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Accurate Approximate Solution of Ambartsumian Delay Differential Equation via Decomposition Method

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Abstract: The Ambartsumian delay equation is used in the theory of surface brightness in the Milky way. The Adomian decomposition method (ADM) is applied in this paper to solve this equation. Two canonical forms are implemented to obtain two types of the approximate solutions. The first solution is provided in the form of a power series which agrees with the solution in the literature, while the second expresses the solution in terms of exponential functions which is viewed as a new solution. A rapid rate of convergence has been achieved and displayed in several graphs. Furthermore, only a few terms of the new approximate solution (expressed in terms of exponential functions) are sufficient to achieve extremely accurate numerical results when compared with a large number of terms of the first solution in the literature. In addition, the residual error using a few terms approaches zero as the delay parameter increases, hence, this confirms the effectiveness of the present approach over the solution in the literature.

Keywords: Adomian decomposition method; Ambartsumian equation; Milky way; series solution

1. Introduction

In astronomy, Ambartsumian [1] derived a delay differential equation (DE) for the surface brightness in the Milky Way. The standard form of this equation is given as [2],

$$y'(t) + y(t) = \frac{1}{q}\, y\left(\frac{t}{q}\right), \quad q > 1, \tag{1}$$

where q is a constant for the given model and Equation (1) is subjected to

$$y(0) = \lambda, \tag{2}$$

where λ is also a constant. Existence and uniqueness has been proved and discussed by Kato and McLeod [3]. It is of great importance to search for accurate solution for Equations (1) and (2) due to its application in astronomy. Very recently, Patade and Bhalekar [2] have obtained a power series solution for this system by applying the Daftardar-Gejji and Jafari Method [4]. They have discussed the convergence for all $|q| > 1$, however, their solution is not valid in the whole domain as will be shown in this paper. In order to overcome such drawback, a new series solution shall be deduced by using the Adomian decomposition method (ADM) which possesses more accuracy and validity.

The ADM was applied to solving algebraic/transcendental/matrix equations [5–9], besides nonlinear integral/differential equations and both IVPs/BVPs, even for irregular boundary contours [10–24]. The solution of this method is an infinite series which converges when choosing an appropriate canonical form. Hence, a few terms achieve good accuracy for the model under consideration.

A very good analysis for delayed differential equations has been presented by [25,26]. In addition, theoretical analysis for the convergence of Adomian's method to differential equations has been discussed earlier by Abbaoui and Cherruault [27]. We remark that a significant advantage of the ADM for solution of differential equations is that it neither invokes the fixed-point theorem to prove convergence nor is the Adomian solution algorithm developed in accordance with its premise. Also, the speed of convergence and the general error estimation of the series solution using the standard ADM have been previously reported by Cherruault and Adomian [28]. Moreover, Rach [29] has introduced an extensive bibliography of the theory, technique, and applications of the Adomian decomposition method.

The objective of this work is to reinvestigate the Ambartsumian delay equation by using the ADM. Two different canonical forms will be constructed. It will be shown that our first solution agrees with the power series solution [2], while the second series solution is expressed in terms of the exponential functions and this can be viewed as a new type of solution for the current problem. Also, it will be demonstrated that the sequence of the approximate solutions of the second type (expressed in terms of the exponential functions) converges faster than the solution in the literatures.

2. Application of the ADM

2.1. Power Series Solution

We rewrite Equation (1) in the following canonical form for the ADM

$$y(t) = \lambda + \int_0^t \left[q^{-1}\, y\left(q^{-1}\tau\right) - y(\tau)\right] d\tau. \tag{3}$$

$y(t)$ is assumed as

$$y(t) = \sum_{i=0}^{\infty} y_i(t). \tag{4}$$

On inserting (4) into (3), we have

$$\begin{aligned} y_0(t) &= \lambda, \\ y_i(t) &= \int_0^t \left[q^{-1}\, y_{i-1}\left(q^{-1}\tau\right) - y_{i-1}(\tau)\right] d\tau, \quad i \geq 1. \end{aligned} \tag{5}$$

Therefore

$$\begin{aligned} y_1 &= \frac{t}{1!}\left(q^{-1}-1\right)\lambda, \\ y_2 &= \frac{t^2}{2!}\left(q^{-1}-1\right)\left(q^{-2}-1\right)\lambda = \frac{t^2}{2!}\prod_{k=1}^{2}\left(q^{-k}-1\right)\lambda, \\ y_3 &= \frac{t^3}{3!}\left(q^{-1}-1\right)\left(q^{-2}-1\right)\left(q^{-3}-1\right)\lambda = \frac{t^3}{3!}\prod_{k=1}^{3}\left(q^{-k}-1\right)\lambda, \\ y_4 &= \frac{1}{4!}\left(q^{-1}-1\right)\left(q^{-2}-1\right)\left(q^{-3}-1\right)\left(q^{-4}-1\right)\lambda = \frac{t^4}{4!}\prod_{k=1}^{4}\left(q^{-k}-1\right)\lambda, \\ &. \\ &. \\ &. \\ y_i &= \frac{t^i}{i!}\left(q^{-1}-1\right)\left(q^{-2}-1\right)\ldots\left(q^{-(i-1)}-1\right)\left(q^{-i}-1\right)\lambda = \frac{t^i}{i!}\prod_{k=1}^{i}\left(q^{-k}-1\right)\lambda. \end{aligned} \tag{6}$$

Hence

$$
\begin{aligned}
y(t) \quad &= y_0 + \sum_{i=1}^{\infty} y_i, \\
&= \lambda + \lambda \sum_{i=1}^{\infty} \frac{t^i}{i!} \prod_{k=1}^{i} \left(q^{-k} - 1\right), \\
&= \lambda \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \left(q^{-k} - 1\right)\right) \frac{t^i}{i!}\right],
\end{aligned}
\tag{7}
$$

and this is the same closed form solution obtained by Patade and Bhalekar [2]. Therefore, the m-term approximate solution of the power series (7) is given by

$$
\psi_m(t) = \lambda \left[1 + \sum_{i=0}^{m-1} \left(\prod_{k=1}^{i+1} \left(q^{-k} - 1\right)\right) \frac{t^{i+1}}{(i+1)!}\right], \quad m \geq 1. \tag{8}
$$

2.2. *Approximate Solution in Terms of Exponential Functions*

Equation (1) can be rewritten in the following canonical form

$$
y(t) = \lambda\, e^{-t} + \frac{1}{q}\, e^{-t} \int_0^t e^{\tau} y\left(\frac{\tau}{q}\right) d\tau. \tag{9}
$$

On using (4) into (9), we obtain the following recurrence scheme

$$
\begin{aligned}
&y_0(t) = \lambda\, e^{-t}, \\
&y_i(t) = \tfrac{1}{q}\, e^{-t} \int_0^t e^{\tau} y_{i-1}\left(\tfrac{\tau}{q}\right) d\tau, \quad i \geq 1,
\end{aligned}
\tag{10}
$$

and hence,

$$
\begin{aligned}
y_1 &= \frac{\lambda}{q-1}\left(e^{-t/q} - e^{-t}\right), \\
y_2 &= \frac{\lambda}{(q^2-1)(q+1)}\left[q\, e^{-t/q^2} - (q+1)e^{-t/q} + e^{-t}\right], \\
y_3 &= \frac{\lambda}{(q^3-1)(q+1)}\left[q^3 e^{-t/q^3} - (q^3+q^2+q)e^{-t/q^2} + (1+q+q^2)e^{-t/q} - e^{-t}\right], \\
&\;\;. \\
&\;\;. \\
&\;\;.
\end{aligned}
\tag{11}
$$

and so on. The n-term approximation $\Phi_n(t)$ for Ambartsumian equation is

$$
\Phi_n(t) = \sum_{i=0}^{n-1} y_i(t). \tag{12}
$$

In the following section, it will be illustrated that the sequence of the approximate solutions in (12) is convergent in a wider range than those in the literature at $q > 1$. Moreover, the present approximate numerical results will be validated by calculating the absolute residual error $|RE_n(t)|$ defined by

$$
|RE_n(t)| = \left|\Phi_n'(t) + \Phi_n(t) - \frac{1}{q}\, \Phi_n\left(\frac{t}{q}\right)\right|, \quad n \geq 1. \tag{13}
$$

In addition, the advantage and the effectiveness of the present low-order approximate analytic solutions for the Ambartsumian delay equation over the existing method in the literature will be proved for certain higher values of the parameter q.

3. Discussion

In the previous section, the ADM was applied to obtain two types of approximate solutions for the system (1) and (2). These two types are investigated in this discussion to stand on their domains of applicability and validity. Usually, we begin by graphically demonstrating the convergence of the approximate solutions $\Phi_n(t)$ in Equation (12). In Figures 1–3, $\Phi_7(t)$, $\Phi_9(t)$, and $\Phi_{11}(t)$ are plotted at a fixed value of $\lambda = 1$ and different values for q, where $q = 1.5$ (Figure 1), $q = 1.6$ (Figure 2), and $q = 2$ (Figure 3). a rapid convergence is observed from these figures using only a few terms of the Adomian approximate solutions of exponential terms (11). The main notice here is that the rate of convergence is increased for higher values of q, where at $q \geq 2$ the 7-term, 9-term, and 11-term are nearly identical. Hence, $\Phi_{11}(t)$ of the decomposition method is sufficient to provide a remarkably accurate solution as will be shown later by discussing the absolute residuals $|RE_7|$, $|RE_9|$, and $|RE_{11}|$, while at the lower values of q in the domain $1 < q < 2$, a higher-order approximate solution such as $\Phi_n(t)$ for $n \geq 11$ is required to achieve a similar high accuracy.

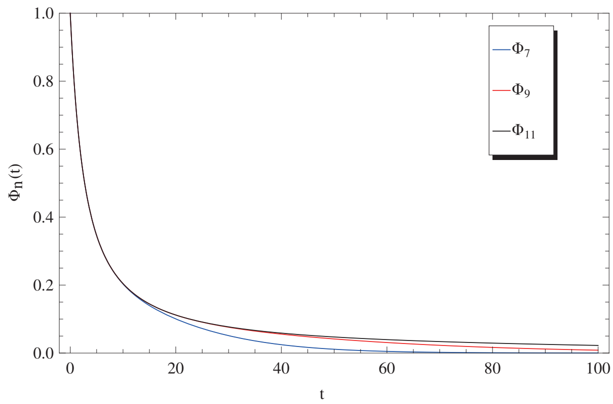

Figure 1. Plots of approximate solutions (12) at $\lambda = 1$ and $q = 1.5$.

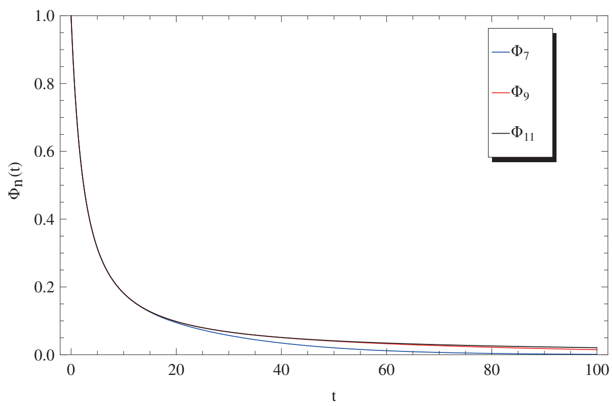

Figure 2. Plots of approximate solutions (12) at $\lambda = 1$ and $q = 1.6$.

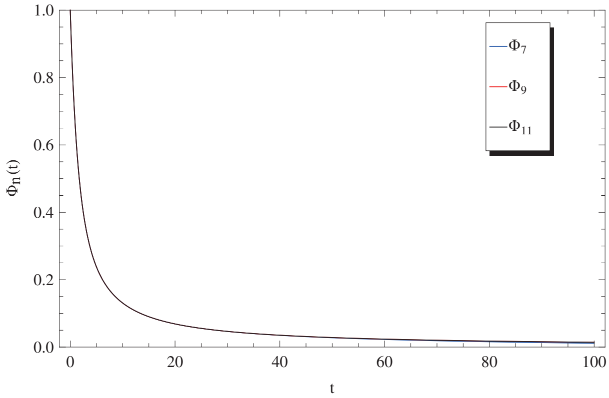

Figure 3. Plots of approximate solutions (12) at $\lambda = 1$ and $q = 2$.

In addition, the approximate solutions $\Phi_7(t)$, $\Phi_9(t)$, and $\Phi_{11}(t)$ of the second type (exponential) are valid in the whole domain of t (≥ 0). However, the approximate solution χ_{100} of the first type (power series) is only valid in sub-domains as shown from Figures 4–6. The comparisons between the two types of the Adomian approximate solutions reveal that the second type posses some advantages over the first type that was expressed as power series. Moreover, a few terms of the second type is sufficient to achieve accurate numerical results in a wider range when compared with the 100-term of the power series solution (8).

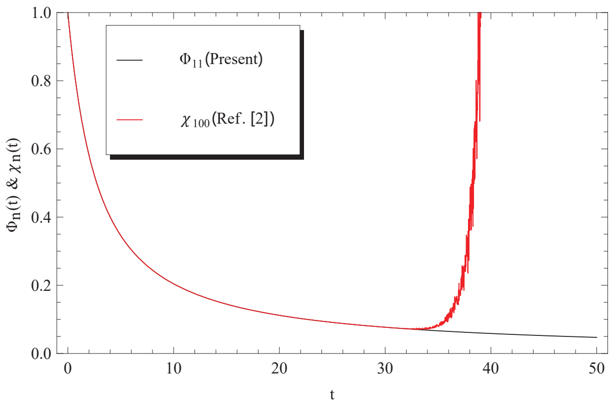

Figure 4. Comparison between approximate solutions (12) and Ref. [2] at $\lambda = 1$ and $q = 1.5$.

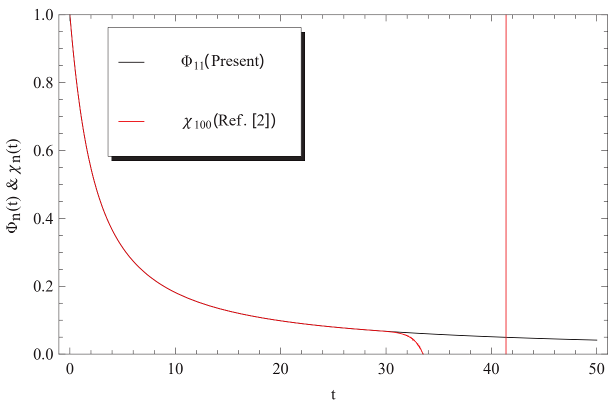

Figure 5. Comparison between approximate solutions (12) and Ref. [2] at $\lambda = 1$ and $q = 1.6$.

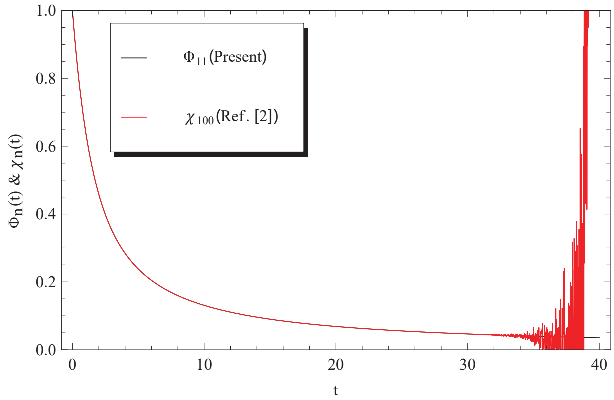

Figure 6. Comparison between approximate solutions (12) and Ref. [2] at $\lambda = 1$ and $q = 2$.

The impacts of the initial condition λ and the delay parameter q on the approximation $\Phi_{11}(t)$ for the fluctuations of the surface brightness $y(t)$ are respectively depicted in Figures 7 and 8. It can be seen from Figure 7 that the surface brightness is increased by increasing the given initial condition λ. However, a rapid decrease in the surface brightness has been remarked by increasing the delay parameter q. This latest notice reveals that the curves of $\Phi_{11}(t)$ tend faster to zero at higher values of q. Moreover, as $q \to \infty$ we have from Equation (11) that

$$\lim_{q\to\infty} y_1 = \lim_{q\to\infty} y_2 = \lim_{q\to\infty} y_3 = \ldots \lim_{q\to\infty} y_j = 0, \quad \forall j \geq 1, \tag{14}$$

and accordingly,

$$\lim_{q\to\infty} (u(t)) = y_0(t) = \lambda\, e^{-t}. \tag{15}$$

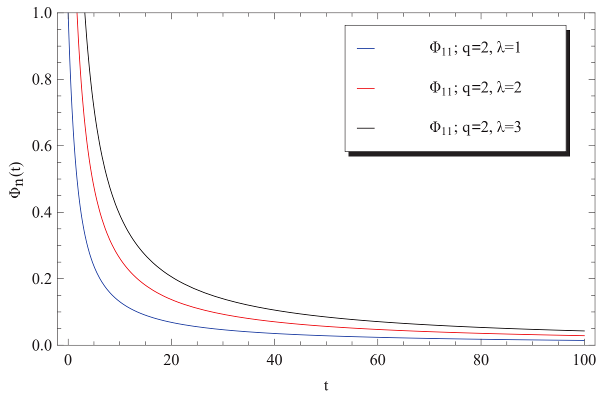

Figure 7. Impact of λ on approximate solution (12) at $q = 2$.

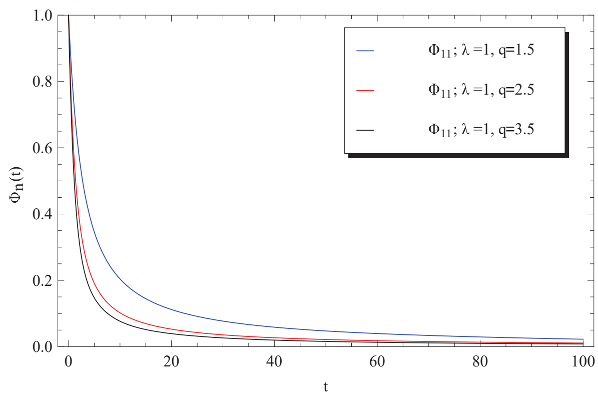

Figure 8. Impact of q on approximate solution (12) at $\lambda = 1$.

For a further validation of the current numerical results, two additional plots for the absolute residual error $|RE_{11}|$ versus t at different values of λ ($q = 2$) and at different values of q ($\lambda = 1$) in Figures 9 and 10, respectively.

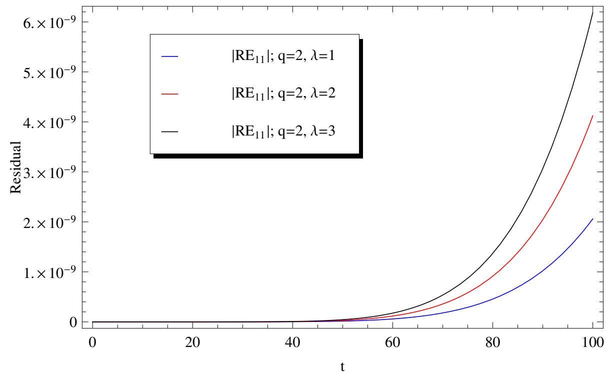

Figure 9. Impact of λ on the absolute remainder error at $q = 2$.

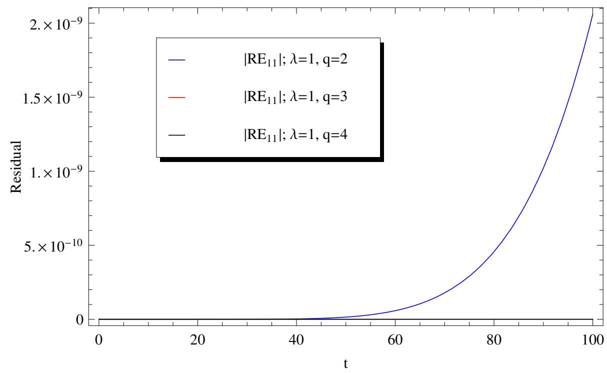

Figure 10. Impact of q on the absolute remainder error at $\lambda = 1$.

The results obtained from these two figures reveal that the approximate solution using only eleven terms of the Adomian's series of exponential orders is highly accurate. Moreover, the absolute residual error $|RE_{11}|$ approaches zero even at higher values of the delay parameter q. This proves the several remarkable advantages of Adomian's method over the existing power series method in the literature [2]. The preceding discussion shows that the ADM can be effectively used to solve similar delay equations. In a future work, some delay equations of higher-orders will be also solved by applying the ADM to prove the effectiveness of this method.

4. Conclusions

In this paper, the Ambartsumian delay equation for the fluctuations of the surface brightness in the Milky way has been analytically solved by using the Adomian decomposition method (ADM). The obtained approximate solutions were of two types. The first was expressed as a power series solution which agreed with a previous solution in the literatures, while the second was expressed in terms of exponential functions. Unfortunately, the solution in the literatures was only effective in sub-domains while our second type solution was valid in the whole domain. Besides, very small absolute residual errors have been achieved using only eleven terms of the Adomian decomposition series. It was also found that the absolute residual errors tend to zero for higher values of the delay parameter q.

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