New Scientific Contribution on the 2-D Subdomain Technique in Polar Coordinates: Taking into Account of Iron Parts

Frédéric Dubas 1,* and Kamel Boughrara 2

1 Département ENERGIE, FEMTO-ST, CNRS, University Bourgogne Franche-Comté, F90000 Belfort, France
2 Laboratoire de Rcherche en Electrotechnique (LRE-ENP), 16200 Algiers, Algeria; kamel.boughrara@g.enp.edu.dz
* Correspondence: frederic.dubas@univ-fcomte.fr; Tel.: +33-3-8457-8203

Received: 22 July 2017; Accepted: 23 October 2017; Published: 25 October 2017

Abstract: This paper presents a new scientific contribution to the two-dimensional red(2-D) subdomain technique in polar coordinates taking into account the finite relative permeability of the ferromagnetic material. The constant relative permeability corresponds to the linear part of the nonlinear \( B(H) \) curve. As in the conventional technique, the separation of variables method and the Fourier series are used for the resolution of magnetostatic Maxwell equations in each region. The general solutions of the magnetic field in subdomains, as well as the boundary conditions (BCs) between regions are different from the conventional method. In the proposed method, the magnetic field solution in each subdomain is a superposition of two magnetic quantities in the two directions (i.e., \( r \)- and \( \Theta \)-axis), and the BCs between two regions are also in both directions. For example, the scientific contribution has been applied to an air- or iron-cored coil supplied by a constant current. The distribution of local quantities (i.e., the magnetic vector potential and flux density) has been validated by a corresponding 2-D finite-element analysis (FEA). The obtained semi-analytical results are in very good agreement with those of the numerical method.

Keywords: air- or iron-cored coil; polar coordinates; Fourier analysis; two-dimensional; subdomain technique

1. Introduction

The full calculation of the magnetic field in electrical engineering applications is the first step for their design and optimization. The methods of magnetic field prediction can be classified into various categories [1]:

- Lehmann’s graphical [2];
- Numerical (i.e., finite-element, finite-difference, boundary-element, etc.) [3–5];
- Equivalent circuit (i.e., electrical, thermal, magnetic, etc.) [6–8];
- Schwarz–Christoffel mapping (i.e., conformal transformation, complex permeance model, etc.) [9];
- Maxwell–Fourier [10–15].

Currently, the works on design are based on (semi-)analytical models (i.e., equivalent circuit, conformal transformation and Maxwell–Fourier methods). This type of model consists of a (non)linear system of \( N \) analytical equations solved analytically or numerically. In comparison with the other methods, under certain geometrical and physical assumptions, these models permit obtaining accurate analytical expressions of the magnetic field and are known as fast for the local/global electromagnetic performances prediction. At present, Maxwell–Fourier methods are one of the most used semi-analytic
approaches with very accurate results (i.e., error less than 5%) on the electromagnetic performances calculation. These models are based on the formal resolution of Maxwell’s equations in Cartesian, cylindrical or spherical coordinates by using the separation of variables method and the Fourier’s series. Taking into account iron parts and/or the effect of local/global saturation is still a scientific challenge in Maxwell–Fourier methods, which is rarely explored in the literature [16–18]. Recently, Dubas et al. (2017) [1] realized an overview of the existing (semi-)analytical models in Maxwell–Fourier methods with the effect of local/global saturation, which can thus be classified as follows:

- Multi-layer models (i.e., Carter’s coefficient [19,20], saturation coefficient [21,22], concept wave impedance [23–26] and convolution theorem [27–30]);
- Eigenvalues model, viz., the method of truncation region eigenfunction expansions (TREE) [31,32];
- Subdomain technique [1,33,34];
- Hybrid models, viz., the analytical solution combined with numerical methods [35,36] or (non)linear magnetic equivalent circuit [37–39].

The consideration of the effect of local/global saturation appears in hybrid models, where the solution is established analytically in concentric regions of very low permeability (e.g., air-gap and magnets), and other methods (e.g., numerical or magnetic equivalent circuit) are sought in regions where the saturation effect cannot be neglected. The other models (i.e., multi-layers models, TREE method and subdomain technique) are more focused on the global saturation. Some details and (dis)advantages of these techniques can be found in [1]. In most semi-analytical models based on the subdomain technique, the iron parts are considered to be infinite permeable due to the variation of material proprieties in the various directions, so that the saturation effect is neglected [16–18]. The first paper introducing the iron parts in the magnetic field calculation by using the subdomain technique is [1], where the authors solve partial differential equations (PDEs) of the magnetic potential vector in Cartesian coordinates in which the subdomains connection is performed directly in both directions (i.e., x- and y-edges). The 2-D magnetostatic model has been applied to an air- or iron-cored coil supplied by a constant current. In [33], the authors propose a 2-D semi-analytical model in spoke-type magnet synchronous machines based on the subdomain technique in polar coordinates with the Taylor polynomial of degree three by focusing on the consideration of iron. The iron magnetic permeability is supposed constant corresponding to the linear zone of the nonlinear \( B(H) \) curve. The subdomains’ connection is carried out in both directions (i.e., \( r \)- and \( \Theta \)-edges). The general solution of the magnetic field is obtained by using the traditional boundary condition (BCs), in addition to new radial BCs (e.g., between the magnets and the rotor teeth, between the teeth and the slots of the stator), which are traduced into a system of linear equations according to Taylor series expansion. In [34], this semi-analytical model has been extended taking into account the initial magnetization curve in each soft-magnetic subdomain by an iterative procedure.

In the literature, to the authors’ knowledge, there exists no exact 2-D subdomain technique in polar coordinates taking into account iron parts with(out) the nonlinear \( B(H) \) curve and not using the Taylor polynomial to satisfy the \( r \)-edges BCs. Thus, in this paper, the research work contributes to the continuous improvement of the 2-D subdomain technique. Moreover, it is an extension of [1] in polar coordinates \((r, \Theta)\). Section 2 presents this new scientific contribution. By applying the principle of superposition on the magnetic quantities in order to respect the BCs on the various edges, the general solution of the magnetic field is decomposed in Fourier’s series into two general solutions in both directions (i.e., \( r \)- and \( \Theta \)-edges). It allows the evaluation of the local distribution of flux densities in the iron parts with a global saturation, does not have numerical convergence problems contrary to others models and would easily introduce the current penetration effect in the conductive materials. The semi-analytical solution is exact as in [1] and does not use the Taylor polynomial to satisfy the \( r \)-edges BCs contrary to [33,34]. For example, it was applied to an air- or iron-cored coil supplied by a constant current. The iron magnetic permeability is constant corresponding to the linear zone of the nonlinear \( B(H) \) curve [1,33]. Nevertheless, as in [29,30,34], the saturation effect could be taken into account by an iterative calculation considering, at each iteration, a constant relative magnetic
permeability according to the nonlinear $B(H)$ curve. However, this is beyond the scope of the paper. In Section 3, in order to confirm the effectiveness of the proposed technique, all semi-analytical results are then compared to those found by 2-D finite-element analysis (FEA) [40]. The comparisons are very satisfying in amplitudes and waveforms.

2. A 2-D Subdomain Technique of the Magnetic Field in Polar Coordinates

2.1. Model Description and Assumptions

Figure 1 represents the physical and geometrical parameters of an air- or iron-cored coil with $N_t$ turns of copper wire supplied by a constant current $I$. The electromagnetic device is surrounded by an infinite box with a null value of magnetic vector potential at it boundaries.

The analytical prediction of the magnetic field based on the 2-D subdomain technique is done by solving magnetostatic Maxwell equations in polar coordinates $(r, \theta)$ with the following assumptions:

- The magnetic vector potential has only one component along the $z$-axis (i.e., $\mathbf{A} = \{0; 0; A_z\}$), and then, the end-effects are not considered;
- All materials are isotropic, and the permeabilities are supposed as constants in both directions (i.e., $r$- and $\theta$-axis);
- All electrical conductivities of materials are supposed as nulls (i.e., the eddy-currents induced in the copper/iron are neglected).

2.2. Problem Discretization in Regions

In Figure 2, we present the studied electromagnetic device, which is divided into seven regions with $\mu = C^{st}$, viz.,

- Region 1 $\{\forall \theta \land r \in [r_1, r_2]\}$, with $\mu_1 = \mu_0$;
• Region 2 \( \{ \forall \Theta \land r \in [r_3, r_4] \} \), with \( \mu_2 = \mu_0 \);
• Region 3 \( \{ \Theta \in [\Theta_1, \Theta_2] \land r \in [r_2, r_3] \} \), with \( \mu_3 = \mu_0 \);
• Region 4 \( \{ \Theta \in [\Theta_5, \Theta_6] \land r \in [r_2, r_3] \} \), with \( \mu_4 = \mu_0 \);
• Region 5 (i.e., the air or iron in the middle of the coil) \( \{ \Theta \in [\Theta_2, \Theta_3] \land r \in [r_2, r_3] \} \), with \( \mu_5 = \mu_0 \) for the air or \( \mu_5 = \mu_{\text{iron}} \) for the iron;
• Region 6 (i.e., the forward conductor) \( \{ \Theta \in [\Theta_2, \Theta_3] \land r \in [r_2, r_3] \} \), with \( \mu_6 = \mu_0 \);
• Region 7 (i.e., the return conductor) \( \{ \Theta \in [\Theta_4, \Theta_5] \land r \in [r_2, r_3] \} \), with \( \mu_7 = \mu_0 \).

Figure 2. Definition of regions in the air-or iron-cored coil.

2.3. Governing PDEs in Polar Coordinates: Laplace’s and Poisson’s Equations

According to Equation (A1) (see Appendix A), the distribution of the magnetic vector potential in polar coordinates \( (r, \Theta) \) is governed by:

\[
\Delta A_{zj} = \frac{\partial^2 A_{zj}}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial A_{zj}}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 A_{zj}}{\partial \Theta^2} = 0 \quad \text{for} \quad j = \{1, \ldots, 5\} \quad \text{(Laplace’s equation),} \tag{1a}
\]

\[
\Delta A_{zk} = \frac{\partial^2 A_{zk}}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial A_{zk}}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 A_{zk}}{\partial \Theta^2} = -\mu_k \cdot J_{zk} \quad \text{for} \quad k = \{6, 7\} \quad \text{(Poisson’s equation),} \tag{1b}
\]

where \( J_{zk} \) is the current density of the coil defined by:

\[
J_{zk} = C_k \cdot \frac{N_l \cdot I}{S_c}, \tag{2}
\]

in which \( S_c \) is the conductor surface and \( C_k \) (with \( C_6 = 1 \) and \( C_7 = -1 \)) is the coefficient that represents the current direction in the conductor.

According to Appendix A, the resolution of Laplace’s and Poisson’s equations by using the separation of variables method and Fourier’s series permit obtaining two potentials in both directions, viz., \( A_{zj}^\Theta \) for the \( \Theta \)-edges (in Equation (A2b)) and \( A_{zk}^r \) for the \( r \)-edges (in Equation (A2c)). The spatial frequency (or periodicity) of \( A_{zj}^\Theta \) and \( A_{zk}^r \) is respectively defined by \( \beta_{zj}^\Theta \) and \( \lambda_{zk}^r \) with \( \beta \) and \( \lambda \) the spatial harmonic orders.
2.4. Definition of Boundary Conditions

In electromagnetics, the general solutions of various regions depend on the BCs at the interface of two surfaces, which are defined by the continuity of the normal flux density \( \mathbf{B}_\perp \) and parallel field intensity \( \mathbf{H}_\parallel \) [1]. On the outer BCs for \((\Theta_1 \land \Theta_6, \forall \rho)\) and \((\forall \Theta, \, r_1 \land r_4)\), \(A_2\) satisfies the Dirichlet BC (see Figure 2), viz., \(A_z = 0\).

Figure 3 represents the respective BCs at the interface between the various regions in both directions (i.e., \(r\) and \(\Theta\)-edges).

2.5. General Solutions of Various Regions

2.5.1. Region 1

The solutions of \(A_{c1}, B_1,\) and \(B_{\Theta 1}\) are determined by Case-Study No. 1 (i.e., \(A_c\) imposed on all edges of a region) in Appendix B. The BCs on the \(r\)-edges of the region (see Figure 3a) are met by posing \(c_h^\Theta = 0\) in Equation (A11). Therefore, \(A_{c1}\) satisfying the BCs of Figure 3a and the solution of Equation (1a) is given by:

\[
A_{c1} = -\sum_{h_1=1}^{\infty} d_{1h_1}^\Theta \cdot \frac{r_2}{\beta_{1h_1}} \cdot \frac{E_f(\beta_{1h_1}, r_1)}{P_f(\beta_{1h_1}, r_2, r_1)} \cdot \sin[\beta_{1h_1} \cdot (\Theta - \Theta_1)],
\]

the components of \(B_1 = \{B_{r1}, B_{\Theta 1}; 0\}\) by:

\[
B_{r1} = -\sum_{h_1=1}^{\infty} d_{1h_1}^\Theta \cdot \frac{r_2}{r} \cdot \frac{E_f(\beta_{1h_1}, r_1)}{P_f(\beta_{1h_1}, r_2, r_1)} \cdot \cos[\beta_{1h_1} \cdot (\Theta - \Theta_1)],
\]

\[
B_{\Theta 1} = \sum_{h_1=1}^{\infty} d_{1h_1}^\Theta \cdot \frac{r_2}{r} \cdot \frac{P_f(\beta_{1h_1}, r_1)}{P_f(\beta_{1h_1}, r_2, r_1)} \cdot \sin[\beta_{1h_1} \cdot (\Theta - \Theta_1)],
\]

where \(E_f(w, x, y)\) and \(P_f(w, x, y)\) are defined in Equation (A9), \(h_1\) the spatial harmonic orders in Region 1, \(d_{1h_1}^\Theta\) the integration constant and \(\beta_{1h_1} = h_1 \cdot \pi / \Theta_6\) with \(\Theta_6 = \Theta_\perp - \Theta_\parallel\).

Using a Fourier series expansion of \(F_1(\Theta)\) (see Figure 3a) over the interval \(\Theta = [\Theta_1, \Theta_6] = [\Theta_1, \Theta_1 + \Theta_6]\), the integration constant \(d_{1h_1}^\Theta\) is determined in Appendix C with:

\[
d_{1h_1}^\Theta = \frac{2}{\tau_{\Theta 1}} \cdot \int_{\Theta_1}^{\Theta_1 + \Theta_6} F_1(\Theta) \cdot \sin[\beta_{1h_1} \cdot (\Theta - \Theta_1)] \cdot d\Theta.
\]

2.5.2. Region 2

The same method as Region 1 is used to define the general solution in Region 2. By posing \(d_{h}^\Theta = 0\) in Equation (A11) (see Appendix B), \(A_{c2}\) satisfying the BCs of Figure 3b and the solution of Equation (1a) is given by:

\[
A_{c2} = \sum_{h_2=1}^{\infty} c_{2h_2}^\Theta \cdot \frac{r_3}{\beta_{2h_2}} \cdot \frac{E_f(\beta_{2h_2}, r_4)}{P_f(\beta_{2h_2}, r_3, r_4)} \cdot \sin[\beta_{2h_2} \cdot (\Theta - \Theta_1)],
\]

the components of \(B_2 = \{B_{r2}, B_{\Theta 2}; 0\}\) by:

\[
B_{r2} = \sum_{h_2=1}^{\infty} c_{2h_2}^\Theta \cdot \frac{r_3}{r} \cdot \frac{E_f(\beta_{2h_2}, r_4)}{P_f(\beta_{2h_2}, r_3, r_4)} \cdot \cos[\beta_{2h_2} \cdot (\Theta - \Theta_1)],
\]

\[
B_{\Theta 2} = \sum_{h_2=1}^{\infty} c_{2h_2}^\Theta \cdot \frac{r_3}{r} \cdot \frac{P_f(\beta_{2h_2}, r_4)}{P_f(\beta_{2h_2}, r_3, r_4)} \cdot \sin[\beta_{2h_2} \cdot (\Theta - \Theta_1)],
\]
where \( h_2 \) is the spatial harmonic orders in Region 2, \( c S_{h_2}^{\Theta} \) the integration constant and \( \beta_2 h_2 = h_2 \cdot \pi / \tau_{\Theta_2} \) with \( \tau_{\Theta_2} = \Theta_6 - \Theta_1 \).

Figure 3. Boundary Conditions (BCs) in both directions (i.e., \( r \)- and \( \Theta \)-edges): (a) Region 1; (b) Region 2; (c) Region 3; (d) Region 4; (e) Region 5; (f) Region 6; and (g) Region 7.
Using a Fourier series expansion of $C_2$ ($\Theta$) (see Figure 3b) over the interval $\Theta = [\Theta_1, \Theta_2] = [\Theta_1, \Theta_1 + \tau_\Theta]$, the integration constant $c_{h_2}^2$ is determined in Appendix C with:

$$c_{h_2}^2 = \frac{2}{\tau_\Theta} \Theta_1 \sum_{\Theta_1} G_2(\Theta) \cdot \sin[\beta 2h_2 \cdot (\Theta - \Theta_1)] \cdot d\Theta. \tag{10}$$

### 2.5.3. Region 3

The solutions of $A_{33}, B_{33}$ and $B_{33}$ are determined by the Case-Study No. 1 (i.e., $A_2$ imposed on all edges of a region) in Appendix B. The BCs on the $\Omega$-edges of the region (see Figure 3c) are met by posing $r_\Omega = 0$ in Equations (A6)–(A8). Therefore, $A_{33}$ satisfying the BCs of Figure 3c and the solution of Equation (1a) is given by:

$$A_{33} = A_{33}^0 + A_{33}^r, \tag{11a}$$

where $A_{33}^0 = \sum_{h_3=1}^{\infty} n h_3 \cdot x_{r_3} \cdot e_{r_3}^2 \cdot \frac{E_{r_3}(\beta h_3, r_3)}{E_{r_3}(\beta h_3, r_3, r_2)} + d_{h_3} \cdot x_{r_3} \cdot E_{r_3}(\beta h_3, r_3, r_2) \cdot \sin[\beta h_3 \cdot (\Theta - \Theta_1)], \tag{11b}$$

and the $r$-component of $B_3$ by:

$$B_{33} = B_{33}^0 + B_{33}^r, \tag{12a}$$

where $B_{33}^0 = \sum_{h_3=1}^{\infty} \beta h_3 \cdot \sum_{h_3=1}^{\infty} n h_3 \cdot x_{r_3} \cdot f_{r_3} \cdot \frac{r_2}{r} \cdot \frac{\sin[\beta 3h_3 \cdot (\Theta - \Theta_1)]}{\sin[\beta h_3 \cdot (\Theta - \Theta_1)]} \cdot \sin[\lambda 3h_3 \cdot \ln(r_2)], \tag{12b}$$

and the $\Theta$-component of $B_3$ by:

$$B_{33} = B_{33}^0 + B_{33}^r, \tag{13a}$$

where $B_{33}^0 = \sum_{h_3=1}^{\infty} \beta h_3 \cdot \sum_{h_3=1}^{\infty} n h_3 \cdot x_{r_3} \cdot f_{r_3} \cdot \frac{r_2}{r} \cdot \frac{\sin[\beta 3h_3 \cdot (\Theta - \Theta_1)]}{\sin[\beta h_3 \cdot (\Theta - \Theta_1)]} \cdot \sin[\lambda 3h_3 \cdot \ln(r_2)], \tag{13b}$$

where $h_3$ and $n_3$ are the spatial harmonic orders in Region 3; $c_{h_3}^\Theta$, $d_{h_3}^\Theta$ and $f_{r_3}^\Theta$ the integration constants; $\beta h_3 = h_3 \cdot n_3 \cdot \pi/\tau_\Theta$ with $\tau_\Theta = \Theta_2 - \Theta_1$; and $\lambda h_3 = h_3 \cdot n_3 \cdot \pi/\tau_3$ with $\tau_3 = \ln(r_2/r_3)$.

Using Fourier series expansion of $A_{13} \mid_{\Theta_1 = \Theta_2}$ and $A_{23} \mid_{\Theta_1 = \Theta_2}$ (see Figure 3c) over the interval $\Theta = [\Theta_1, \Theta_2] = [\Theta_1, \Theta_1 + \tau_\Theta]$, the integration constants $c_{h_3}^\Theta$ and $d_{h_3}^\Theta$ are determined in Appendix C with:

$$c_{h_3}^\Theta = \frac{2}{\tau_\Theta} \Theta_1 \sum_{\Theta_1} \frac{A_{21} \mid_{r_2} \cdot \sin[\beta 2h_3 \cdot (\Theta - \Theta_1)] \cdot d\Theta, \tag{14a}}$$

$$d_{h_3}^\Theta = \frac{2}{\tau_\Theta} \Theta_1 \sum_{\Theta_1} \frac{A_{21} \mid_{r_2} \cdot \sin[\beta 2h_3 \cdot (\Theta - \Theta_1)] \cdot d\Theta, \tag{14b}}$$

With a weighting function $g(r) = r^{-1}$ and using a Fourier series expansion of $A_{36} \mid_{\Theta_1 = \Theta_2}$ (see Figure 3c) over the interval $r = [r_2, r_3]$, the integration constant $f_{r_3}^\Theta$ is determined in Appendix C with:
\[ f^{3}_{n3} = \frac{2}{r_{3}} \int_{r_{2}}^{r_{3}} \frac{1}{r} \cdot \frac{A_{26} \mid \theta = \Theta_{2}}{r_{2}} \sin \left[ \lambda 3_{n3} \cdot \ln \left( \frac{r}{r_{2}} \right) \right] \cdot dr. \] (15)

2.5.4. Region 4

The solution in Region 4 is obtained using the same development as Region 3. By posing \( f''_{n} = 0 \) in Equations (A6)–(A8) (see Appendix B), \( A_{24} \) satisfying the BCs of Figure 3d and the solution of Equation (1a) is given by:

\[ A_{24} = A_{24}^\Theta + A_{24}^\tau, \] (16a)

\[ A_{24}^\Theta = \sum_{h=1}^{\infty} \sum_{r_{2}}^{\infty} E_{f} (\beta_{4h4}, r_{3}, r_{2}) + E_{2} (\beta_{4h4}, r_{2}) \cdot r_{3} \cdot E_{f} (\beta_{4h4}, r_{3}, r_{2}) \cdot \sin [\beta_{4h4} \cdot (\Theta - \Theta_{5})], \] (16b)

\[ A_{24}^\tau = \sum_{n=1}^{\infty} E_{t_{n4}} \cdot r_{2} \cdot \frac{1}{r} \cdot \frac{\lambda_{4n4} \cdot (\Theta_{5} - \Theta)}{\sin (\lambda_{4n4} \cdot \tau_{4})} \cdot \ln \left( \frac{r_{2}}{r_{2}} \right), \] (16c)

the \( r \)-component of \( B_{4} \) by:

\[ B_{4} = B_{4}^\Theta + B_{4}^\tau, \] (17a)

\[ B_{4}^\Theta = \sum_{h=1}^{\infty} \beta_{4h4} \cdot \left[ c_{4h4} \cdot r_{2} \cdot \frac{E_{f} (\beta_{4h4}, r_{3}, r_{2})}{E_{f} (\beta_{4h4}, r_{3}, r_{2})} + d_{4h4} \cdot r_{2} \cdot \frac{E_{2} (\beta_{4h4}, r_{3}, r_{2})}{E_{2} (\beta_{4h4}, r_{3}, r_{2})} \right] \cdot \cos [\beta_{4h4} \cdot (\Theta - \Theta_{5})], \] (17b)

\[ B_{4}^\tau = -\sum_{n=1}^{\infty} \lambda_{4n4} \cdot e_{4n4} \cdot r_{2} \cdot \frac{1}{r} \cdot \frac{\sin (\lambda_{4n4} \cdot (\Theta_{5} - \Theta))}{\sin (\lambda_{4n4} \cdot \tau_{4})} \cdot \ln \left( \frac{r_{2}}{r_{2}} \right), \] (17c)

the \( \Theta \)-component of \( B_{4} \) by:

\[ B_{4} = B_{4}^\Theta + B_{4}^\tau, \] (18a)

\[ B_{4}^\Theta = \sum_{h=1}^{\infty} \beta_{4h4} \cdot \left[ c_{4h4} \cdot r_{2} \cdot \frac{P_{f} (\beta_{4h4}, r_{3}, r_{2})}{E_{f} (\beta_{4h4}, r_{3}, r_{2})} - d_{4h4} \cdot r_{2} \cdot \frac{P_{2} (\beta_{4h4}, r_{3}, r_{2})}{E_{2} (\beta_{4h4}, r_{3}, r_{2})} \right] \cdot \sin [\beta_{4h4} \cdot (\Theta - \Theta_{5})], \] (18b)

\[ B_{4}^\tau = -\sum_{n=1}^{\infty} \lambda_{4n4} \cdot e_{4n4} \cdot r_{2} \cdot \frac{1}{r} \cdot \frac{\sin (\lambda_{4n4} \cdot (\Theta_{5} - \Theta))}{\sin (\lambda_{4n4} \cdot \tau_{4})} \cdot \cos \left( \frac{\lambda_{4n4} \cdot \ln \left( \frac{r_{2}}{r_{2}} \right)}{\tau_{4}} \right), \] (18c)

where \( h_{4} \) and \( n_{4} \) are the spatial harmonic orders in Region 4; \( c_{4h4}^\Theta, d_{4h4}^\Theta \) and \( e_{4n4}^\Theta \) the integration constants; \( \beta_{4h4} = h \cdot \pi / \tau_{4} \) with \( \tau_{4} = \Theta_{5} - \Theta_{3} \) and \( \lambda_{4n4} = n_{4} \cdot \pi / \tau_{4} \) with \( \tau_{4} = \ln (r_{3} / r_{2}) \).

Using Fourier series expansion of \( A_{21}^{\Theta} \mid \theta = \Theta_{5} \) and \( A_{22}^{\Theta} \mid \theta = \Theta_{5} \) (see Figure 3d) over the interval \( \Theta = [\Theta_{5}, \Theta_{6}] = [\Theta_{5}, \Theta_{5} + \tau_{4}] \), the integration constants \( c_{4h4}^\Theta \) and \( d_{4h4}^\Theta \) are determined in Appendix B with:

\[ c_{4h4}^\Theta = \frac{2}{\tau_{4}} \cdot \int_{\Theta_{5}}^{\Theta_{6}} A_{21} \mid \theta = \Theta_{2} \cdot \sin [\beta_{4h4} \cdot (\Theta - \Theta_{5})] \cdot d\Theta, \] (19a)

\[ d_{4h4}^\Theta = \frac{2}{\tau_{4}} \cdot \int_{\Theta_{5}}^{\Theta_{6}} A_{22} \mid \theta = \Theta_{2} \cdot \sin [\beta_{4h4} \cdot (\Theta - \Theta_{5})] \cdot d\Theta. \] (19b)

With a weighting function \( g(r) = r^{-1} \) and using a Fourier series expansion of \( A_{27}^{\Theta} \mid \theta = \Theta_{5} \) (see Figure 3d) over the interval \( r = [r_{2}, r_{3}] \), the integration constant \( e_{4n4}^\Theta \) is determined in Appendix C with:

\[ e_{4n4}^\Theta = \frac{2}{\tau_{4}} \cdot \int_{r_{2}}^{r_{3}} \frac{1}{r} \cdot A_{27} \mid \theta = \Theta_{5} \cdot \sin \left[ \lambda_{4n4} \cdot \ln \left( \frac{r}{r_{2}} \right) \right] \cdot dr. \] (20)
2.5.5. Region 5

For Region 5, the general solution is given according to the BCs of Case-Study No. 1 (i.e., $A_z$ imposed on all edges of a region) in Appendix B. Therefore, $A_{z5}$ satisfying the BCs of Figure 3e and the solution of Equation (1a) is given by:

$$A_{z5} = A_{z5}^\Theta + A_{z5}^\tau,$$

where

$$A_{z5}^\Theta = \sum_{h5=1}^{\infty} \left[ c_{50}^{\Theta} \cdot r^2 \cdot \frac{E_f(\beta_{5h5}, r, 3)}{E_f(\beta_{5h5}, r, r_2)} + d_{50}^{\Theta} \cdot r^2 \cdot \frac{E_f(\beta_{5h5}, r, r_2)}{E_f(\beta_{5h5}, r, r_2)} \right] \cdot \sin [\beta_{5h5} \cdot (\Theta - \Theta_3)],$$

$$A_{z5}^\tau = \sum_{n5=1}^{\infty} \left\{ e_{5n}^\tau \cdot \frac{\cdots}{\sin(\lambda_{5n} \cdot \tau_{5})} + f_{5n}^\tau \cdot \frac{\cdots}{\sin(\lambda_{5n} \cdot \tau_{5})} \right\} \cdot r^2 \cdot \sin [\lambda_{5n} \cdot \ln \left( \frac{r}{r_2} \right)],$$

the $r$-component of $B_5$ by:

$$B_{r5} = B_{r5}^\Theta + B_{r5}^\tau,$$

where

$$B_{r5}^\Theta = \sum_{h5=1}^{\infty} \beta_{5h5} \cdot \left[ c_{50}^{\Theta} \cdot \frac{r^2}{2} \cdot \frac{E_f(\beta_{5h5}, r, r_2)}{E_f(\beta_{5h5}, r, r_2)} - d_{50}^{\Theta} \cdot \frac{r^2}{2} \cdot \frac{E_f(\beta_{5h5}, r, r_2)}{E_f(\beta_{5h5}, r, r_2)} \right] \cdot \cos [\beta_{5h5} \cdot (\Theta - \Theta_3)],$$

$$B_{r5}^\tau = \sum_{n5=1}^{\infty} \lambda_{5n} \cdot \left\{ -e_{5n}^\tau \cdot \frac{\cdots}{\sin(\lambda_{5n} \cdot \tau_{5})} + f_{5n}^\tau \cdot \frac{\cdots}{\sin(\lambda_{5n} \cdot \tau_{5})} \right\} \cdot \frac{r^2}{2} \cdot \cos [\lambda_{5n} \cdot \ln \left( \frac{r}{r_2} \right)],$$

and

$$B_{\Theta5} = B_{\Theta5}^\Theta + B_{\Theta5}^\tau,$$

where $h5$ and $n5$ are the spatial harmonic orders in Region 5; $c_{50}^{\Theta}$, $d_{50}^{\Theta}$, $e_{5n}^\tau$ and $f_{5n}^\tau$ are the integration constants; $\beta_{5h5} = h5 \cdot \pi / \tau_{5}$ with $\tau_{5} = \Theta_4 - \Theta_3$; and $\lambda_{5n} = n5 \cdot \pi / \tau_{5}$ with $\tau_{5} = \ln (r_3 / r_2)$.

Using Fourier series expansion of $A_{z1}|_{\Theta=\tau=r_2}$ and $A_{z2}|_{\Theta=\tau=r_3}$ (see Figure 3e) over the interval $\Theta = [\Theta_3, \Theta_4] = [\Theta_3, \Theta_3 + \tau_{5}]$, the integration constants $c_{50}^{\Theta}$ and $d_{50}^{\Theta}$ are determined in Appendix C with:

$$c_{50}^{\Theta} = \frac{2}{\tau_{5}} \cdot \Theta_{3} + \Theta_{5} \cdot \int_{\Theta_3}^{\Theta_5} A_{z1}|_{\tau=r_2} \cdot \sin [\beta_{5h5} \cdot (\Theta - \Theta_3)] \cdot d\Theta,$$

$$d_{50}^{\Theta} = \frac{2}{\tau_{5}} \cdot \Theta_{3} + \Theta_{5} \cdot \int_{\Theta_3}^{\Theta_5} A_{z2}|_{\tau=r_3} \cdot \sin [\beta_{5h5} \cdot (\Theta - \Theta_3)] \cdot d\Theta.$$

With a weighting function $g(r) = r^{-1}$ and using a Fourier series expansion of $A_{z6}|_{\Theta=\Theta_{5} \& \tau=r}$ and $A_{z7}|_{\Theta=\Theta_{5} \& \tau=r}$ (see Figure 3e) over the interval $r = [r_2, r_3]$, the integration constants $e_{5n}^\tau$ and $f_{5n}^\tau$ are determined in Appendix C with:

$$e_{5n}^\tau = \frac{2}{\tau_{5}} \cdot \int_{r_2}^{r_3} \frac{1}{r} \cdot A_{z6}|_{\Theta=\Theta_{3}} \cdot \sin [\lambda_{5n} \cdot \ln \left( \frac{r}{r_2} \right)] \cdot dr,$$

$$f_{5n}^\tau = \frac{2}{\tau_{5}} \cdot \int_{r_2}^{r_3} \frac{1}{r} \cdot A_{z7}|_{\Theta=\Theta_{3}} \cdot \sin [\lambda_{5n} \cdot \ln \left( \frac{r}{r_2} \right)] \cdot dr.$$
2.5.6. Region 6

For Region 6, the general solution is given according to the BCs of Case-Study No. 2 (i.e., $B_r$ and $A_z$ are respectively imposed on $r$- and $\Theta$-edges of a region) in Appendix B. Therefore, $A_{z6}$ satisfying the BCs of Figure 3f and the solution of Equation (1b) is given by:

$$A_{z6} = A_{z6}^\Theta + A_{z6}^\rho + A_{z6}^\rho,$$  \hspace{1cm} (26a)

$$A_{z6}^\Theta = \left[ -\frac{1}{2} \cdot r^2 \cdot \mu_6 \cdot J_{z6} \right].$$  \hspace{1cm} \text{(26d)}

The $r$-component of $B_6$ is defined by:

$$B_{r6} = B_{r6}^\Theta + B_{r6}^\rho + B_{r6}^\rho,$$  \hspace{1cm} (27a)

$$B_{r6}^\Theta = \left[ -\frac{1}{2} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \right] \cos \left[ \beta \theta_{66} \cdot (\Theta - \Theta_2) \right],$$  \hspace{1cm} (27b)

$$B_{r6}^\rho = \left( c_{06}^\rho \cdot \beta \theta_{66} \cdot \Theta \cdot \Theta_2 \right) + d_{66}^\rho \cdot \frac{\beta \theta_{66} \cdot \Theta \cdot \Theta_2}{r^2},$$  \hspace{1cm} (27c)

and the $\Theta$-component of $B_6$ by:

$$B_{\Theta6} = B_{\Theta6}^\Theta + B_{\Theta6}^\rho + B_{\Theta6}^\rho,$$  \hspace{1cm} (28a)

$$B_{\Theta6}^\Theta = \left[ -\frac{1}{2} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \right] \cos \left[ \beta \theta_{66} \cdot (\Theta - \Theta_2) \right],$$  \hspace{1cm} (28b)

$$B_{\Theta6}^\rho = \left( c_{06}^\Theta \cdot \beta \theta_{66} \cdot (\Theta \cdot \Theta_2) \right) - f_{66}^\rho \cdot \frac{\beta \theta_{66} \cdot (\Theta \cdot \Theta_2)}{r^2},$$  \hspace{1cm} (28c)

$$B_{\Theta6}^\rho = \left[ -\frac{1}{2} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \cdot \frac{1}{r^2} \cdot \frac{1}{r} \right] \cos \left[ \beta \theta_{66} \cdot (\Theta - \Theta_2) \right],$$  \hspace{1cm} (28d)

where $h_6$ and $n_6$ are the spatial harmonic orders in Region 6; $c_{06}^\Theta, d_{06}^\Theta, c_{06}^\rho, d_{06}^\rho, c_{n_6}^\Theta \Theta_2$ and $f_{n_6}^\rho$ are the integration constants; $\beta \theta_{66} = h_6 \cdot \pi / \tau_{66}$ with $\tau_{66} = \Theta_3 - \Theta_2$; and $\lambda \theta_{n_6} = n_6 \cdot \pi / \tau_{66}$ with $\tau_{66} = \ln (r_3 / r_2)$.

Using Fourier series expansion of $A_{z11} |_{\Theta = \Theta_2}$ and $A_{z12} |_{\Theta = \Theta_2}$ (see Figure 3f) over the interval $\Theta = [\Theta_2, \Theta_3] = [\Theta_2, \Theta_2 + \tau_{66}]$, the integration constants $c_{06}^\Theta$ and $c_{06}^\rho$ and $d_{06}^\Theta$ and $d_{06}^\rho$ are determined in Appendix C with:

$$c_{06}^\Theta = \frac{1}{\Theta_{66}} \cdot \left[ A_{z11} |_{r = r_2} - A_{z12} |_{r = r_2} \right] \cdot d\Theta,$$  \hspace{1cm} (29a)
Using a Fourier series expansion of $\mu_6/\mu_5 \cdot B_{r5}|_{\Theta=\Theta_2}$ and $\mu_6/\mu_3 \cdot B_{r3}|_{\Theta=\Theta_2}$ (see Figure 3f) over the interval $r = [r_2, r_3]$, the integration constants $e6'_{n6}$ and $f6'_{n6}$ are determined in Appendix C with:

$$e6'_{n6} = \frac{2}{r^{n6}_6} \cdot \int_{r_2}^{r_3} \frac{1}{r^{n6}_6} \cdot \left[ A_{z1}|_{r=r_2} - A_{z1}|_{r=r_3} \right] \cdot \cos \left[ \beta6_{n6} \cdot (\Theta - \Theta_2) \right] \cdot d\Theta,$$

$$f6'_{n6} = \frac{2}{r^{n6}_6} \cdot \int_{r_2}^{r_3} \frac{1}{r^{n6}_6} \cdot \left[ A_{z2}|_{r=r_2} - A_{z2}|_{r=r_3} \right] \cdot \cos \left[ \beta6_{n6} \cdot (\Theta - \Theta_2) \right] \cdot d\Theta.$$

2.5.7. Region 7

The solution in Region 7 is using the same development as Region 6. Thus, $A_{z7}$ satisfying the BCs of Figure 3g and the solution of Equation (2) is defined by:

$$A_{z7} = A^Q_{z7} + A^T_{z7} + A_{z7r7}$$

$$A^Q_{z7} = \begin{bmatrix} c7^Q_0 \cdot r_2 \cdot \frac{\ln(r_2/r)}{\ln(r_2/r_7)} + d7^Q_0 \cdot r_3 \cdot \frac{\ln(r_3/r)}{\ln(r_3/r_7)} \\ \vdots + \sum_{h=1}^{\infty} c7^Q_h \cdot r_2 \cdot \frac{E_2(\beta7^Q_{h7},r_2)}{E_2(\beta7^Q_{h7},r_7)} + d7^Q_h \cdot r_3 \cdot \frac{E_2(\beta7^Q_{h7},r_3)}{E_2(\beta7^Q_{h7},r_7)} \end{bmatrix} \cdot \cos \left[ \beta7_{h7} \cdot (\Theta - \Theta_4) \right],$$

$$A^T_{z7} = \sum_{n7=1}^{\infty} \left\{ c7^T_{n7} \cdot \frac{\sinh(\lambda7^T_{n7}r_7)}{\sinh(\lambda7^T_{n7}r_7)} + f7^T_{n7} \cdot \frac{\sinh(\lambda7^T_{n7}r_7)}{\sinh(\lambda7^T_{n7}r_7)} \right\} \cdot \frac{\lambda7^T_{n7}}{\lambda7^T_{n7}} \cdot \sin \left[ \lambda7_{n7} \cdot \ln \left( \frac{r_2}{r_7} \right) \right],$$

$$A_{z7r7} = -\frac{1}{4} \cdot r^2 \cdot \mu_7 \cdot J_{z7}.$$

The $r$-component of $B_7$ is defined by:

$$B_{r7} = B^Q_{r7} + B^T_{r7} + B_{r7r7},$$

$$B^Q_{r7} = -\sum_{h=1}^{\infty} \beta7^Q_{h7} \cdot \left[ c7^Q_{h7} \cdot \frac{\zeta7_{h7}}{\eta7_{h7}} + d7^Q_{h7} \cdot \frac{\zeta7_{h7}}{\eta7_{h7}} \right] \cdot \sin \left[ \beta7_{h7} \cdot (\Theta - \Theta_4) \right],$$

$$B^T_{r7} = \sum_{n7=1}^{\infty} \left\{ c7^T_{n7} \cdot \frac{\sinh(\lambda7^T_{n7}r_7)}{\sinh(\lambda7^T_{n7}r_7)} + f7^T_{n7} \cdot \frac{\sinh(\lambda7^T_{n7}r_7)}{\sinh(\lambda7^T_{n7}r_7)} \right\} \cdot \frac{\zeta7_{n7}}{\lambda7_{n7}} \cdot \sin \left[ \lambda7_{n7} \cdot \ln \left( \frac{r_2}{r_7} \right) \right],$$

$$B_{r7r7} = \frac{1}{r} \cdot \frac{\partial A_{z7r7}}{\partial \Theta} = 0.$$
and the Θ-component of $B_7$ by:

$$B_{0\Theta} = B_{0\Theta}^0 + B_{0\Theta}^r + B_{0\Theta}^\tau,$$

(33a)

$$B_{0\Theta}^0 = \left[ c_{r0}^r \cdot \frac{1}{\ln(r/r_l)} - d_{r0}^r \cdot \frac{1}{\ln(r/r_l)} \right] \cdot \frac{1}{r} \cdot \lambda_{r0}^r \cdot \frac{P_l(\lambda_{r0}^r r)}{P_l(\lambda_{r0}^r r)} \cdot \cos \left[ \beta_{r0}^r \cdot (\Theta - \Theta_4) \right],$$

(33b)

$$B_{0\Theta}^r = - \sum_{n7=1}^2 \left\{ c_{n7}^r \cdot \left( \frac{\lambda_{n7}^r (\Theta - \Theta_4)}{\sin(\lambda_{n7}^r \cdot \tau_{r7})} \right) - f_{n7}^r \cdot \left( \frac{\lambda_{n7}^r \cdot (\Theta_3 - \Theta_5)}{\sin(\lambda_{n7}^r \cdot \tau_{r7})} \right) \right\} \cdot \frac{\tau_{r7}}{\tau_{r7}} \cdot \cos \left[ \lambda_{n7}^r \cdot \ln \left( \frac{r}{r_2} \right) \right],$$

(33c)

$$B_{0\Theta}^\tau = - \frac{dA_{0\Theta}}{dr} = \frac{1}{2} \cdot r \cdot \mu_7 \cdot J_{27},$$

(33d)

where $h_7$ and $n_7$ are the spatial harmonic orders in Region 7; $c_{r0}^r$, $d_{r0}^r$, $c_{n7}^r$, $d_{n7}^r$, $f_{n7}^r$ and $f_{n7}^r$ the integration constants; $\beta_{r0}^r = h_7 \cdot \pi / \tau_0 \Theta_7$ for $\Theta_{07} = \Theta_1 - \Theta_4$ and $\lambda_{n7}^r = n_7 \cdot \pi / \tau_7$ with $\tau_7 = \ln \left( r_3 / r_2 \right)$.

Using Fourier series expansion of $A_{21}|_{\Theta_4 \wedge r=r_2}$ and $A_{21}|_{\Theta_1 \wedge r=r_2}$ (see Figure 3g) over the interval $\Theta = [\Theta_4, \Theta_5] = [\Theta_4, \Theta_3 + \tau_0 \Theta]$, the integration constants $c_{r0}^r$ and $d_{r0}^r$ and $d_{n7}^r$ and $d_{n7}^r$ are determined in Appendix C with:

$$c_{r0}^r = \frac{1}{\Theta_7} \cdot \int_{\Theta_4}^{\Theta_5} \frac{1}{r_2} \cdot \left[ A_{21}|_{r=r_2} - A_{21}|_{r=r_3} \right] \cdot d\Theta,$$

(34a)

$$c_{n7}^r = \frac{2}{\Theta_7} \cdot \int_{\Theta_4}^{\Theta_5} \frac{1}{r_2} \cdot \left[ A_{21}|_{r=r_2} - A_{21}|_{r=r_3} \right] \cdot \cos \left[ \beta_{n7}^r \cdot (\Theta - \Theta_4) \right] \cdot d\Theta,$$

(34b)

$$d_{r0}^r = \frac{1}{\Theta_7} \cdot \int_{\Theta_4}^{\Theta_5} \frac{1}{r_3} \cdot \left[ A_{21}|_{r=r_3} - A_{21}|_{r=r_2} \right] \cdot d\Theta,$$

(34c)

$$d_{n7}^r = \frac{2}{\Theta_7} \cdot \int_{\Theta_4}^{\Theta_5} \frac{1}{r_3} \cdot \left[ A_{21}|_{r=r_3} - A_{21}|_{r=r_4} \right] \cdot \cos \left[ \beta_{n7}^r \cdot (\Theta - \Theta_4) \right] \cdot d\Theta.$$

(34d)

Using a Fourier series expansion of $\mu_7 / \mu_4 \cdot B_{14}|_{\Theta=\Theta_0 \wedge r}$ and $\mu_7 / \mu_5 \cdot B_{15}|_{\Theta=\Theta_1 \wedge r}$ (see Figure 3g) over the interval $r = [r_2, r_3]$, the integration constants $c_{n7}^r$ and $f_{n7}^r$ are determined in Appendix C with:

$$c_{n7}^r = \frac{2}{r_3} \cdot \int_{r_2}^{r_3} \frac{1}{r_2} \cdot \left[ \frac{\lambda_{n7}^r}{\mu_4} \cdot B_{14}|_{\Theta=\Theta_5} - B_{14}|_{\Theta=\Theta_5} \right] \cdot \sin \left[ \lambda_{n7}^r \cdot \ln \left( \frac{r}{r_2} \right) \right] \cdot d\tau,$$

(35a)

$$f_{n7}^r = \frac{2}{r_3} \cdot \int_{r_2}^{r_3} \frac{1}{r_2} \cdot \left[ \frac{\lambda_{n7}^r}{\mu_5} \cdot B_{15}|_{\Theta=\Theta_4} - B_{15}|_{\Theta=\Theta_4} \right] \cdot \sin \left[ \lambda_{n7}^r \cdot \ln \left( \frac{r}{r_2} \right) \right] \cdot d\tau.$$  

(35b)
3. Validation of the Semi-Analytic Method with Finite-Element Analysis

3.1. Introduction

The objective of this section is to validate the 2-D subdomain technique in polar coordinates \((r, \Theta)\) on the magnetic field distribution in relation to the numerical method. The physical and geometrical parameters of studied electromagnetic device are given in Table 1.

<table>
<thead>
<tr>
<th>Parameters, Symbols (Units)</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of turns of the coil, (N_t) (–)</td>
<td>60</td>
</tr>
<tr>
<td>Supply current, (I) (A)</td>
<td>20</td>
</tr>
<tr>
<td>Conductor surface, (S_c) (mm(^2))</td>
<td>120</td>
</tr>
<tr>
<td>Current density of the coil, (J_{zk}) (A/mm(^2))</td>
<td>±10</td>
</tr>
<tr>
<td>Effective axial length, (L_z) (mm)</td>
<td>60</td>
</tr>
<tr>
<td>Geometrical parameters in the (\Theta)-axis, ({\Theta_1; \Theta_2; \Theta_3; \Theta_4; \Theta_5; \Theta_6}) (deg.)</td>
<td>{0; 17; 21; 29; 33; 50}</td>
</tr>
<tr>
<td>Geometrical parameters in the (r)-axis, ({r_1; r_2; r_3; r_4}) (mm)</td>
<td>{21; 81; 100; 160}</td>
</tr>
<tr>
<td>Relative magnetic permeability of the iron, (\mu_{iron}) (–)</td>
<td>1,500</td>
</tr>
<tr>
<td>Number of harmonics for Region 1, (H_{1_{max}}) (–)</td>
<td>260</td>
</tr>
<tr>
<td>Number of harmonics for Region 2, (H_{2_{max}}) (–)</td>
<td>260</td>
</tr>
<tr>
<td>Number of harmonics for Region 3, ({H_{3_{max}}; N_{3_{max}}}) (–)</td>
<td>{88; 124}</td>
</tr>
<tr>
<td>Number of harmonics for Region 4, ({H_{4_{max}}; N_{4_{max}}}) (–)</td>
<td>{88; 124}</td>
</tr>
<tr>
<td>Number of harmonics for Region 5, ({H_{5_{max}}; N_{5_{max}}}) (–)</td>
<td>{42; 124}</td>
</tr>
<tr>
<td>Number of harmonics for Region 6, ({H_{6_{max}}; N_{6_{max}}}) (–)</td>
<td>{21; 124}</td>
</tr>
<tr>
<td>Number of harmonics for Region 7, ({H_{7_{max}}; N_{7_{max}}}) (–)</td>
<td>{21; 124}</td>
</tr>
</tbody>
</table>

For this validation, the air- or iron-cored coil has been modeled using Cedrat’s Flux2D (Version 10.2.1, Altair Engineering, Meylan Cedex, France) software package (i.e., an advanced finite-element method-based numeric field analysis program) [40]. FEA is done with the same assumptions as the semi-analytical model (see Section 2.1). The linear system is given in Appendix C and has been implemented in MATLAB\textsuperscript{®} (R2015a, MathWorks, Natick, MA, USA) by using the sparse matrix/vectors. A discussion of the numerical problems (viz., harmonics and ill-conditioned systems) of such semi-analytical models has been clarified in [1]. The Maxwell–Fourier methods exhibit a similar problem to the numerical methods due to the periodicity of Fourier series and, consequently, to the finite number of harmonics. Hence, \(A_z\) and \(B = \{B_r; B_\Theta; 0\}\) in the various regions (see Section 2.5) have been computed with a finite number of spatial harmonics terms \(H_{1_{max}}\)–\(H_{7_{max}}\) (for the \(\Theta\)-edges) and \(N_{3_{max}}\)–\(N_{7_{max}}\) (for the \(r\)-edges). As indicated in [41,42], these spatial harmonics terms, given in Table 1, have been imposed according to an optimal ratio, i.e., for \(H_{1_{max}}\) given,

\[
H_{\bullet_{max}} = H_{1_{max}} \cdot \frac{\Theta_{\bullet}}{\Theta_{1}} \quad \text{and} \quad N_{\bullet_{max}} = H_{\bullet_{max}} \cdot \frac{\Theta_{\bullet}}{\Theta_{\bullet}}. \tag{36}
\]

The linear system size depends on the number of: (i) regions; (ii) BCs; and (iii) harmonics of each subdomain. In our study, the linear system named Equation (A17) consists of 2036 elements, which is much smaller than the 2-D FEA mesh having 3,081 surfaces elements of second order (viz., the triangles number of system) with the number of excellent quality elements equal to 100%. For information, the 2-D FEA mesh for an air- or iron-cored coil is illustrated in Figure 4. The personal computer used for this comparison has the following characteristics: HP Z800 Intel(R) Xeon(R) CPU @ 2.4 GHz (with two processors) RAM 16 Go 64 bits. The computation time of 2-D subdomain model is divided by two (viz., 0.5 s for 2-D subdomain model and 1 s for the 2-D FEA).
Figure 4. 2-D Finite-Element Analysis (FEA) mesh for the air- or iron-cored coil.

3.2. Results Discussion

The validation paths of $A_z$ and $B = \{B_r, B_\Theta; 0\}$ for the semi-analytic and numeric comparison are given in Figure 5.

Figure 5. Validation paths for the semi-analytic and numeric comparison.

The waveforms of global quantities are shown on different paths in Figure 6 for $A_z$ and in Figures 7–11 for the components of $B$. The solid lines represent the global quantities computed by the 2-D FEA, and the circles correspond to the 2-D subdomain model. Comparing those results with 2-D FEA, it can be shown that a very good evaluation is obtained for $A_z$ and for the components of $B$, whatever the paths, for both the air- and iron-core. This confirms that the effect of global saturation can be taken into account accurately. According to the concept of symmetry, it can be seen that in polar coordinates, there is only one symmetry of $A_z$ on Path 5 (viz., $B_r \neq 0$ and $B_\Theta = 0$ in Figure 11) unlike the same electromagnetic device in Cartesian coordinates. Indeed, in Cartesian coordinates, there exist two symmetry axes of $A_z$ on Path 2 and Path 5 [1]. In polar coordinates, Path 2 does not correspond to a symmetry axis of $A_z$; consequently, $B_r \neq 0$ and $B_\Theta \neq 0$ in Figure 8. It will be noted that the $r$-component of $B$ in Region 5 is more intense with the magnetic core (see Figure 8a) and that the magnetic leakages in the middle of the device in the $\Theta$-axis are equivalent for an air- and iron-cored coil (see Figure 8b). It is interesting to note that numerical peaks appear in the FEA results.
(see Figures 6e, 7, 8b and 11b), which are mainly due to the mesh. The relative error is less than 1.5% for the various global quantities (see Figure 6a,c for the maximum error).

Figure 6. Waveforms of $A_z$ for: (a) Path 1; (b) Path 2; (c) Path 3; (d) Path 4; and (e) Path 5.
Figure 7. Waveforms of $B$ for Path 1: (a) $r$- and (b) $\Theta$-component.

Figure 8. Waveforms of $B$ for Path 2: (a) $r$- and (b) $\Theta$-component.

Figure 9. Waveforms of $B$ for Path 3: (a) $r$- and (b) $\Theta$-component.
4. Conclusions

It has been demonstrated that there exists no exact semi-analytical model based on the 2-D subdomain technique in polar coordinates taking into account iron parts with(out) the nonlinear $B(H)$ curve. An improved 2-D subdomain method in polar coordinates $(r, \Theta)$ to study the magnetic field distribution in the iron parts with a finite relative permeability has been presented in this paper. Nevertheless, the research work is an extension of [1] in polar coordinates $(r, \Theta)$.

The proposed new subdomain model is applied to an air- or iron-cored coil supplied by a constant current. The magnetic field solutions in the subdomains and the BCs between regions are carried out in the two directions (i.e., $r$- and $\Theta$-axis). The iron relative permeability used in this model is constant and corresponds to the linear part of the nonlinear $B(H)$ curve. However, the whole $B(H)$ curve of the magnetic material can be applied with an iterative algorithm as in [29,30,34]. The proposed subdomain method in polar coordinates $(r, \Theta)$ takes less computing time than the FEA (approximately two-fold versus to FEA). It is very suitable for the design and optimization of the electromechanical systems in general and electrical machines in particular. The semi-analytical results have been validated with FEA, and good agreement has been obtained in both amplitudes and waveforms.

The major scientific contribution could be applied to rotating electrical machines (e.g., radial-flux machines) in polar coordinates with(out) magnets supplied by a direct or alternate current (with any waveforms). Moreover, one advantage of this technique would be their exploitation in...
which leads to:

Author Contributions: The work presented here was carried out in cooperation among all authors, which have written the paper and have gave advice for the manuscripts.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. The 2-D General Solution of PDEs (i.e., Laplace’s and Poisson’s Equations) in Polar Coordinates

Using the magnetostatic Maxwell’s equations (viz., the Maxwell–Ampere law, the Maxwell–Thomson law and the magnetic material equation) [1], the general PDEs in terms of magnetic vector potential \( A = \{0;0;A_z\} \) with \( \mu = C \) can be expressed in polar coordinates \((r,\Theta)\) by:

\[
\Delta A_z = \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial A_z}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 A_z}{\partial \Theta^2} = ES, \quad (A1a)
\]

\[
ES = - \left[ \mu \cdot J_z + \mu_0 \cdot \left( M_\Theta + r \cdot \frac{\partial M_\Theta}{\partial r} - \frac{\partial M_r}{\partial \Theta} \right) \right]. \quad (A1b)
\]

where \( J = \{0;0;J_z\} \) is the current density (due to supply currents) vector, \( M = \{M_r;M_\Theta;0\} \) is the magnetization vector (with \( M = 0 \) for the vacuum/iron or \( M \neq 0 \) for the magnets according to the magnetization direction [43]) and \( \mu = \mu_0 \cdot \mu_r \) is the absolute magnetic permeability of the magnetic material in which \( \mu_0 \) and \( \mu_r \) are respectively the vacuum permeability and the relative permeability of the magnetic material (with \( \mu_r = 1 \) for the vacuum or \( \mu_r = 0 \) for the magnets/iron).

The magnetic vector potential \( A_z \) is governed by Poisson’s equation (i.e., \( ES \neq 0 \)) or Laplace’s equation (i.e., \( ES = 0 \)). Using the separation of variables method, the 2-D general solution of \( A_z \) in both directions (i.e., \( r \)- and \( \Theta \)-edges) can be written as Fourier’s series:

\[
A_z = A^0_z + A^r_z + A^{zp}_z, \quad (A2a)
\]

\[
A^0_z = \left[ C^0 + D^0 \cdot \ln (r) \right] \cdot \left( E^0 \cdot F^0 \cdot \Theta \right) + \sum_{h=1}^{\infty} \left( C^h \cdot r^\beta_h \right) \cdot \left( E^h \cdot \cos (\beta_h \cdot \Theta) \right), \quad (A2b)
\]

\[
A^r_z = \left[ C^0 + D^0 \cdot \ln (r) \right] \cdot \left( E^0 \cdot F^0 \cdot \Theta \right) + \sum_{h=1}^{\infty} \left( C^h \cdot \cos [\lambda_n \cdot \ln (r)] \right) \cdot \left( E^h \cdot \cos (\lambda_n \cdot \Theta) \right), \quad (A2c)
\]

where \( A^{zp}_z \) is the particular solution of \( A_z \) respecting the second member \( ES \) in Equation \((A1)\), \( C^0 - D^0 \) and \( C^h - F^h \) the integration constants, \( \beta_h \) and \( \lambda_n \) the spatial frequency (or periodicity) of \( A^0_z \) and \( A^r_z \) and \( h \) and \( n \) the spatial harmonic orders.

Using \( B = \nabla \times A \), the components of magnetic flux density \( B = \{B_r;B_\Theta;0\} \) can be deduced by:

\[
B_r = \frac{1}{r} \cdot \frac{\partial A_z}{\partial \Theta} \quad \text{and} \quad B_\Theta = -\frac{\partial A_z}{\partial r}, \quad (A3)
\]

which leads to:

\[
B_r = B^0_r + B^r_r + \frac{1}{r} \cdot \frac{\partial A^{zp}_z}{\partial \Theta}, \quad (A4a)
\]

\[
B^0_r = \left[ \frac{r^0}{r^2} \cdot \left( C^0 + D^0 \cdot \ln (r) \right) \right] \cdot \left[ -E^0 + F^0 \cdot \Theta \right] + \sum_{h=1}^{\infty} \left[ \frac{h_0}{r^2} \cdot \left( C^h \cdot r^\beta_h \right) \right] \cdot \left[ -E^h + F^h \cdot \Theta \right], \quad (A4b)
\]
$$B'_r = \left[ \frac{F'_r}{r} \right] \left[ C'_n + D'_n \cdot \ln(r) \right] + \sum_{n=1}^{\infty} \frac{\lambda_n}{r} \left\{ \begin{array}{l} C'_n \cdot \cos [\lambda_n \cdot \ln(r)] \\ \cdots + D'_n \cdot \sin [\lambda_n \cdot \ln(r)] \end{array} \right\} \cdot \left\{ \begin{array}{l} E'_n \cdot \sin (\lambda_n \cdot \Theta) \\ \cdots + F'_n \cdot \cos (\lambda_n \cdot \Theta) \end{array} \right\},$$

(A4c)

and:

$$B_\Theta = B_\Theta^0 + B_\Theta' - \frac{\partial A_{zp}}{\partial r},$$

(A5a)

$$B_\Theta^0 = \left[ \frac{D^\Theta_0}{r} \right] \left( E^\Theta_0 + F^\Theta_0 \cdot \Theta \right) + \sum_{h=1}^{\infty} \frac{\beta_h}{r^2} \left\{ C^\Theta_h \cdot r^2 \beta_h \\ \cdots - D^\Theta_h \cdot r^2 \beta_h \right\} \cdot \left\{ \begin{array}{l} E^\Theta_h \cdot \cos (\beta_h \cdot \Theta) \\ \cdots + F^\Theta_h \cdot \sin (\beta_h \cdot \Theta) \end{array} \right\},$$

(A5b)

$$B_\Theta' = \left[ \frac{D^\Theta_0}{r} \right] \left( E^\Theta_0 + F^\Theta_0 \cdot \Theta \right) + \sum_{n=1}^{\infty} \frac{\lambda_n}{r^2} \left\{ -C'_n \cdot \sin [\lambda_n \cdot \ln(r)] \\ \cdots + D'_n \cdot \cos [\lambda_n \cdot \ln(r)] \right\} \cdot \left\{ \begin{array}{l} E'_n \cdot \cos (\lambda_n \cdot \Theta) \\ \cdots + F'_n \cdot \sin (\lambda_n \cdot \Theta) \end{array} \right\}. $$

(A5c)

**Appendix B. Simplification of Laplace’s Equations According to Imposed BCs**

**Appendix B.1. Case-Study No. 1: A_z Imposed on all Edges of a Region**

Figure A1a shows a region (for $\Theta \in [\Theta_0, \Theta_1]$ and $r \in [r_l, r_f]$) whose $A_z$ is imposed on all edges. By respecting the BCs and applying the principle of superposition on the magnetic quantities, Figure A1a is redefined by Figure A1b.

![Figure A1](image-url)

**Figure A1.** $A_z$ imposed on all edges of a region: (a) general and (b) principle of superposition.

In Case-Study No. 1, $A_z = A_z^\Theta + A_z'$, i.e., Equation (A2), is redefined by:

$$A_z^\Theta = \sum_{h=1}^{\infty} \left\{ c^\Theta_h \cdot \frac{E^\wedge_f (\beta_h, r_f)}{r_f} \cdot d^\Theta_h \cdot \frac{E^\wedge_f (\beta_h, r_l)}{r_l} \right\} \cdot \sin [\beta_h \cdot (\Theta - \Theta_l)],$$

(A6a)

$$A_z' = \sum_{n=1}^{\infty} \left\{ c'_n \cdot \frac{E^\wedge_f (\beta_h, r_f)}{r_f} \cdot d'_n \cdot \frac{E^\wedge_f (\beta_h, r_l)}{r_l} \right\} \cdot \sin [\lambda_n \cdot \ln \left( \frac{r_f}{r_l} \right)],$$

(A6b)

the component $B_r = B_r^\Theta + B_r'$ of $B$, i.e., Equation (A4), by:

$$B_r^\Theta = \sum_{h=1}^{\infty} \beta_h \cdot \left\{ c^\Theta_h \cdot \frac{r_l}{r_f} \cdot \frac{E^\wedge_f (\beta_h, r_f)}{E^\wedge_f (\beta_h, r_l)} \cdot d^\Theta_h \cdot \frac{r_f}{r_l} \cdot \frac{E^\wedge_f (\beta_h, r_f)}{E^\wedge_f (\beta_h, r_l)} \right\} \cdot \sin [\beta_h \cdot (\Theta - \Theta_l)],$$

(A7a)
\begin{equation}
B'_r = \sum_{n=1}^{\infty} \lambda_n \cdot \left\{ -c_n' \cdot \frac{\chi \left[ \lambda_n \cdot (\Theta - \Theta) \right]}{\sinh \left( \lambda_n \cdot \tau_\Theta \right)} + f_n' \cdot \frac{\chi \left[ \lambda_n \cdot (\Theta - \Theta) \right]}{\sinh \left( \lambda_n \cdot \tau_\Theta \right)} \right\} \cdot \frac{r_l}{r} \cdot \sin \left[ \lambda_n \cdot \ln \left( \frac{r_r}{r_l} \right) \right], \tag{A7b} \end{equation}

and the component \( B_\Theta = B_\Theta^O + B_\Theta^r \) of \( B \), i.e., Equation (A5), by:

\begin{align}
B_\Theta^O &= -\sum_{h=1}^{\infty} \beta_h \cdot \left\{ -c_h^O \cdot \frac{r_l}{r} \cdot \frac{P_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} + d_h^O \cdot \frac{r_l}{r} \cdot \frac{P_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} \right\} \cdot \sin \left[ \beta_h \cdot (\Theta - \Theta) \right], \tag{A8a} \\
B_\Theta^r &= -\sum_{n=1}^{\infty} \lambda_n \cdot \left\{ c_n' \cdot \frac{\sinh \left( \lambda_n \cdot (\Theta - \Theta) \right)}{\sinh \left( \lambda_n \cdot \tau_\Theta \right)} + f_n' \cdot \frac{\sinh \left( \lambda_n \cdot (\Theta - \Theta) \right)}{\sinh \left( \lambda_n \cdot \tau_\Theta \right)} \right\} \cdot \frac{r_l}{r} \cdot \cos \left[ \lambda_n \cdot \ln \left( \frac{r_r}{r_l} \right) \right], \tag{A8b} \end{align}

where \( c_h^O, d_h^O, c_n' \) and \( f_n' \) are new integration constants; \( \beta_h = h \cdot \pi / \tau_\Theta \) with \( \tau_\Theta = \Theta - \Theta \); \( \lambda_n = n \cdot \pi / \tau_r \) with \( \tau_r = \ln (r_l/r_r) \); and \( E_f(w, x, y) \) and \( P_f(w, x, y) \) are [44]:

\begin{equation}
E_f(w, x, y) = \left( \frac{x}{y} \right)^w - \left( \frac{y}{x} \right)^w \quad \text{and} \quad P_f(w, x, y) = \left( \frac{x}{y} \right)^w + \left( \frac{y}{x} \right)^w, \tag{A9} \end{equation}

with:

\begin{align}
\frac{\partial E_f(w, x, y)}{\partial x} &= \frac{w}{x} \cdot P_f(w, x, y) \quad \text{and} \quad \frac{\partial E_f(w, x, y)}{\partial y} = -\frac{w}{y} \cdot P_f(w, x, y), \tag{A10a} \\
\frac{\partial P_f(w, x, y)}{\partial x} &= \frac{w}{x} \cdot E_f(w, x, y) \quad \text{and} \quad \frac{\partial P_f(w, x, y)}{\partial y} = -\frac{w}{y} \cdot E_f(w, x, y). \tag{A10b} \end{align}

When \( A_z = 0 \) on \( \Theta \)-edges and \( A_z \) imposed on \( r \)-edges (see Figure A2), \( A_z \) with \( A_z^r = 0 \) in Equation (A6) is expressed by:

\begin{equation}
A_z = \sum_{h=1}^{\infty} \left[ c_h^O \cdot r_l \cdot \frac{E_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} + d_h^O \cdot r_l \cdot \frac{E_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} \right] \cdot \sin \left[ \beta_h \cdot (\Theta - \Theta) \right], \tag{A11a} \end{equation}

the \( r \)-component of \( B \) with \( B'_r = 0 \) in Equation (A7) by:

\begin{equation}
B_r = \sum_{h=1}^{\infty} \beta_h \cdot \left[ c_h^O \cdot \frac{r_l}{r} \cdot \frac{E_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} + d_h^O \cdot \frac{r_l}{r} \cdot \frac{E_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} \right] \cdot \cos \left[ \beta_h \cdot (\Theta - \Theta) \right], \tag{A11b} \end{equation}

the \( \Theta \)-component of \( B \) with \( B'_\Theta = 0 \) in Equation (A8) by:

\begin{equation}
B_\Theta = -\sum_{h=1}^{\infty} \beta_h \cdot \left[ -c_h^O \cdot \frac{r_l}{r} \cdot \frac{P_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} + d_h^O \cdot \frac{r_l}{r} \cdot \frac{P_f(\beta_h, r, r)}{E_f(\beta_h, r, r)} \right] \cdot \sin \left[ \beta_h \cdot (\Theta - \Theta) \right]. \tag{A11c} \end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figA2.png}
\caption{Figure A2. Particular case: \( A_z = 0 \) on \( \Theta \)-edges and \( A_z \) imposed on \( r \)-edges of a region.}
\end{figure}
Appendix B.2. Case-Study No. 2: $B_r$ and $A_z$ Are Respectively Imposed on $r$- and $\Theta$-Edges of a Region

Figure A3a shows a region (for $\Theta \in [\Theta_r, \Theta_l]$ and $r \in [r_t, r_l]$) whose $B_r$ and $A_z$ are respectively imposed on $r$- and $\Theta$-edges. By respecting the BCs and applying the principle of superposition on the magnetic quantities, Figure A3a is redefined by Figure A3b.

Figure A3. $B_r$ imposed on $r$-edges and $A_z$ imposed on $\Theta$-edges of a region: (a) general and (b) principle of superposition.

In Case-Study No. 2, $A_z = A_z^\Theta + A_z^\tau$, i.e., Equation (A2), is redefined by:

$$A_z^\Theta = \sum_{n=1}^{\infty} \left\{ \frac{\epsilon_n^\Theta}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\Theta \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

$$A_z^\tau = \sum_{n=1}^{\infty} \left\{ \frac{\epsilon_n^\tau}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\tau \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

the component $B_r = B_r^\Theta + B_r^\tau$ of B, i.e., Equation (A4), by:

$$B_r^\Theta = -\sum_{n=1}^{\infty} \beta_n \left\{ \frac{\epsilon_n^\Theta}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\Theta \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

$$B_r^\tau = \sum_{n=1}^{\infty} \left\{ \frac{\epsilon_n^\tau}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\tau \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

and the component $B_\Theta = B_\Theta^\Theta + B_\Theta^\tau$ of B, i.e., Equation (A5), by:

$$B_\Theta^\Theta = \sum_{n=1}^{\infty} \beta_n \left\{ \frac{\epsilon_n^\Theta}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\Theta \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

$$B_\Theta^\tau = -\sum_{n=1}^{\infty} \left\{ \frac{\epsilon_n^\tau}{\lambda_n} \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} - \delta_n^\tau \cdot \frac{\ln(r_t/r)}{\ln(r_t/r)} \right\} \cdot \frac{\lambda_n}{\lambda_n} \cdot \sin \left[ \frac{\lambda_n \cdot \ln \left( \frac{r}{r_l} \right)}{} \right],$$

where $\epsilon_n^\Theta, \delta_n^\Theta, \beta_n, \epsilon_n^\tau$ and $f_n^\tau$ are new integration constants.
Appendix C. Solving of the Linear System

Appendix C.1. Calculation of General Integrals

For the determination of Fourier’s series coefficients, it is required to calculate general integrals of the form:

\[ F_1^\Theta = \int_{l_1}^{l_1+w} \sin [\alpha_s \cdot (l - l_s)] \cdot dl, \]  
(A15a)

\[ F_2^\Theta = \int_{l_1}^{l_1+w} \cos [\alpha_s \cdot (l - l_s)] \cdot sin [\alpha_s \cdot (l - l_s)] \cdot dl, \]  
(A15b)

\[ F_3^\Theta = \int_{l_1}^{l_1+w} \sin [\alpha_{s1} \cdot (l - l_{s1})] \cdot sin [\alpha_{s2} \cdot (l - l_{s2})] \cdot dl, \]  
(A15c)

\[ F_4^\Theta = \int_{l_1}^{l_1+w} \cosh [\alpha_{ch} \cdot (l - l_{ch})] \cdot sin [\alpha_s \cdot (l - l_s)] \cdot dl, \]  
(A15d)

\[ F_5^\Theta = \int_{l_1}^{l_1+w} \sinh [\alpha_{sh} \cdot (l - l_{sh})] \cdot sin [\alpha_s \cdot (l - l_s)] \cdot dl, \]  
(A15e)

\[ F_1' = \int_{r_1}^{r_1+w} \frac{1}{r} \cdot \sin \left[ \alpha_{s1} \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot \sin \left[ \alpha_{s2} \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr, \]  
(A15f)

\[ F_2' = \int_{r_1}^{r_1+w} r \cdot \sin \left[ \alpha_s \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr, \]  
(A15g)

\[ F_3' = \int_{r_1}^{r_1+w} \frac{1}{r} \cdot \frac{\ln (r_1/r)}{\ln (r_1/r)} \cdot \sin \left[ \alpha_s \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr, \]  
(A15h)

\[ F_4' = \int_{r_1}^{r_1+w} \frac{1}{r} \cdot \frac{\ln (r/r_1)}{\ln (r_1/r)} \cdot \sin \left[ \alpha_s \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr, \]  
(A15i)

\[ F_5' = \int_{r_1}^{r_1+w} \frac{1}{r} \cdot \frac{E_f(w, r_1, r)}{E_f(w, r_1, r)} \cdot \sin \left[ \alpha_s \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr, \]  
(A15j)

\[ F_6' = \int_{r_1}^{r_1+w} \frac{1}{r} \cdot \frac{E_f(w, r_1, r)}{E_f(w, r_1, r)} \cdot \sin \left[ \alpha_s \cdot \ln \left( \frac{r}{r_1} \right) \right] \cdot dr. \]  
(A15k)

The Equations (A15) will be used in the expression of the integration constants. The expressions of Equations (A15a)–(A15e) have been given in [1,44]. The development of Equations (A15f)–(A15k) gives:

\[ F_1' (\alpha_{s1}, \alpha_{s2}, r_1, r_1) = \frac{\ln (r_1/r_1)}{2} \cdot \left\{ \text{sinc} \left[ (\alpha_{s1} - \alpha_{s2}) \cdot \ln \left( \frac{r_1}{r_1} \right) \right] - \text{sinc} \left[ (\alpha_{s1} + \alpha_{s2}) \cdot \ln \left( \frac{r_1}{r_1} \right) \right] \right\}, \]  
(A16a)

\[ F_2' (\alpha_s, r_1, r_1) = r_1^2 \cdot \frac{\alpha_s}{\alpha_s^2 + 4} \cdot \left\{ 2 \cdot \ln \left( \frac{r_1}{r_1} \right) \cdot \text{sinc} \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] + \left( \frac{r_1}{r_1} \right)^2 - \cos \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] \right\}, \]  
(A16b)
which consists of:

\[ \frac{F'_\ell (\alpha_s, r, r)}{\kappa} = \frac{1}{\kappa} \left\{ 1 - \text{sinc} \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r} \right) \right] \right\}, \]  
(A16c)

\[ F''_\ell (\alpha_s, r, r) = \frac{1}{\kappa} \left\{ \text{sinc} \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r} \right) \right] - \cos \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] \right\}, \]  
(A16d)

\[ F''_\ell (\alpha_s, r, r) = -\frac{\alpha_s}{2 \pi^2 + \alpha_s^2} \left\{ \left\{ w \cdot \frac{2}{E'_f (w, r, r)} \cdot \ln \left( \frac{r_1}{r_1} \right) \cdot \text{sinc} \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] - 1 \right\}, \]  
(A16e)

\[ F''_\ell (\alpha_s, r, r) = \frac{\alpha_s}{w^2 + \alpha_s^2} \cdot \left\{ w \cdot \text{ln} \left( \frac{r_1}{r} \right) \cdot \frac{p_{r_1} (w, r_1, r)}{E'_f (w, r, r)} \cdot \text{sinc} \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] - \cos \left[ \alpha_s \cdot \ln \left( \frac{r_1}{r_1} \right) \right] \right\}. \]  
(A16f)

Appendix C.2. Determination of Integral Constants

The integration constants are determined by solving:

\[ [IC] = [BC]^{-1} \cdot [ES] \quad (i.e., \text{Cramer's system}) \]  
(A17)

which consists of:

\[ X_{\max} = \begin{bmatrix} H_{1_{\max}} + H_{2_{\max}} + 2 \cdot H_{3_{\max}} + N_{3_{\max}} + 2 \cdot H_{4_{\max}} + N_{4_{\max}} \\ \vdots + 2 \cdot (H_{5_{\max}} + N_{5_{\max}}) + 2 \cdot (H_{6_{\max}} + N_{6_{\max}} + 1) + 2 \cdot (H_{7_{\max}} + N_{7_{\max}} + 1) \end{bmatrix} \]  
(A18)

equations and unknowns \([1]\), where \(H_{1_{\max}} - H_{7_{\max}}\) (for the \(\Theta\)-edges) and \(N_{3_{\max}} - N_{7_{\max}}\) (for the \(r\)-edges) are the maximal number of spatial harmonics in the various regions for the computation of \(A_2\) and \(B = \{B_{ij}; B_{ij} = 0\}\). To solve Equation (A17), a numerical matrix inversion is required for the calculation of \([IC]\). This set is implemented in MATLAB\textsuperscript{\textcircled{R}} (R2015a, MathWorks, Natick, MA, USA) by using the sparse matrix/vectors \([1]\). Usually, the two reasons for the possibility of including a finite number of harmonics is a limiting computational time and numerical accuracy \([45]\).

The integration constants vector \([IC]\) (of dimension \(X_{\max} \times 1\)) is defined by:

\[ [IC] = \left[ [IC1] \quad [IC2] \quad [IC3] \quad [IC4] \quad [IC5] \quad [IC6] \quad [IC7] \right]^T, \]  
(A19a)

\[ [IC1] = \begin{bmatrix} d_{1h1}^\Theta \\ [IC2] = \begin{bmatrix} c_{2h2}^\Theta \\ [IC3] = \begin{bmatrix} c_{3h3}^\Theta \\ [IC4] = \begin{bmatrix} c_{4h4}^\Theta \\ [IC5] = \begin{bmatrix} e_{5h5}^\Theta \\ [IC6] = \begin{bmatrix} c_{6h6}^\Theta \\ [IC7] = \begin{bmatrix} c_{7h7}^\Theta \\ \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{bmatrix} \]  
(A19b)

(A19c)

(A19d)

(A19e)

(A19f)

(A19g)

(A19h)

The structure of the electromagnetic sources vector \([ES]\) (of dimension \(X_{\max} \times 1\)), as well as the BCs matrix \([BC]\) (of dimension \(X_{\max} \times X_{\max}\)) is given in \([1]\) (see Section 2.6). The novel corresponding elements in \([ES]\) and \([BC]\) are defined as follows for Region 1:

\[ Q_{13} c_{h1h3} = -\frac{2 \beta_3 h_3}{\tau_{\Theta 1}} \cdot \frac{P_d (\beta_3 h_3, r_3, r_2)}{E_d (\beta_3 h_3, r_3, r_2)} \cdot F^\Theta (\beta_3 h_3, \beta_1 h_1, \Theta_1, \Theta_1, \tau_{\Theta 3}), \]  
(A20a)
\[
Q_{13}h_{1,3} = \frac{2 \cdot \beta_3}{\theta_1} \cdot \frac{\mu_1}{\mu_3} \cdot \frac{r_3}{r_2} \cdot \frac{2}{E_f} \left( f_3^\Theta \left( \frac{\beta_3 h_3}{\beta_1 h_1}, \Theta_1, \Theta_1, \Theta_1, \Theta_3 \right) \right), \tag{A20b}
\]
\[
Q_{13}f_{1,3} = \frac{2 \cdot \lambda_3}{\theta_1} \cdot \frac{\mu_1}{\mu_3} \cdot csch \left( \lambda_3 h_3 - \Theta_3 \right) \cdot f_3^\Theta \left( \lambda_3 h_3, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_3 \right), \tag{A20c}
\]
\[
Q_{14}h_{1,4} = -\frac{2 \cdot \beta_4}{\theta_1} \cdot \frac{\mu_1}{\mu_4} \cdot \frac{P^f}{E_f} \left( \beta_4 h_4, r_3, r_2 \right) \cdot f_3^\Theta \left( \beta_4 h_4, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_4 \right), \tag{A20d}
\]
\[
Q_{14}f_{1,4} = -\frac{2 \cdot \lambda_4}{\theta_1} \cdot \frac{\mu_1}{\mu_4} \cdot csch \left( \lambda_4 h_4 - \Theta_4 \right) \cdot f_3^\Theta \left( \lambda_4 h_4, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_4 \right), \tag{A20e}
\]
\[
Q_{15}h_{1,5} = -\frac{2 \cdot \beta_5}{\theta_1} \cdot \frac{\mu_1}{\mu_5} \cdot \frac{P^f}{E_f} \left( \beta_5 h_5, r_3, r_2 \right) \cdot f_3^\Theta \left( \beta_5 h_5, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_5 \right), \tag{A20f}
\]
\[
Q_{15}f_{1,5} = -\frac{2 \cdot \lambda_5}{\theta_1} \cdot \frac{\mu_1}{\mu_5} \cdot csch \left( \lambda_5 h_5 - \Theta_5 \right) \cdot f_3^\Theta \left( \lambda_5 h_5, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_5 \right), \tag{A20g}
\]
\[
Q_{16}h_{1,6} = -\frac{2 \cdot \mu_1}{\theta_1} \cdot \frac{1}{\mu_6} \cdot \frac{1}{\left( \frac{\mu_6}{\beta_6 h_6} \right)} \cdot f_1^\Theta \left( \beta_1 h_1, \Theta_1, \Theta_2, \Theta_6 \right) \quad \text{for } h6 = 0 \tag{A20h}
\]
\[
Q_{16}f_{1,6} = -\frac{2 \cdot \mu_1}{\theta_1} \cdot \frac{1}{\mu_6} \cdot csch \left( \lambda_6 h_6 - \Theta_6 \right) \cdot f_4^\Theta \left( \lambda_6 h_6, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_6 \right), \tag{A20i}
\]
\[
Q_{17}h_{1,7} = -\frac{2 \cdot \mu_1}{\theta_1} \cdot \frac{1}{\mu_7} \cdot \frac{1}{\left( \frac{\mu_7}{\beta_7 h_7} \right)} \cdot f_1^\Theta \left( \beta_1 h_1, \Theta_1, \Theta_4, \Theta_7 \right) \quad \text{for } h7 = 0 \tag{A20j}
\]
\[
Q_{17}f_{1,7} = -\frac{2 \cdot \mu_1}{\theta_1} \cdot \frac{1}{\mu_7} \cdot csch \left( \lambda_7 h_7 - \Theta_7 \right) \cdot f_4^\Theta \left( \lambda_7 h_7, \beta_1 h_1, \Theta_1, \Theta_1, \Theta_7 \right), \tag{A20k}
\]
\[
ES_{16} + ES_{17} = \mu_1 \cdot \frac{r_2}{\theta_1} \cdot \left[ f_6 \cdot f_1^\Theta \left( \beta_1 h_1, \Theta_1, \Theta_2, \Theta_6 \right) + f_7 \cdot f_1^\Theta \left( \beta_1 h_1, \Theta_1, \Theta_4, \Theta_7 \right) \right], \tag{A20l}
\]
\[
Q_{23}c_{1,3} = \frac{2 \cdot \beta_3}{\theta_2} \cdot \frac{\mu_2}{\mu_3} \cdot \frac{r_2}{r_3} \cdot \frac{2}{E_f} \left( f_3^\Theta \left( \beta_3 h_3, \beta_2 h_2, \Theta_1, \Theta_1, \Theta_3 \right) \right), \tag{A21a}
\]
\[
Q23_{h,2, h}^{3} = \frac{2}{\tau_2} \cdot \beta_3^{h,3} \cdot \frac{P_{\mathcal{E}} (\beta_3^{h,3}, r_3, r_2)}{\lambda_2^{h}} \cdot \frac{F_{\mathcal{E}} (\beta_3^{h,3}, \beta_2^{h,2}, \Theta_1, \Theta_1, \tau_{\Theta_3})}{F_{\mathcal{E}} (\beta_3^{h,3}, r_3, r_2)},
\]

\[
Q23_{h,2, n}^{3} = \frac{2}{\tau_2} \cdot \lambda_3^{h,3} \cdot \frac{P_{\mathcal{E}} (\lambda_3^{h,3}, r_3)}{\lambda_2^{h}} \cdot \frac{F_{\mathcal{E}} (\lambda_3^{h,3}, \beta_2^{h,2}, \Theta_1, \Theta_1, \tau_{\Theta_3})}{F_{\mathcal{E}} (\lambda_3^{h,3}, r_3, r_2)},
\]

\[
Q24_{h,2, h}^{4} = -\frac{2}{\tau_2} \cdot \beta_4^{h,4} \cdot \frac{P_{\mathcal{E}} (\beta_4^{h,4}, r_3, r_2)}{\lambda_2^{h}} \cdot \frac{F_{\mathcal{E}} (\beta_4^{h,4}, \beta_2^{h,2}, \Theta_1, \Theta_1, \tau_{\Theta_3})}{F_{\mathcal{E}} (\beta_4^{h,4}, r_3, r_2)},
\]

\[
Q24_{h,2, n}^{4} = -\frac{2}{\tau_2} \cdot \lambda_4^{h,4} \cdot \frac{P_{\mathcal{E}} (\lambda_4^{h,4}, r_3)}{\lambda_2^{h}} \cdot \frac{F_{\mathcal{E}} (\lambda_4^{h,4}, \beta_2^{h,2}, \Theta_1, \Theta_1, \tau_{\Theta_3})}{F_{\mathcal{E}} (\lambda_4^{h,4}, r_3, r_2)},
\]

for Region 3:

\[
Q31_{h,3, h}^{1} = \frac{2}{\tau_3} \cdot \frac{1}{\beta_1^{h}} \cdot \frac{E_{\mathcal{E}} (\beta_1^{h,1}, r_2, r_1)}{P_{\mathcal{E}} (\beta_1^{h,1}, r_2, r_1)} \cdot \frac{F_{\mathcal{E}} (\beta_1^{h,1}, \beta_3^{h,3}, \Theta_1, \Theta_1, \tau_{\Theta_3})}{F_{\mathcal{E}} (\beta_1^{h,1}, \beta_3^{h,3}, \Theta_1, \Theta_1, \tau_{\Theta_3})},
\]

for Region 3:
for Region 4:

\[
Q_{32c_{h3,h2}} = -\frac{2}{\tau_{53}} \cdot \frac{1}{\beta_{21h}} \cdot \frac{E_f (\beta_{2h2}, r_4, r_3)}{P_f (\beta_{2h2}, r_4, r_3)} \cdot F_3^\Theta (\beta_{2h2}, \beta_{3h5}, \Theta_1, \Theta_3, \tau_{53}), \tag{A22b}
\]

\[
Q_{36c_{h3,h6}} = -\frac{2}{\tau_{53}} \cdot \frac{1}{\lambda_{6h6}} \cdot \cos (\lambda_{6h6} \cdot \tau_{53}) \cdot F_5 (\lambda_{3h3}, r_2, r_3) \quad \text{for } h6 = 0
\]

\[
Q_{36d_{h3,h6}} = -\frac{2}{\tau_{53}} \cdot \frac{1}{\lambda_{6h6}} \cdot \sin (\lambda_{6h6} \cdot \tau_{53}) \cdot F_6 (\lambda_{3h3}, r_2, r_3) \quad \text{for } h6 \neq 0
\]

\[
Q_{36e_{h3,h6}} = -\frac{2}{\tau_{53}} \cdot \frac{1}{\lambda_{6h6}} \cdot \cosh (\lambda_{6h6} \cdot \tau_{53}) \cdot F_1 (\lambda_{6h6}, \lambda_{3h3}, r_2, r_3), \tag{A22c}
\]

\[
Q_{36f_{h3,h6}} = \frac{2}{\tau_{53}} \cdot \frac{1}{\lambda_{6h6}} \cdot \coth (\lambda_{6h6} \cdot \tau_{53}) \cdot F_1 (\lambda_{6h6}, \lambda_{3h3}, r_2, r_3), \tag{A22d}
\]

\[
ES_{36n_3} = -\mu_6 \cdot \frac{1}{2} \cdot \frac{1}{r_2} \cdot j_s \cdot F_2 (\lambda_{3n_3}, r_2, r_3), \tag{A22g}
\]

for Region 5:

\[
Q_{41d_{h4,h1}} = \frac{2}{\tau_{64}} \cdot \frac{1}{\beta_{1h1}} \cdot \frac{E_f (\beta_{1h1}, r_2, r_1)}{P_f (\beta_{1h1}, r_2, r_1)} \cdot F_3^\Theta (\beta_{1h1}, \beta_{4h4}, \Theta_1, \Theta_5, \tau_{64}), \tag{A23a}
\]

\[
Q_{42c_{h4,h2}} = -\frac{2}{\tau_{54}} \cdot \frac{1}{\beta_{21h}} \cdot \frac{E_f (\beta_{2h2}, r_4, r_3)}{P_f (\beta_{2h2}, r_4, r_3)} \cdot F_3^\Theta (\beta_{2h2}, \beta_{4h4}, \Theta_1, \Theta_5, \tau_{54}), \tag{A23b}
\]

\[
Q_{47e_{h4,h7}} = -\frac{2}{\tau_{54}} \cdot \frac{1}{\lambda_{7h7}} \cdot \cos (\lambda_{7h7} \cdot \tau_{54}) \cdot F_5 (\lambda_{4h4}, r_2, r_3) \quad \text{for } h7 = 0
\]

\[
Q_{47d_{h4,h7}} = -\frac{2}{\tau_{54}} \cdot \frac{1}{\lambda_{7h7}} \cdot \sin (\lambda_{7h7} \cdot \tau_{54}) \cdot F_6 (\lambda_{7h7}, \lambda_{4h4}, r_2, r_3) \quad \text{for } h7 \neq 0
\]

\[
Q_{47e_{h4,n7}} = -\frac{2}{\tau_{54}} \cdot \frac{1}{\lambda_{7h7}} \cdot \cosh (\lambda_{7h7} \cdot \tau_{54}) \cdot F_1 (\lambda_{7h7}, \lambda_{4h4}, r_2, r_3), \tag{A23c}
\]

\[
Q_{47f_{h4,n7}} = \frac{2}{\tau_{54}} \cdot \frac{1}{\lambda_{7h7}} \cdot \sinh (\lambda_{7h7} \cdot \tau_{54}) \cdot F_1 (\lambda_{7h7}, \lambda_{4h4}, r_2, r_3), \tag{A23d}
\]

\[
ES_{47n_4} = -\mu_7 \cdot \frac{1}{2} \cdot \frac{1}{r_2} \cdot j_s \cdot F_2 (\lambda_{4n_4}, r_2, r_3), \tag{A23g}
\]
for Region 6:

\[ Q_{57}c_{h6,h1} = -\frac{2}{\tau_5} \left\{ \begin{array}{ll} \frac{F_5^\Theta(\lambda_{h5}, r_2, r_3)}{\tau_5} & \text{for } h_7 = 0 \\ \frac{F_5^\tau(\beta_1 h, \lambda_{h5}, r_2, r_3)}{\tau_5} & \text{for } h_7 \neq 0 \end{array} \right. \quad (A24g) \]

\[ Q_{57}d_{h6,h7} = -\frac{2}{\tau_5} \left\{ \begin{array}{ll} \frac{F_4^\Theta(\lambda_{h5}, r_2, r_3)}{\tau_5} & \text{for } h_7 = 0 \\ \frac{F_6^\tau(\beta_1 h, \lambda_{h5}, r_2, r_3)}{\tau_5} & \text{for } h_7 \neq 0 \end{array} \right. \quad (A24h) \]

\[ Q_{57}e_{h5,n7} = -\frac{2}{\tau_5} \cdot \frac{1}{\lambda_{h7}^n} \cdot \csc(h_{77} \cdot \tau_{67}) \cdot F_1^\Theta(\lambda_{77}, \lambda_{h5}, r_2, r_3), \quad (A24i) \]

\[ Q_{57}f_{h5,n7} = \frac{2}{\tau_5} \cdot \frac{1}{\lambda_{h7}^n} \cdot \coth(h_{77} \cdot \tau_{67}) \cdot F_1^\tau(\lambda_{77}, \lambda_{h5}, r_2, r_3), \quad (A24j) \]

\[ ES_{66} = -\mu \cdot \frac{1}{\tau_5} \cdot \frac{1}{r_2} \cdot J_{26} \cdot \frac{F_2^\tau(\lambda_{h5}, r_2, r_3)}{\mu}\quad (A24k) \]

\[ ES_{77} = -\mu \cdot \frac{1}{\tau_5} \cdot \frac{1}{r_2} \cdot J_{27} \cdot \frac{F_2^\tau(\lambda_{h5}, r_2, r_3)}{\mu}\quad (A24l) \]

for Region 7:

\[ Q_{71}d_{h7,h1} = -\frac{1}{\tau_7} \cdot \frac{1}{\beta_1 h_1} \cdot E_{(\beta_1 h, r_2, r_1)} \left\{ \begin{array}{ll} \frac{F_1^\Theta(\beta_1 h_1, \Theta_1, \Theta_2, \tau_{67})}{\tau_7} & \text{for } h_7 = 0 \\ \frac{F_2^\tau(\beta_1 h_2, \beta_1 h_1, \Theta_1, \Theta_2, \tau_{67})}{\tau_7} & \text{for } h_7 \neq 0 \end{array} \right. \quad (A26a) \]

\[ Q_{72}c_{h7,h2} = -\frac{1}{\tau_7} \cdot \frac{1}{\beta_2 h_2} \cdot E_{(\beta_2 h, r_2, r_3)} \left\{ \begin{array}{ll} \frac{F_1^\Theta(\beta_2 h_2, \Theta_1, \Theta_4, \tau_{67})}{\tau_7} & \text{for } h_7 = 0 \\ \frac{F_2^\tau(\beta_2 h_3, \beta_2 h_2, \Theta_1, \Theta_4, \tau_{67})}{\tau_7} & \text{for } h_7 \neq 0 \end{array} \right. \quad (A26b) \]
\begin{align}
Q74c_{n7,4} &= -\frac{2}{r_7} \cdot \frac{\beta_4 h_{44}}{\mu_7} \cdot \frac{\mu_7}{\mu_4} \cdot F_5^\tau (\beta_4 h_{44}, \lambda_7 n_7, r_2, r_3), \quad (A26c) \\
Q74d_{n7,4} &= -\frac{2}{r_7} \cdot \frac{\beta_4 h_{44}}{\mu_7} \cdot \frac{\mu_7}{\mu_4} \cdot \frac{r_3}{r_2} \cdot F_6^\tau (\beta_4 h_{44}, \lambda_7 n_7, r_2, r_3), \quad (A26d) \\
Q74c_{n7,n4} &= \frac{2}{r_7} \cdot \frac{\lambda_4 h_{44}}{\tau_7} \cdot \frac{\mu_7}{\mu_4} \cdot \coth (\lambda_4 n_4 \cdot \tau_4) \cdot F_7^\tau (\lambda_4 n_4, \lambda_7 n_7, n_2, n_3), \quad (A26e) \\
Q75c_{n7,n5} &= -\frac{2}{r_7} \cdot \frac{\beta_5 h_{55}}{\tau_7} \cdot \frac{\mu_7}{\mu_5} \cdot (\lambda_7 n_7, r_2, r_3), \quad (A26f) \\
Q75d_{n7,n5} &= -\frac{2}{r_7} \cdot \frac{\beta_5 h_{55}}{r_7} \cdot \frac{r_3}{r_2} \cdot (\lambda_7 n_7, r_2, r_3), \quad (A26g) \\
Q75e_{n7,n5} &= \frac{2}{r_7} \cdot \frac{\lambda_5 h_{55}}{\tau_7} \cdot \frac{\mu_7}{\mu_5} \cdot \operatorname{csch} (\lambda_5 n_5 \cdot \tau_5) \cdot F_1^\Omega (\lambda_5 n_5, \lambda_7 n_7, n_2, n_3), \quad (A26h) \\
Q75f_{n7,n5} &= \frac{2}{r_7} \cdot \frac{\lambda_5 h_{55}}{\tau_7} \cdot \frac{\mu_7}{\mu_5} \cdot \coth (\lambda_5 n_5 \cdot \tau_5) \cdot F_1^\Omega (\lambda_5 n_5, \lambda_7 n_7, n_2, n_3), \quad (A26i) \\
ES71_0 &= \frac{1}{4} \cdot \mu_7 \cdot J_7 \cdot r_2, \quad (A26j) \\
ES72_0 &= \frac{1}{4} \cdot \mu_7 \cdot J_7 \cdot r_3. \quad (A26k)
\end{align}

References


