z+1/2)+(a-b)} \\
& \times \left[\left(\frac{b}{a}\right)^{q-p+1}\frac{\prod_{k=0}^{p-1}(a+pz+k)}{\prod_{k=0}^{q-1}(b+qz+k)}\right]^z \exp\biggl\{\frac{p-q}{2}+(b-a) \\
& + \int_0^\infty \frac{e^{-zt}}{t}\biggl[\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{a}{p}\right)\left(\frac{1-e^{-t}}{1-e^{-t/p}}e^{-at/p}-pe^{-t}\right) \\
& - \left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{b}{q}\right)\left(\frac{1-e^{-t}}{1-e^{-t/q}}e^{-bt/q}-qe^{-t}\right) \\
& - \left(\frac{e^{-t/p}}{p(1-e^{-t/p})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/p}}e^{-at/p} \\
& + \left(\frac{e^{-t/q}}{q(1-e^{-t/q})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/q}}e^{-bt/q}+\frac{p-q}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\}.
\end{aligned}
$$

Proof. Making use of the integral representation (3) leads to

$$
\begin{aligned}
\prod_{k=0}^{p-1} C\left(\frac{k+a}{p},1;z\right) = {} & \prod_{k=0}^{p-1}\left[\frac{1}{\Gamma\left(\frac{k+a}{p}\right)}\left(\frac{p}{k+a}\right)^z\frac{\left(z+\frac{k+a}{p}\right)^z}{(z+1)^{z+1-\frac{k+a}{p}}}\right. \\
& \times \left.\exp\left[\sum_{k=0}^{p-1}\left(1-\frac{k+a}{p}\right)+\sum_{k=0}^{p-1}\int_0^\infty\frac{1}{t}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{k+a}{p}\right)\left(e^{-\frac{k+a}{p}t}-e^{-t}\right)e^{-zt}\,\mathrm{d}\,t\right]\right] \\
= {} & \frac{1}{\prod_{k=0}^{p-1}\Gamma\left(\frac{k+a}{p}\right)}\frac{1}{\left[\prod_{k=0}^{p-1}(k+a)\right]^z}\frac{\prod_{k=0}^{p-1}(pz+k+a)^z}{(z+1)^{p(z+1)}}(z+1)^{a+\frac{\sum_{k=0}^{p-1}k}{p}} \\
& \times \exp\left[p-a-\frac{\sum_{k=0}^{p-1}k}{p}+\int_0^\infty\frac{e^{-zt}}{t}\sum_{k=0}^{p-1}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{k+a}{p}\right)\left(e^{-\frac{k+a}{p}t}-e^{-t}\right)\mathrm{d}\,t\right] \\
= {} & \frac{1}{\prod_{k=0}^{p-1}\Gamma\left(\frac{a}{p}+\frac{k}{p}\right)}\left[\frac{\Gamma(a)}{\Gamma(p+a)}\right]^z\left[\frac{\Gamma(p(z+1)+a)}{\Gamma(pz+a)}\right]^z\frac{(z+1)^{a+(p-1)/2}}{(z+1)^{p(z+1)}} \\
& \times \exp\left[p-a-\frac{p-1}{2}+\int_0^\infty\frac{e^{-zt}}{t}\sum_{k=0}^{p-1}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{k+a}{p}\right)\left(e^{-\frac{k+a}{p}t}-e^{-t}\right)\mathrm{d}\,t\right] \\
= {} & \frac{p^{a-1/2}}{(2\pi)^{(p-1)/2}\Gamma(a)}\left[\frac{\Gamma(a)\Gamma(p)}{\Gamma(p+a)}\right]^z\left[\frac{\Gamma(p(z+1)+a)}{\Gamma(pz+a)\Gamma(p)}\right]^z\frac{1}{(z+1)^{p(z+1/2)-a+1/2}} \\
& \times \exp\biggl\{\frac{p}{2}-a+\frac{1}{2}+\int_0^\infty\frac{e^{-zt}}{t}\biggl[\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{a}{p}\right)\left(\sum_{k=0}^{p-1}e^{-\frac{k+a}{p}t}-pe^{-t}\right) \\
& \qquad -\sum_{k=0}^{p-1}\frac{k}{p}e^{-\frac{k+a}{p}t}+\frac{p-1}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\} \\
= {} & \frac{p^{a-1/2}}{(2\pi)^{(p-1)/2}\Gamma(a)}\left[\frac{B(a,p)}{B(a+pz,p)}\right]^z\frac{1}{(z+1)^{p(z+1/2)-a+1/2}} \\
& \times \exp\left\{\frac{p}{2}-a+\frac{1}{2}+\int_0^\infty\frac{e^{-zt}}{t}\left[\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{a}{p}\right)\left(\sum_{k=0}^{p-1}e^{-\frac{k+a}{p}t}-pe^{-t}\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{k=0}^{p-1}\frac{k}{p}e^{-\frac{k+a}{p}t}+\frac{p-1}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\}\\
&=\frac{p^{a-1/2}}{(2\pi)^{(p-1)/2}\Gamma(a)}\biggl[\frac{B(a,p)}{B(a+pz,p)}\biggr]^{z}\frac{1}{(z+1)^{p(z+1/2)-a+1/2}}\\
&\times\exp\biggl\{\frac{p}{2}-a+\frac{1}{2}+\int_0^\infty\frac{e^{-zt}}{t}\biggl[\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{a}{p}\biggr)\biggl(e^{-at/p}\frac{1-e^{-t}}{1-e^{-t/p}}-pe^{-t}\biggr)\\
&-\frac{p(e^{-t/p}-1)e^{-t}+(1-e^{-t})e^{-t/p}}{p(1-e^{-t/p})^2}e^{-at/p}+\frac{p-1}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\},
\end{aligned}
$$

where $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the classical beta function. Similarly, we also have

$$
\begin{aligned}
\prod_{k=0}^{q-1}C\biggl(\frac{k+b}{q},1;z\biggr)&=\frac{q^{b-1/2}}{(2\pi)^{(q-1)/2}\Gamma(b)}\biggl[\frac{B(b,q)}{B(b+qz,q)}\biggr]^{z}\frac{1}{(z+1)^{q(z+1/2)-b+1/2}}\\
&\times\exp\biggl\{\frac{q}{2}-b+\frac{1}{2}+\int_0^\infty\frac{e^{-zt}}{t}\biggl[\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{b}{q}\biggr)\biggl(e^{-bt/q}\frac{1-e^{-t}}{1-e^{-t/q}}-qe^{-t}\biggr)\\
&-\frac{q(e^{-t/q}-1)e^{-t}+(1-e^{-t})e^{-t/q}}{q(1-e^{-t/q})^2}e^{-bt/q}+\frac{q-1}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\frac{\prod_{k=0}^{p-1}C\bigl(\frac{k+a}{p},1;z\bigr)}{\prod_{k=0}^{q-1}C\bigl(\frac{k+b}{q},1;z\bigr)}&=(2\pi)^{(q-p)/2}(z+1)^{(q-p)(z+1/2)+(a-b)}\frac{\Gamma(b)}{\Gamma(a)}\frac{p^{a-1/2}}{q^{b-1/2}}\\
&\times\biggl[\frac{B(a,p)}{B(a+pz,p)}\frac{B(b+qz,q)}{B(b,q)}\biggr]^{z}\exp\biggl\{\frac{p-q}{2}+(b-a)\\
&+\int_0^\infty\frac{e^{-zt}}{t}\biggl[\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{a}{p}\biggr)\biggl(e^{-at/p}\frac{1-e^{-t}}{1-e^{-t/p}}-pe^{-t}\biggr)\\
&-\biggl(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{b}{q}\biggr)\biggl(e^{-bt/q}\frac{1-e^{-t}}{1-e^{-t/q}}-qe^{-t}\biggr)\\
&-\frac{p(e^{-t/p}-1)e^{-t}+(1-e^{-t})e^{-t/p}}{p(1-e^{-t/p})^2}e^{-at/p}\\
&+\frac{q(e^{-t/q}-1)e^{-t}+(1-e^{-t})e^{-t/q}}{q(1-e^{-t/q})^2}e^{-bt/q}+\frac{p-q}{2}e^{-t}\biggr]\,\mathrm{d}\,t\biggr\}.
\end{aligned}
$$

Substituting this into (5) and simplifying yields the integral representation in Theorem 2. The proof of Theorem 2 is complete. □

By the Gauss multiplication formula (8) and an integral representation for the logarithm of the gamma function $\Gamma(z)$, we can acquire the second integral representation of the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$, which is seemingly simpler than the one in Theorem 2.

Theorem 3. *For $a,b>0$ and $x\ge 0$, when $p,q\in\mathbb{N}$, we have*

$$
\begin{aligned}
Q(a,b;p,q;x)&=(2\pi)^{(q-p)/2}e^{(p-q)/2+b-a}\frac{\Gamma(b)}{\Gamma(a)}\biggl(\frac{b}{a}\biggr)^{(q-p+1)x}\\
&\times(x+1)^{(q-p)(x+1/2)}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}\frac{\prod_{k=0}^{p-1}\bigl(x+\frac{a+k}{p}\bigr)^{x+(a+k)/p-1/2}}{\prod_{k=0}^{q-1}\bigl(x+\frac{b+k}{q}\bigr)^{x+(b+k)/q-1/2}}
\end{aligned}
$$

$$\times \exp\left\{\int_0^\infty \beta(t)\left[(q-p)e^{-t}+\frac{1-e^{-t}}{1-e^{-t/p}}e^{-at/p}-\frac{1-e^{-t}}{1-e^{-t/q}}e^{-bt/q}\right]e^{-xt}\,\mathrm{d}\,t\right\}, \tag{9}$$

where

$$\beta(t)=\frac{1}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right). \tag{10}$$

Proof. By Formula (8) and

$$\ln\Gamma(z)=\ln\left(\sqrt{2\pi}\,z^{z-1/2}e^{-z}\right)+\int_0^\infty \beta(t)e^{-zt}\,\mathrm{d}\,t \tag{11}$$

in ([34] (3.22)), we have

$$\begin{aligned}
&\ln\left([\Gamma(x+1)]^{q-p}\frac{\Gamma(px+a)}{\Gamma(qx+b)}\right)=(q-p)\ln\Gamma(x+1)+\ln\frac{\Gamma\left(p\left(x+\frac{a}{p}\right)\right)}{\Gamma\left(q\left(x+\frac{b}{q}\right)\right)}\\
&=(q-p)\ln\Gamma(x+1)+\ln\frac{\frac{p^{px+a-1/2}}{(2\pi)^{(p-1)/2}}\prod_{k=0}^{p-1}\Gamma\left(x+\frac{a+k}{p}\right)}{\frac{q^{qx+b-1/2}}{(2\pi)^{(q-1)/2}}\prod_{k=0}^{q-1}\Gamma\left(x+\frac{b+k}{q}\right)}\\
&=(q-p)\ln\Gamma(x+1)+\ln\left[(2\pi)^{(q-p)/2}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}\right]\\
&\quad+\sum_{k=0}^{p-1}\ln\Gamma\left(x+\frac{a+k}{p}\right)-\sum_{k=0}^{q-1}\ln\Gamma\left(x+\frac{b+k}{q}\right)\\
&=(q-p)\ln\left[\sqrt{2\pi}\,(x+1)^{x+1/2}e^{-(x+1)}\right]+(q-p)\int_0^\infty\beta(t)e^{-(x+1)t}\,\mathrm{d}\,t\\
&\quad+\ln\left[(2\pi)^{(q-p)/2}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}\right]-\sum_{k=0}^{q-1}\int_0^\infty\beta(t)e^{-[x+(b+k)/q]t}\,\mathrm{d}\,t\\
&\quad+\sum_{k=0}^{p-1}\ln\left[\sqrt{2\pi}\left(x+\frac{a+k}{p}\right)^{x+(a+k)/p-1/2}e^{-[x+(a+k)/p]}\right]\\
&\quad-\sum_{k=0}^{q-1}\ln\left[\sqrt{2\pi}\left(x+\frac{b+k}{q}\right)^{x+(b+k)/q-1/2}e^{-[x+(b+k)/q]}\right]+\sum_{k=0}^{p-1}\int_0^\infty\beta(t)e^{-[x+(a+k)/p]t}\,\mathrm{d}\,t\\
&=\ln\left[(2\pi)^{(q-p)/2}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}(x+1)^{(q-p)(x+1/2)}\frac{\prod_{k=0}^{p-1}\left(x+\frac{a+k}{p}\right)^{x+(a+k)/p-1/2}}{\prod_{k=0}^{q-1}\left(x+\frac{b+k}{q}\right)^{x+(b+k)/q-1/2}}\right]\\
&\quad+(p-q)(x+1)-\sum_{k=0}^{p-1}\left(x+\frac{a+k}{p}\right)+\sum_{k=0}^{q-1}\left(x+\frac{b+k}{q}\right)\\
&\quad+(q-p)\int_0^\infty\beta(t)e^{-(x+1)t}\,\mathrm{d}\,t+\int_0^\infty\beta(t)e^{-(x+a/p)t}\sum_{k=0}^{p-1}e^{-kt/p}\,\mathrm{d}\,t\\
&\quad-\int_0^\infty\beta(t)e^{-(x+b/q)t}\sum_{k=0}^{q-1}e^{-kt/q}\,\mathrm{d}\,t\\
&=\ln\left[(2\pi)^{(q-p)/2}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}(x+1)^{(q-p)(x+1/2)}\frac{\prod_{k=0}^{p-1}\left(x+\frac{a+k}{p}\right)^{x+(a+k)/p-1/2}}{\prod_{k=0}^{q-1}\left(x+\frac{b+k}{q}\right)^{x+(b+k)/q-1/2}}\right]\\
&\quad+\frac{p-q}{2}+b-a+(q-p)\int_0^\infty\beta(t)e^{-(x+1)t}\,\mathrm{d}\,t\\
&\quad+\int_0^\infty\beta(t)e^{-(x+a/p)t}\frac{1-e^{-t}}{1-e^{-t/p}}\,\mathrm{d}\,t-\int_0^\infty\beta(t)e^{-(x+b/q)t}\frac{1-e^{-t}}{1-e^{-t/q}}\,\mathrm{d}\,t
\end{aligned}$$

$$
\begin{aligned}
&= \ln\left[(2\pi)^{(q-p)/2}\frac{p^{px+a-1/2}}{q^{qx+b-1/2}}(x+1)^{(q-p)(x+1/2)}\frac{\prod_{k=0}^{p-1}\left(x+\frac{a+k}{p}\right)^{x+(a+k)/p-1/2}}{\prod_{k=0}^{q-1}\left(x+\frac{b+k}{q}\right)^{x+(b+k)/q-1/2}}\right]\\
&\quad+\frac{p-q}{2}+b-a+\int_0^\infty\beta(t)\left[(q-p)e^{-t}+\frac{1-e^{-t}}{1-e^{-t/p}}e^{-at/p}-\frac{1-e^{-t}}{1-e^{-t/q}}e^{-bt/q}\right]e^{-xt}\,\mathrm{d}\,t.
\end{aligned}
$$

Substituting this into (5) leads to the integral representation (9). □

Only by the integral representation (11), we can establish the third integral representation of the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$, which is seemingly simpler than the previous ones.

Theorem 4. *For $a,b,p,q>0$ and $x\ge 0$, we have*

$$
\begin{aligned}
Q(a,b;p,q;x)&=\frac{\Gamma(b)}{\Gamma(a)}(2\pi)^{(q-p)/2}e^{p-q+b-a}\left(\frac{b}{a}\right)^{(q-p+1)x}(x+1)^{(q-p)(x+1/2)}\frac{(px+a)^{px+a-1/2}}{(qx+b)^{qx+b-1/2}}\\
&\quad\times\exp\left\{\int_0^\infty\beta(t)\left[(q-p)e^{-(x+1)t}-e^{-(qx+b)t}+e^{-(px+a)t}\right]\mathrm{d}\,t\right\},
\end{aligned}\tag{12}
$$

where $\beta(t)$ is defined by (10)*.*

Proof. By virtue of (11), we obtain

$$
\begin{aligned}
\ln\left([\Gamma(x+1)]^{q-p}\frac{\Gamma(px+a)}{\Gamma(qx+b)}\right)&=(q-p)\ln\Gamma(x+1)+\ln\Gamma(px+a)-\ln\Gamma(qx+b)\\
&=(q-p)\ln\left[\sqrt{2\pi}\,(x+1)^{x+1/2}e^{-(x+1)}\right]+(q-p)\int_0^\infty\beta(t)e^{-(x+1)t}\,\mathrm{d}\,t\\
&\quad+\ln\left[\sqrt{2\pi}\,(px+a)^{px+a-1/2}e^{-(px+a)}\right]+\int_0^\infty\beta(t)e^{-(px+a)t}\,\mathrm{d}\,t\\
&\quad-\ln\left[\sqrt{2\pi}\,(qx+b)^{qx+b-1/2}e^{-(qx+b)}\right]-\int_0^\infty\beta(t)e^{-(qx+b)t}\,\mathrm{d}\,t\\
&=p-q+b-a+\ln\left[(2\pi)^{(q-p)/2}(x+1)^{(q-p)(x+1/2)}\frac{(px+a)^{px+a-1/2}}{(qx+b)^{qx+b-1/2}}\right]\\
&\quad+\int_0^\infty\beta(t)\left[(q-p)e^{-(x+1)t}+e^{-(px+a)t}-e^{-(qx+b)t}\right]\mathrm{d}\,t.
\end{aligned}
$$

Substituting this into (5) leads to the integral representation (12). The proof of Theorem 4 is complete. □

Remark 4. *From* (6) *and the integral representation in Theorem 2, we obtain*

$$
\begin{aligned}
A_n(p,r)&=\frac{1}{\sqrt{2\pi}}r\frac{p^{r-1/2}}{(p-1)^{r+1/2}}\frac{1}{(n+1)^{n+3/2}}\left[\frac{\prod_{k=0}^{p-1}(k+r+pv)}{\prod_{k=0}^{p-2}(k+r+(p-1)v+1)}\right]^n\\
&\quad\times\exp\left\{\frac{3}{2}+\int_0^\infty\frac{e^{-vt}}{t}\left[\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{r}{p}\right)\left(e^{-\frac{r}{p}t}\frac{1-e^{-t}}{1-e^{-t/p}}-pe^{-t}\right)\right.\right.\\
&\quad-\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{r+1}{p-1}\right)\left(e^{-\frac{r+1}{p-1}t}\frac{1-e^{-t}}{1-e^{-t/(p-1)}}-(p-1)e^{-t}\right)\\
&\quad-\left(\frac{e^{-t/p}}{p(1-e^{-t/p})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/p}}e^{-\frac{r}{p}t}\\
&\quad\left.\left.+\left(\frac{e^{-t/(p-1)}}{(p-1)(1-e^{-t/(p-1)})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/(p-1)}}e^{-\frac{r+1}{p-1}t}+\frac{1}{2}e^{-t}\right]\mathrm{d}\,t\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
C(n,p-1) = {} & \frac{1}{\sqrt{2\pi}}\left(\frac{p}{p-1}\right)^{p+1/2}\frac{1}{(n+1)^{n+1/2}}\left[\frac{\prod_{k=1}^{p-1}(k+pn)}{\prod_{k=0}^{p-2}(k+(p-1)n+2)}\right]^{n} \\
& \times \exp\left\{\frac{3}{2}+\int_0^\infty \frac{e^{-(n-1)t}}{t}\left[\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-1\right)\left(\frac{1-e^{-t}}{1-e^{-t/p}}-p\right)e^{-t}\right.\right. \\
& -\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{p+1}{p-1}\right)\left(e^{-\frac{p+1}{p-1}t}\frac{1-e^{-t}}{1-e^{-t/(p-1)}}-(p-1)e^{-t}\right) \\
& -\left(\frac{e^{-t/p}}{p(1-e^{-t/p})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/p}}e^{-t} \\
& \left.\left.+\left(\frac{e^{-t/(p-1)}}{(p-1)(1-e^{-t/(p-1)})}-\frac{e^{-t}}{1-e^{-t}}\right)\frac{1-e^{-t}}{1-e^{-t/(p-1)}}e^{-\frac{p+1}{p-1}t}+\frac{1}{2}e^{-t}\right]\mathrm{d}\,t\right\}.
\end{aligned}
$$

Remark 5. *When $p = q$, the integral representation in Theorem 2 is reduced to*

$$
\begin{aligned}
Q(a,b;q,q;z) = {} & \frac{\Gamma(b)}{\Gamma(a)}q^{a-b}(z+1)^{a-b}\left[\frac{b}{a}\prod_{k=0}^{q-1}\frac{a+qz+k}{b+qz+k}\right]^{z} \\
& \times \exp\left\{b-a+\int_0^\infty \frac{e^{-zt}}{t}\left[(a-b)e^{-t}+\left(\frac{b}{q}e^{-bt/q}-\frac{a}{q}e^{-at/q}\right)\frac{1-e^{-t}}{1-e^{-t/q}}\right.\right. \\
& \left.\left.+\left(\frac{1+e^{-t}}{1-e^{-t}}-\frac{e^{-t/q}}{q(1-e^{-t/q})}-\frac{1}{t}\right)\frac{1-e^{-t}}{1-e^{-t/q}}\left(e^{-at/q}-e^{-bt/q}\right)\right]\mathrm{d}\,t\right\}.
\end{aligned}
$$

Remark 6. *By the integral representation (12) and the second relation in (6), we find*

$$
\begin{aligned}
A_n(p,r) = {} & \frac{e^2}{\sqrt{2\pi}}r(n+1)^{-(n+1/2)}\frac{(pn+r)^{pn+r-1/2}}{[(p-1)n+r+1]^{(p-1)n+r+1/2}} \\
& \times \exp\left\{\int_0^\infty \beta(t)\left[e^{-(pn+r)t}-e^{-((p-1)n+r+1)t}-e^{-(n+1)t}\right]\mathrm{d}\,t\right\}.
\end{aligned}
$$

Remark 7. *The function $Q(a,b;p,q;z)$ defined by (5) can be rewritten as*

$$
Q(a,b;p,q;z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^{(q-p+1)z}\frac{G(b,q,z)}{G(a,p,z)},
$$

where

$$
G(c,r,z) = \frac{[\Gamma(z+1)]^r}{\Gamma(rz+c)}.
$$

Taking the logarithm of $G(c,r,z)$ and differentiating gives

$$
\ln G(c,r,z) = r\ln\Gamma(z+1) - \ln\Gamma(rz+c)
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}\,z}[\ln G(c,r,z)] = r[\psi(z+1)-\psi(rz+c)].
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,z}\left[\ln\frac{G(b,q,z)}{G(a,p,z)}\right] &= q[\psi(z+1)-\psi(qz+b)] - p[\psi(z+1)-\psi(pz+a)] \\
&= (q-p)\psi(z+1) - [q\psi(qz+b)-p\psi(pz+a)]
\end{aligned}
$$

and

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}\,z^{k+1}}\left[\ln\frac{G(b,q,z)}{G(a,p,z)}\right]=(q-p)\psi^{(k)}(z+1)-q^{k+1}\psi^{(k)}(qz+b)+p^{k+1}\psi^{(k)}(pz+a)$$

for $k\in\{0\}\cup\mathbb{N}$. Further making use of (15) arrives at

$$(-1)^{k+1}\frac{\mathrm{d}^{k+1}}{\mathrm{d}\,z^{k+1}}\left[\ln\frac{G(b,q,z)}{G(a,p,z)}\right]=\int_0^\infty\frac{t^k}{1-e^{-t}}\left[(q-p)e^{-(z+1)t}-q^{k+1}e^{-(qz+b)t}+p^{k+1}e^{-(pz+a)t}\right]\mathrm{d}\,t$$

for $k\in\mathbb{N}$. Consequently, for $k\in\mathbb{N}$, we have

$$(-1)^{k+1}\frac{\mathrm{d}^{k+1}\ln Q(a,b;p,q;z)}{\mathrm{d}\,z^{k+1}}=\int_0^\infty\frac{t^k}{1-e^{-t}}\left[(q-p)e^{-(z+1)t}-q^{k+1}e^{-(qz+b)t}+p^{k+1}e^{-(pz+a)t}\right]\mathrm{d}\,t.$$

5. Properties of the Fuss–Catalan–Qi Function

Recall from ([35] Chapter XIII), ([36] Chapter 1), and ([37] Chapter IV) that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0\le(-1)^kf^{(k)}(x)<\infty$ on I for all $k\ge0$.

Recall from [38,39] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0\le(-1)^k[\ln f(x)]^{(k)}<\infty$ holds on I for all $k\in\mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [40–43] and plenty of references therein.

Recall from [38] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of the k-th order on an interval I.

Stimulated by the above definitions and main results in [44], we now introduce the concept of logarithmically completely monotonic functions of the k-th order.

Definition 1. *For a positive function $f(x)$ on an interval I, if $[\ln f(x)]^{(k)}$ for some nonnegative integer k is completely monotonic on an interval I but $[\ln f(x)]^{(k-1)}$ is not completely monotonic on I, then we call $f(x)$ a logarithmically completely monotonic function of the k-th order on I.*

In terms of the terminology of logarithmically completely monotonic functions of the k-th order, we can state the main results of this section as the following theorem.

Theorem 5. *The function*

$$Q(a,b;q,q;x)=\frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x\frac{\Gamma(qx+a)}{\Gamma(qx+b)},\quad a,b,p>0,\quad x\ge0$$

satisfies the following conclusions:

1. *if $a<b$ and $q\le\frac{\ln b-\ln a}{\psi(b)-\psi(a)}$, the function $Q(a,b;q,q;x)$ is increasing on $[0,\infty)$;*
2. *if $a>b$ and $q\le\frac{\ln b-\ln a}{\psi(b)-\psi(a)}$, the function $Q(a,b;q,q;x)$ is decreasing on $[0,\infty)$;*
3. *if $a<b$ and $q>\frac{\ln b-\ln a}{\psi(b)-\psi(a)}$, the function $Q(a,b;q,q;x)$ has a unique minimum on $(0,\infty)$;*
4. *if $a>b$ and $q>\frac{\ln b-\ln a}{\psi(b)-\psi(a)}$, the function $Q(a,b;q,q;x)$ has a unique maximum on $(0,\infty)$;*
5. *if and only if $a\lessgtr b$, the function $[Q(a,b;q,q;x)]^{\pm}$ is logarithmically completely monotonic of the second order on $[0,\infty)$; in particular, if and only if $a\lessgtr b$, the function $[Q(a,b;q,q;x)]^{\pm1}$ is logarithmically convex on $[0,\infty)$.*

Proof. Taking the logarithm on both sides of Equation (5) and differentiating with respect to x yields

$$\frac{\mathrm{d}[\ln Q(a,b;p,q;x)]}{\mathrm{d}\,x}=(q-p+1)\ln\frac{b}{a}+(q-p)\psi(x+1)+p\psi(px+a)-q\psi(qx+b)$$

and

$$\frac{\mathrm{d}^2[\ln Q(a,b;p,q;x)]}{\mathrm{d}\,x^2} = (q-p)\psi'(x+1) + p^2\psi'(px+a) - q^2\psi'(qx+b).$$

It is not difficult to see that

$$\lim_{x\to 0^+} \frac{\mathrm{d}[\ln Q(a,b;p,q;x)]}{\mathrm{d}\,x} = (q-p+1)\ln\frac{b}{a} + (q-p)\psi(1) + p\psi(a) - q\psi(b) \tag{13}$$

and

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathrm{d}[\ln Q(a,b;p,q;x)]}{\mathrm{d}\,x} &= (q-p+1)\ln\frac{b}{a} + \lim_{x\to\infty}\Bigg\{(q-p)\Bigg[\ln(x+1) \\
&\quad - \frac{1}{2(x+1)} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(x+1)^{2n}}\Bigg] + p\Bigg[\ln(px+a) - \frac{1}{2(px+a)} \\
&\quad - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(px+a)^{2n}}\Bigg] - q\Bigg[\ln(qx+b) - \frac{1}{2(qx+b)} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(qx+b)^{2n}}\Bigg]\Bigg\} \\
&= (q-p+1)\ln\frac{b}{a} + \lim_{x\to\infty}[(q-p)\ln(x+1) + p\ln(px+a) - q\ln(qx+b)] \\
&\quad + \lim_{x\to\infty}\Bigg\{(p-q)\Bigg[\frac{1}{2(x+1)} + \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(x+1)^{2n}}\Bigg] \\
&\quad - p\Bigg[\frac{1}{2(px+a)} + \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(px+a)^{2n}}\Bigg] + q\Bigg[\frac{1}{2(qx+b)} + \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(qx+b)^{2n}}\Bigg]\Bigg\} \\
&= (q-p+1)\ln\frac{b}{a} + \lim_{x\to\infty}[(q-p)\ln(x+1) + p\ln(px+a) - q\ln(qx+b)] \\
&= (q-p+1)\ln\frac{b}{a} + p\ln p - q\ln q \\
&= \ln\left[\left(\frac{b}{a}\right)^{q-p+1}\frac{p^p}{q^q}\right],
\end{aligned} \tag{14}$$

where the asymptotic expansion

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}$$

as $z \to \infty$ in $|\arg z| < \pi$ (see [33] p. 259, 6.3.18) was used, and B_k stands for the Bernoulli numbers that are defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty}\frac{B_k x^k}{k!}.$$

When $p = q$, making use of

$$\psi^{(k)}(z) = (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}}e^{-zt}\,\mathrm{d}\,t, \quad \Re(z) > 0, \quad k \in \mathbb{N} \tag{15}$$

in ([33] p. 260, 6.4.1) leads to

$$\frac{\mathrm{d}^2[\ln Q(a,b;q,q;x)]}{\mathrm{d}\,x^2} = q^2[\psi'(qx+a) - \psi'(qx+b)] = q^2\int_0^\infty \frac{t}{1-e^{-t}}e^{-qxt}\left(e^{-at} - e^{-bt}\right)\mathrm{d}\,t,$$

which means that the derivative $\pm\frac{d^2[\ln Q(a,b;q,q;x)]}{d\,x^2}$ is completely monotonic on $[0,\infty)$ if and only if $a \lesssim b$. Hence, the first derivative $\pm\frac{d[\ln Q(a,b;q,q;x)]}{d\,x}$ is increasing on $[0,\infty)$ if and only if $a \lesssim b$. Meanwhile, the limits (13) and (14) become

$$\lim_{x\to0^+}\frac{d[\ln Q(a,b;q,q;x)]}{d\,x} = \ln\frac{b}{a} + q[\psi(a)-\psi(b)] = [\psi(a)-\psi(b)]\left[q - \frac{\ln b - \ln a}{\psi(b)-\psi(a)}\right]$$

and

$$\lim_{x\to\infty}\frac{d[\ln Q(a,b;q,q;x)]}{d\,x} = \ln\frac{b}{a}.$$

As a result,

1. if $a < b$ and $q \le \frac{\ln b - \ln a}{\psi(b)-\psi(a)}$, the first derivative $\frac{d[\ln Q(a,b;q,q;x)]}{d\,x}$ is non-negative on $[0,\infty)$;
2. if $a < b$ and $q > \frac{\ln b - \ln a}{\psi(b)-\psi(a)}$, the first derivative $\frac{d[\ln Q(a,b;q,q;x)]}{d\,x}$ has a unique zero, which is a minimum point of $\ln Q(a,b;q,q;x)$, on $(0,\infty)$;
3. if $a > b$ and $q \le \frac{\ln b - \ln a}{\psi(b)-\psi(a)}$, the first derivative $\frac{d[\ln Q(a,b;q,q;x)]}{d\,x}$ is non-positive on $[0,\infty)$;
4. if $a > b$ and $q > \frac{\ln b - \ln a}{\psi(b)-\psi(a)}$, the first derivative $\frac{d[\ln Q(a,b;q,q;x)]}{d\,x}$ has a unique zero, which is a maximum point of $\ln Q(a,b;q,q;x)$, on $(0,\infty)$.

Therefore, the conclusions on $Q(a,b;q,q;x)$ are thus proved. The proof of Theorem 5 is complete. □

6. Remarks

Finally, we list additional several remarks.

Remark 8. *Combining Theorem 5 and the last relation in (6), we obtain that the Catalan–Qi function $C(a,b;x)$ is a logarithmically completely monotonic function of the second order.*

Remark 9. *Similar to the introduction of the Catalan–Qi function $C(a,b;z)$ in [22], we had better leave the combinatorial interpretation of the Fuss–Catalan–Qi function $Q(a,b;p,q;z)$ to combinatorialists and number theorists.*

Remark 10. *In recent years, there were many results on the Catalan numbers C_n and the Catalan–Qi numbers $C(a,b;n)$ in [45–53] and the closely related references therein.*

Remark 11. *This paper is a corrected and revised version of the preprint [54].*

7. Conclusions

In this paper, we introduce a unified generalization of the Catalan numbers, the Fuss numbers, the Fuss–Catalan numbers, and the Catalan–Qi function, and discover some properties of the unified generalization, including a product-ratio expression of the unified generalization in terms of the Catalan–Qi functions, three integral representations of the unified generalization, and the logarithmically complete monotonicity of the second order for a special case of the unified generalization.

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