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The Modeling and Calculation of the Heading Machine Based on Differential Geometry

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Abstract: The kinematic sketch of the heading machine’s cutting part is plotted and the kinematic relation is analyzed. The pose-attitude model of the cutting part is derived from the geometry method, and the velocity and acceleration relations are derived by the differential geometry method. According to the recurrence relation among the pose-attitude, the velocities and the accelerations, the numerical solving strategy is designed. The nonlinear part of the kinematics model is solved by the Newton iterative method. The kinematics model is simulated by MATLAB. The trigonometric functions are avoided by using the differential geometry method, and the derivation process and the results are simplified simultaneously. The simulation results give the curves of each kinematic parameter which verifies the validity of the kinematic model.

Keywords: the heading machine; spatial differential geometry; spatial mechanism; nonlinear; Newton iterative

1. Introduction

The horizontal axis heading machine is a piece of high-power excavating equipment in the coal mine. The cutting part is an open loop link consisting of the turntable, the cutting arm, the slip sleeve and the cutting head, which connects together in series by the rotation pairs and the sliding pairs. The cutting work is realized by the movement of the cutting teeth, which are driven by the four parts, so the kinematic model of the cutting teeth under the driving of the four parts is significant to realize the robotized working of the heading machine.

The main works regarding the mathematical modeling of the heading machine are as follows. Guo Y. F. et al. [1] analyzed the yawn motion of the cutting head and simulated a mathematical model of the mounting position of the cutting teeth on the cutting head of the horizontal axis. The heading machine is also analyzed [2]. Zhao L. J. et al. [3] analyzed the optimal design of the yawn motion parameters of the vertical axis heading machine. Li J. L. et al. [4] analyzed the kinematic problems and positioning of the trunk of the robotized heading machine. Chen H. Y. et al. [5] analyzed the multi-body dynamics problem of the drilling manipulator which is on the heading machine. However, the mathematical models in the above explorations are all based on the triangle functions. The solution pattern would be an appropriate analytical solution, but the expression, derivational processes and solving methods have a high complexity.

Recently, the development of computation geometry mechanisms makes the modeling method based on the matrix gradually a simple and efficient new method which can replace the mathematical modeling method based on the triangle function. The differential geometry method is used in many explorations such as: robotics [6], mechanism [7], snake like robot [8], parallel robot [9], hydraulic

In this paper, the kinematic equation of the cutting part of the horizontal axis heading machine is built up by the differential geometry method in space. The numerical method which can solve the implicit equation in the mathematical model is also constructed. The main character of the mathematical modeling method is that the matrix operator and vector operator are used to replace the algebra, which makes the mathematical model clearer and simpler. The kinematical relation of the cutting tooth is derived and the trajectory of the cutting tooth is calculated. This mathematical model offers the basis for the dynamics modeling and the control of the robotized heading machine.

2. The Mechanism Analysis of the Horizontal Axis Heading Machine

Figure 1 is the mechanism diagram of the horizontal axis heading machine. The components of the cutting part are numbered 1 to 8, the lower pairs are allocated the letters A to J, so the freedom is 4, which can be calculated by \( F_1 = 3 \times 8 - 2 \times 10 = 4 \). The four freedoms correspond to the four drives of the cutting part. The cutting head 8 rotates along the revolute pair \( F \), the actuated motor acts on the cutting head through a screw gearing mechanism, which is a linear drive. The slip sleeve 5 slides relative to the cutting arm 4 along the sliding pair \( E \), which is also a linear drive. The cutting arm rotates along the revolute joint \( D \) relative to the turn Table 1, and the rotation is realized by the motion of the sliding pair \( B \). The length of BC has a nonlinear relation with angle \( a \). The drive cylinder will rotate at point \( A \) with the motion of the hydraulic rod. The turntable rotates at point \( L \), which drives by the hydraulic system, the displacement then translates to the rack connected to the hydraulic rod 7. The rack is engaged with the gear on the turntable by \( G \), which is used to drive the turntable. For the linear displacement of \( H \) equal to the circle displacement of the turntable’s gear, we rotate along the point \( I \), so the displacement along \( H \) and the rotation along \( I \) has a linear relation.

![Figure 1. The schematic diagram of mechanism of the horizontal axis type heading machine.](image)

According to the analysis, the motion of the cutting teeth in space are influenced by the combined action of the four actuator, so the trajectory of the cutting teeth has a highly nonlinear character. Simultaneously, the cutting arm rotates along \( A \) which is driven by the actuator \( B \), which also has a high nonlinearity, which is also needed in order for it to be effectively calculated.
The relation between $\mathbf{R}$ of the rod in plane following derivation. Equation (2) can be used to simplify the derivation process. The rotation matrix in Equation (5), frame, the rotation matrix satisfies the following property the two elements in the two columns respect the projection of the axis of $\mathbf{Ox}$ expressed as the matrix type $[22]$ as rotate $\theta$ the axises from these two frames concide together at the initial moment, then the rod anticlockwise $\mathbf{Ox}$ in the inertial frame $\mathbf{SO}(3)$ is needed. The differential geometry in space $\mathbf{SO}(3)$ can be explained by the triangle function and the differential geometry on $\mathbf{SO}(2)$. Define the inertial frame $\mathbf{Ox}\mathbf{y}\mathbf{z}$ and the body frame which can rotate with the rod, as in Figure 2. Supposing that the axises from these two frames coincide together at the initial moment, then the rod anticlockwise rotate $\theta$ along the point $\mathbf{O}$, the projection of the axises of the body frame on the inertial frame can be expressed as the matrix type $[22]$ as

$$
\mathbf{R} = \begin{bmatrix}
\cos\theta & -\sin\theta \\
\sin\theta & \cos\theta
\end{bmatrix}
$$

(1)

the two elements in the two columns respect the projection of the axis of $\mathbf{Ox}$ and $\mathbf{Oy}$ in the inertial frame, the rotation matrix satisfies the following property

$$
\mathbf{SO}(2) = \left\{ \mathbf{R} \bigg| \mathbf{RR^T} = \mathbf{I}_{2 \times 2}, \det \mathbf{R} = 1 \right\}.
$$

(2)

According to Equation (1), the rotation matrix $\mathbf{R}$ can be written as

$$
\mathbf{R} = \cos\theta \mathbf{I}_{2 \times 2} + \sin\theta \mathbf{S} (1) = \begin{bmatrix} p & q \\ q & -p \end{bmatrix},
$$

(3)
in Equation (3), $p, q$ satisfy $p^2 + q^2 = 1$, $\mathbf{I}_{2 \times 2}$ is the unit matrix, $\mathbf{S} (1)$ is the skew matrix

$$
\mathbf{S} (1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
$$

(4)

The first order derivative of $\mathbf{R}$ is

$$
\dot{\mathbf{R}} = \begin{bmatrix}
-\sin\theta & -\cos\theta \\
\cos\theta & -\sin\theta
\end{bmatrix}, \dot{\theta} = \mathbf{RS} (\omega) = \mathbf{\omega RS} (1),
$$

(5)
in Equation (5), $\dot{\theta} = \omega$. The properties of (2), (3), (5) avoid the triangle function that occurs in the following derivation. Equation (2) can be used to simplify the derivation process. The rotation matrix of the rod in plane $\mathbf{R}$ can be expanded to be a three dimensional matrix $\mathbf{R}_z$ which can express the rotation along the $z$ axis in space. The relation between $\mathbf{R}$ and $\mathbf{R}_z$ is
\[ R_z = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, \]  
(6)

if the rod rotates along the z axis, the vector on the z axis has no projection on the x and y axis, so the first two elements in the third column of the rotation matrix \( R_z \) has no change. According to the Equation (5), the first order differential of the matrix \( R_z \) is

\[ \dot{R}_z = \begin{bmatrix} \dot{R} & 0 \\ 0 & 1 \end{bmatrix} = \omega R S_z \]  
(7)

the skew matrix in Equation (7) is

\[ S_z(1) = \begin{bmatrix} S(1) & 0 \\ 0 & 0 \end{bmatrix}. \]  
(8)

When the rod rotate \( \alpha \) along the y axis, the rotation matrix is

\[ R_y = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}, \]  
(9)

the projection of the unit vector on the y axis has no projection on the x and z axis, so the second column of \( R_y \) is 1. The first order differential of \( R_y \) is

\[ \dot{R}_y = \omega_y R_y S_y(1), \]  
(10)

\[ \omega_y = \dot{\alpha} \]  
the type of the skew matrix is

\[ S_y(1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \]  
(11)

The above derivation gives the rotation matrix and the differential of the matrix of the rigid body in the three dimensional space which is rotated along the z axis and the y axis. In practice, the rod has some geometrical character, so the coordinate at any position in the rod needs to be defined. Supposing that the position vector of the point A (Figure 2) in \( Ox_1y_1 \) is \( I_A \), then the projection of point A in \( Ox_0y_0 \) after the rotation can be expressed as

\[ \Gamma = RI_A. \]  
(12)

The velocity vector of point A in \( Ox_0y_0 \) is

\[ \dot{\Gamma} = \omega RS(1) I_A. \]  
(13)

In addition, there are many sliding pairs, supposing that the coordinates of the two end points of the sliding pairs are \( m, n \), then the distance between the two points can be expressed as

\[ l = ||m - n||, \]  
(14)
So the position vector of the cutting teeth in the inertial frame which is at the point of cutting head along the z axis of the frame sleeve’s extending and shrinking. The driving cylinder of the turntable is fixed with the main engine.

4. The Pose and Attitude Analysis

The geometry relation graph is shown in Figure 3. Supposing that the turntable is rotated along the frame $O_1$, the cutting arm rotates along the frame $O_2$, the cutting head rotates along the frame $O_3$, the drive cylinder of the cutting arm is hinged with the turntable and the cutting arm by the point $O_4$ and $O_5$, respectively. The distance between the points $O_6$ and $O_7$ represents the length of the slip sleeve’s extending and shrinking. The driving cylinder of the turntable is fixed with the main engine by the point $O_8$, the gear of the turntable is engaged with the rack by $O_9$. Constructing the body frame and the inertial frame at these points respectively, the rotation direction of the body frames is all along the anticlockwise direction [23].

According to the coupling relation of each motion component, the analysis of the motion is represented by the sequence from the cutting head to the cutting arm, then to the turntable. Define the position vector of the cutting tooth in the body frame at $O_3$ to be $r_1$, and the rotation angle of the cutting head along the z axis of the frame $O_3$ to be $\theta_3$, then the rotation matrix is as Equation (15). So the position vector of the cutting teeth in the inertial frame which is at the point of $O_3$ is $R_3r_1$.

$$
R_3 = \begin{bmatrix}
\cos\theta_3 & -\sin\theta_3 & 0 \\
\sin\theta_3 & \cos\theta_3 & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

Figure 2. The kinematic diagram of differential geometry.

Figure 3. The geometry relation of the heading machine.
The relation of the slip sleeve and the cutting tooth is analyzed as follows. Define the coordinate vector of the O3 relative to the O2 to be \( r_2 \), then the coordinate value of the cutting tooth relative to the frame O2 to be \( r_2 + R_3 r_1 \). Define the distance between O6 and O7 is \( l_5 \), and the x axis of the frame O6 and O7 are coincide with the line O6O7, so the coordinate of O7 relative to point O6 to be \( l_5 e_1 \), then the position of the cutting tooth relative to the frame O6 is

\[
r_7^6 = l_5 e_1 + r_2 + R_3 r_1. \tag{16}\]

Similarly, the distance between the points of O6 and O2 is \( l_f \), so the coordinate of O6 in the frame of O2 is \( l_f e_1 \), the coordinate of the cutting tooth in the frame O2 is

\[
r_7^2 = (l_f + l_5) e_1 + r_2 + R_3 r_1. \tag{17}\]

The about kinematics analysis of the cutting tooth is based on the body frame O2, when the cutting arm rotates \( \theta_2 \) along the z axis of the frame O2, the rotation matrix is

\[
R_2 = \begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 & 0 \\
\sin \theta_2 & \cos \theta_2 & 0 \\
0 & 0 & 1
\end{bmatrix}, \tag{18}
\]

so the position of the cutting tooth in the inertial frame at the point O2 is

\[
r_7^2a = R_2 \left[ (l_f + l_5) e_1 + r_2 + R_3 r_1 \right]. \tag{19}\]

Define the distance between O2 and O1 to be \( l_Z \), the position vector of O2 relative to frame O1 is \( l_Z e_1 \), then the position vector of the cutting tooth in frame O1 is

\[
r_7^1 = l_Z e_1 + R_2 \left[ (l_f + l_5) e_1 + r_2 + R_3 r_1 \right]. \tag{20}\]

Define the position of O3 in the body frame at point O2 to be \( r_3 \), the point O3 rotate with the body frame at point O2, so the position vector of point O3 is \( R_2 r_3 \), then the position of O5 in the frame at O1 is

\[
r_5^3 = l_Z e_1 + R_2 r_3. \tag{21}\]

Define the position vector of point O4 in the frame O1 to be \( r_4 \), then the distance between the two points O4 and O5 can be expressed as

\[
l_Y = l_Z e_1 + R_2 r_3 - r_4, \tag{22}\]

according to Equation (14), the distance can be calculated by the distance vector as Equation (23).

\[
l_Y = ||l_Y|| = ||l_Z e_1 + R_2 r_3 - r_4||, \tag{23}\]

Dragging Equation (22) into Equation (23) to obtain the following equation

\[
2l_Z e_1^T R_2 r_3 - 2r_4^T R_2 r_3 = l_Y^2 - K, \tag{24}\]

in Equation (24), \( K = l_Z^2 + r_4^T r_4 - 2l_Z r_4^T e_1 + r_3^T r_3 \). The Equation (24) is the pose and attitude relation between the length of the hydraulic rod O4 O5 and the rotation angle of the cutting arm. Supposing that the rotation angle of the rod rotate along the point is \( \theta_4 \), the rotation matrix is
\[
R_4 = \begin{bmatrix}
\cos \theta_4 & -\sin \theta_4 & 0 \\
\sin \theta_4 & \cos \theta_4 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

(25)

Define the initial position of the rod \(O_4O_5\) to coincide with the x axis of the frame \(O_4\), the coordinate of the point \(O_5\) in the frame \(O_1\) after the rotation can be expressed as

\[
l_y R_4 e_1 = R_2 r_3 + l_y e_1 - r_4,
\]

(26)

With Equation (26), the rotation matrix \(R_4\) can be derived. Equation (26) is the expression of the rotation motion of the hydraulic cylinder which accompany with the extending and shrinking of the hydraulic cylinder \(O_4O_5\). When the turntable rotates \(\theta_1\) along the y axis of frame \(O_1\), the rotation matrix is

\[
R_1 = \begin{bmatrix}
\cos \theta_1 & 0 & \sin \theta_1 \\
0 & 1 & 0 \\
-\sin \theta_1 & 0 & \cos \theta_1
\end{bmatrix},
\]

(27)

According to Equation (20), the coordinate of the cutting tooth \(r_j\) in the space is

\[
r_j = f (R_1, R_2, R_3, l_5) = R_1 l_y e_1 + R_1 R_2 R_3 r_1 + R_1 R_2 (l_3 + l_5) e_1 + R_1 R_2 r_2,
\]

(28)

5. The Velocity Analysis

Based on the pose and attitude relation, the velocity response of the cutting tooth can be derived by the differential calculation. The first order differential of the rotation matrixes are

\[
\dot{R}_1 = \omega_1 R_1 s_2 (1), \dot{R}_2 = \omega_2 R_2 s_2 (1), \dot{R}_3 = \omega_3 R_3 s_2 (1), \dot{R}_4 = \omega_4 R_4 s_2 (1),
\]

(29)

\(\omega_1, \omega_2, \omega_3, \omega_4\) are the angular velocities of the turntable, the cutting arm, the cutting head and the hydraulic rod \(O_4O_5\) respectively. Differentiate Equation (24), replace \(R_2\) by the Equation (29), the relation between \(l_y\) and \(\omega_2\) is

\[
\omega_2 \left( l_y e_1^T - r_4^T \right) R_2 s_2 (1) r_3 = l_y \dot{l}_y,
\]

(30)

The hydraulic rod is the actuator, the cutting arm is the driven part, so \(\omega_2\) can be expressed as

\[
\omega_2 = \frac{l_y \dot{l}_y}{(l_y e_1^T - r_4^T) R_2 s_2 (1) r_3},
\]

(31)

the angular velocity of the cutting arm and the velocity of the rod \(O_4O_5\) have a nonlinear relation. Differentiate Equation (26), the following relation can be obtained.

\[
\dot{l}_y R_4 e_1 + \omega_4 l_y R_4 s_2 (1) e_1 = \omega_2 R_2 s_2 (1) r_3,
\]

(32)

\(\omega_4\) is the angular velocity of \(O_4O_5\) rotate along \(O_4\), which can be expressed as

\[
\omega_4 = -\omega_2 e_1^T s_2 (1) R_4^T R_2 s_2 (1) r_3 / l_y,
\]

(33)

Differentiate Equation (28), the velocity of the cutting tooth is

\[
r_j = (l_1 + l_5) \left( R_1 R_2 + R_1 R_2 \right) e_1 + \left( R_1 R_2 + R_1 R_2 \right) r_2 + R_1 R_2 R_3 r_1 + R_1 R_2 \left. \left( R_3 r_1 + l_5 e_1 \right) \right) + R_1 \left. \left( l_y e_1 + R_2 r_3 \right) \right),
\]

(34)
bring Equation (29) into (34) to replace the first order differential part, the response of the velocity of the cutting tooth can be expressed as a neatly type.

\[ \dot{r}_j = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3 + \dot{l}_s m_4, \]  

(35)

the expressions of the parameters \( m_1, m_2, m_3, m_4 \) are as follows.

\[
\begin{align*}
    m_1 &= (l_f + l_s) R_1 S_g (1) R_2 e_1 + R_1 S_g (1) R_2 (r_2 + R_3 r_1) + l_{CD} R_1 S_g (1) e_1, \\
    m_2 &= (l_f + l_s) R_1 R_2 S_z (1) e_1 + R_1 R_2 S_z (1) (r_2 + R_3 r_1), \\
    m_3 &= R_1 R_2 R_3 S_z (1) r_1, \\
    m_4 &= R_1 R_2 e_1
\end{align*}
\]

(36)

if the pose and attitudes of each components of the system are known, \( m_1, m_2, m_3, m_4 \) are constant values, so the velocity of the cutting tooth is the linear combination of \( \omega_1, \omega_2, \omega_3, \dot{l}_s \).

6. The Acceleration Analysis of the Motion

The acceleration relations of the system are derived as follows. Firstly, the relation between the velocity of the extending and shrinking of the cylinder \( O_4 O_5 \) and the angular acceleration of the cutting arm is derived. Differentiate Equation (30), the angular acceleration response of the cutting arm is

\[ \dot{\omega}_2 = \frac{\dot{l}_y l_y}{b_1} + \frac{l_y^2}{b_1} - \frac{b_2}{b_1} \omega_2^2, \]  

(37)

the expressions of the parameters \( b_1 \) and \( b_2 \) in Equation (37) are

\[
\begin{align*}
    b_1 &= (l_z e_1 - r_4)^T R_2 S_z (1) r_3, \\
    b_2 &= (l_z e_1 - r_4)^T R_2 S_z^2 (1) r_3
\end{align*}
\]

(38)

the angular acceleration of the cutting arm has a complex nonlinear relation with the extending and shrinking acceleration and velocity of the hydraulic cylinder, also with the angular velocity of the cutting arm. The angular acceleration of the rod during the extending and shrinking is derived by the differentiate calculation of the Equation (32), as

\[ \dot{\omega}_4 = g_1 \omega_2 \omega_4 - g_2 \omega_2^2 - \omega_4 \dot{l}_y / l_y - g_3 \dot{\omega}_2, \]

(39)

\( g_1, g_2, g_3 \) are the values about the pose and attitudes, the expressions are

\[
\begin{align*}
    g_1 &= e_1^T S_z^2 (1) R_1^T R_2 S_z (1) r_3 / l_y, \\
    g_2 &= e_1^T S_z (1) R_1^T R_2 S_z^2 (1) r_3 / l_y, \\
    g_3 &= e_1^T S_z (1) R_1^T R_2 S_z (1) r_3 / l_y
\end{align*}
\]

(40)

The acceleration response of the cutting tooth is derived by the differentiate calculation of the Equation (35), the result is

\[ \ddot{r}_j = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + \dot{l}_s \ddot{n}_4 + \omega_2^2 n_6 + \omega_2^2 n_5 + 2 \omega_1 \omega_2 n_8 + 2 \omega_1 \omega_3 n_9 + , \]

\[
2 \omega_2 \omega_3 n_{10} + 2 \omega_1 \dot{l}_s n_{11} + 2 \omega_2 \dot{l}_s n_{12}
\]

(41)

\( n_i, i = 1, ..., 12 \) is the function about the pose and attitudes, which is expressed as follows. According to Equation (41), each motion of the components in the cutting part has an influence on the acceleration of the cutting tooth, and the motion has a high nonlinear character.
\[ n_1 = R_1 S_y (1) R_2 ((l_j + I_3) e_1 + (r_2 + R_3 r_1)) + l_2 R_1 S_y (1) e_1 \]
\[ n_2 = R_1 R_2 S_z (1) ((l_j + I_3) e_1 + (r_2 + R_3 r_1)) \]
\[ n_3 = R_1 R_2 R_3 S_z (1) r_1, \]
\[ n_4 = R_1 R_2 e_1, \]
\[ n_5 = R_1 R_2 R_3 S_z^2 (1) r_1, \]
\[ n_6 = R_1 S_y (1) ((l_j + I_3) R_2 e_1 + R_2 (R_3 r_1 + r_2) + l_2 e_1) \]
\[ n_7 = R_1 R_2 S_z^2 (1) ((l_j + I_3) e_1 + (r_2 + R_3 r_1)), \]
\[ n_8 = R_1 S_y (1) R_2 S_z (1) ((l_j + I_3) e_1 + (r_2 + R_3 r_1)) \]
\[ n_9 = R_1 S_y (1) R_2 R_3 S_z (1) r_1 \]
\[ n_{10} = R_1 R_2 S_z (1) R_3 S_z (1) r_1 \]
\[ n_{11} = R_1 S_y (1) R_2 e_1 \]
\[ n_{12} = R_1 R_2 S_z (1) e_1 \]

7. The Numerical Calculation Flow

The kinematic model has three parts: (24), (26) and (28) constitutes the pose and attitude relation, (31), (33) and (34) constitutes the velocity and angular velocity relation, (37), (39) and (41) constitutes the acceleration and angular acceleration relation. The kinematic model has a nonlinear character and recurrence connection, so the numerical solution is the only solving method. According to the recurrence relation, the calculation flow is confirmed as follows.

In the pose and attitude response, the Equation \( R_2 = f(l_y) \) is a nonlinear equation which is implicit, the solution needs the help of the character of the rotation matrix to translate it to be a nonlinear equation system which can be solved by the numerical calculation method. According to Equation (3), the rotation matrix \( R_2 \) satisfies the following character:

\[ R_2 = p_2 I_1 + q_2 S_z (1) + I_2, \]  

the parameters in it are
\[ I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ p_2 = \cos \theta_2, \quad q_2 = \sin \theta_2, \quad p_2^2 + q_2^2 = 1. \]

Bring Equation (43) into (24) to obtain the nonlinear equation is

\[ \begin{cases} K_1 p_2 + K_2 q_2 = K_3 \\ p_2^2 + q_2^2 = 1 \end{cases}, \]

the expressions of the parameters \( K_1, K_2, K_3 \) are

\[ \begin{cases} K_1 = 2 (l_2 e_1^T I_1 r_3 - r_4^T I_3 r_3) \\ K_2 = 2 (l_2 e_1^T S_z (1) r_3 - r_4^T S_z (1) r_3) \\ K_3 = l_2^2 - K - 2 l_2 e_1^T I_2 r_3 + 2 r_4^T I_2 r_3 \end{cases} \]

the Equation (45) can be solved by the “fsolve” function in MATLAB [24]. With Equation (43), the Equation (26) can be translate to be the following type as

\[ e_1 p_4 + e_2 q_4 = K_4, \]

in Equation (47), \( K_4 = (R_2 r_3 + l_2 e_1 - r_4) / l_Y \). It is a linear equation with constant coefficient, so the results can be obtained directly.
8. The Simulation Results

According to the structure of the heading machine, the main parameters are as follows. \( r_1 = [79; 591; 515] \), \( r_2 = [1560; 165; 0] \), \( r_3 = [20; -510; 0] \), \( r_4 = [-618; -461; 0] \), \( l_1 = 696 \), \( l_2 = 1270 \) mm. Setting the initial lengths to be \( l_1^0 = 2096 \), \( l_2^0 = 1538 \), the initial attitudes are \( R_1^0 = R_2^0 = I_3 \). The constant rotation angular velocity of the cutting head is set to \( \omega_3 = 2.1 \text{rad/s} \). The acceleration of the drive cylinder of the turntable is 3.75 mm/s\(^2\), the initial and the end positions are 0 and 1500 mm respectively, the acceleration of the slip sleeve is 1 mm/s\(^2\), the initial and the end positions are 1538 mm and 1938 mm respectively, the acceleration of the drive cylinder of the cutting arm is 0.5 mm/s\(^2\), the initial and the end positions are 2096 mm and 2296 mm, respectively. In the first 10 s, the turntable, the slip sleeve and the drive cylinder of the cutting arm are accelerated with their constant accelerations. Then they move with a constant velocity with the next 20 s, and decelerate within 10 s. The whole time of the simulation is 40 s. The simulation results of the kinematic parameters of the heading machine are shown in Figures 4–7.

![Figure 4](image1.png)  
**Figure 4.** The kinematics parameters of the cutting arm and drive cylinder.

![Figure 5](image2.png)  
**Figure 5.** The variation of the attitude and the velocity of the cutting tooth in space.
The variations of the angles, the angular velocities and the angular accelerations of the cutting arm $\theta_2, \omega_2, \alpha_2$ and the drive cylinder of it $\theta_4, \omega_4, \alpha_4$ are expressed in Figure 4. The curves are all smooth, which verifies the correctness of the Equations. With the rotation of the cutting arm, the drive cylinder has a small rotation angle, the angular velocities and the accelerations are all very small which meet the character of the heavy machine.

The variations of the attitude, the angular velocity and the angular acceleration are expressed in Figures 5 and 6. According to Figure 5, the attitudes and the velocities of the cutting tooth which are projected on the x, y, z axis of the inertial frame are changed with the coupling motion of the turntable, the cutting arm, the slip sleeve and the cutting head. Their periodic waves are led by the motion of the cutting head, so the motion of the cutting tooth is the nonlinear coupling of all the motions parts. According to Figure 6, the variation of the acceleration of the cutting tooth on the x and y axis is small, which is mainly influenced by the pitching motion of the cutting arm and the slide motion of the slip sleeve. The variation on the z axis is obvious, which is mainly influenced by the yawn motion of the turntable. The trajectory of the cutting tooth in space is shown in Figure 7, the trajectory is a continuous loop curve which fixes the motion character of the cutting tooth in the practical application.

![Figure 6. The acceleration of the cutting tooth in space.](image1)

![Figure 7. The trajectory of the cutting tooth in space.](image2)

9. Summary and Prospects

In this paper, the kinematic model of the cutting part of the horizontal axis heading machine is established by the differential geometry method. In this modeling method, the triangle function is
not realized and the expressions are simpler than in other methods. The kinematic relation between the linear parts and the cutting tooth are derived, the kinematic equation of the cutting tooth with high nonlinear character and the nonlinear response of other components is derived. A numerical calculation method is designed which can avoid the complex solution process. The correctness of the equations is testified by simulation. However, this paper uses differential equations and geometric concepts from linear algebra, but it does not use differential geometry of the type considered for kinematics of continuous media involving linear connections, torsion, curvature, etc., in reference [15] below. The pose and attitude response and the velocity and acceleration response offer a mathematical basis for the exploration of dynamics and the robotized exploration of the heading machine.

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References


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