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Article

Analytic Properties of the Sum $B_1(h,k)$

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Abstract: In this paper, with the help of the Hardy and Dedekind sums we will give many properties of the sum $B_1(h,k)$, which was defined by Cetin et al. Then we will give the connections of this sum with the other well-known finite sums such as the Dedekind sums, the Hardy sums, the Simsek sums $Y(h,k)$ and the sum $C_1(h,k)$. By using the Fibonacci numbers and two-term polynomial relation, we will also give a new property of the sum $B_1(h,k)$.

Keywords: Hardy sums; Dedekind sums; two-term polynomial relation; greatest integer function; $Y(h,k)$ sums; $C_1(h,k)$ sums; $B_1(h,k)$ sums

1. Introduction

The aim of this paper is to investigate and study the properties of the sum $B_1(h,k)$ which is defined by

$$B_1(h,k) = \sum_{j=1}^{k-1} \left[\frac{hj}{k}\right] (-1)^{\left[\frac{hj}{k}\right]} \tag{1}$$

In Equation (1), $[x]$ indicates the greatest integer function, which is also called floor function or integer value, that gives the largest integer less than or equal to x. Besides, the greatest integer function can also be defined by the help of the sawtooth function $((x))$, as follows:

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

Dedekind sums $DS(h,k)$, which are defined by Richard Dedekind in the nineteeth century, is given with the below equality:

$$DS(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{hj}{k}\right)\right) \left(\left(\frac{j}{k}\right)\right)$$

where h is an integer, and k is a positive integer. The basic introduction to the arithmetic properties of the Dedekind sum can be found in [1–5]. Dedekind defined these sums with the help of the famous Dedekind eta function. Although Dedekind sums arise in the transformation formula for the eta function, they can be defined independently of the eta function. Dedekind sums have many interesting properties but most important one is the reciprocity theorem: When h and k are coprime positive integers, the following reciprocity law holds [6]:

$$DS(h,k) + DS(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right) \tag{2}$$

The first proof of (2) was given by Richard Dedekind in 1892 [6]. After R. Dedekind, Apostol [7] and many authors have given many different proofs [1]. By using contour integration, in 1905, Hardy [8], gave another proof of the reciprocity theorem.

In that work, Hardy also gave some finite arithmetical sums which are called Hardy sums. These Hardy sums are also related to the Dedekind sums and have many useful properties.

We are ready to recall some of the Hardy sums which are needed in the further sections: If h and $k \in \mathbb{Z}$ with $k > 0$, the Hardy sums $HS_1(h,k)$ and $HS_2(h,k)$ are defined by

$$\begin{aligned} HS_1(h,k) &= \sum_{j \bmod k} (-1)^{j+1+[\frac{jh}{k}]}, \\ HS_2(h,k) &= \sum_{j \bmod k} (-1)^{j+[\frac{hj}{k}]} \left(\left(\frac{j}{k}\right)\right) \end{aligned} \tag{3}$$

We also note that some authors have called Hardy sums as Hardy-Berndt sums. For $HS_2(h,k)$, the below equality also holds true:

$$HS_2(h,k) = \frac{1}{k}\sum_{j=1}^{k-1} j(-1)^{j+\left[\frac{hj}{k}\right]} \tag{4}$$

when h and k are odd [9]. Further, following equations will be necessary in the next section [10]:

$$\begin{aligned} \sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}\left(\frac{j}{k}\right) &= HS_2(h,k) - \frac{1}{2}HS_1(h,k), \\ \sum_{j=1}^{h-1}(-1)^{j+\left[\frac{kj}{h}\right]}\left(\frac{j}{h}\right) &= HS_2(k,h) - \frac{1}{2}HS_1(k,h) \end{aligned} \tag{5}$$

Reciprocity law for the $HS_2(h,k)$ is given by the following theorem:

Theorem 1. *Let h and k be coprime positive integers. If h and k are odd, then*

$$HS_2(h,k) + HS_2(k,h) = \frac{1}{2} - \frac{1}{2hk} \tag{6}$$

and if $h+k$ is odd then

$$HS_2(h,k) = HS_2(k,h) = 0 \tag{7}$$

(cf. [9–12] and the references cited in each of these works).

The proof of the next reciprocity theorem was given by Hardy [8] and Berndt [13]:

Theorem 2. *Let h and k be coprime positive integers. Then*

$$HS_1(h,k) + HS_1(k,h) = 1 \quad \textit{if} \quad h+k \textit{ is odd} \tag{8}$$

In the light of Equation (8), Apostol [14] gave the below result:

Theorem 3. *If both h and k are odd and $(h,k) = 1$, then*

$$HS_1(h,k) = HS_1(k,h) = 0 \tag{9}$$

The following two theorems give the relations between the Hardy-Berndt sums and the Dedekind sums $DS(h,k)$:

Theorem 4. *[10] Let $(h,k)=1$. Then*

$$HS_1(h,k)=8DS(h,2k)+8DS(2h,k)-20DS(h,k),\ \textit{if}\ \ h+k \textit{ is odd} \tag{10}$$

$$HS_2(h,k)=-10DS(h,k)+4DS(2h,k)+4DS(h,2k),\ \textit{if}\ h+k \textit{ is even} \tag{11}$$

$$HS_1(h,k)=0,\ \textit{if}\ \ h+k \textit{ is even}$$

$$HS_2(h,k)=0,\ \textit{if}\ \ h+k \textit{ is odd}$$

Theorem 5. *[15] For $h+k$ is odd and $(h,k)=1$ with $k>0$, then we have*

$$HS_1(h,k)=4DS(h,k)-8DS(h+k,2k) \tag{12}$$

Next theorem gives infinite series representation of the Hardy-Berndt sums:

Theorem 6. *[9] Let h and k denote relatively prime integers with $k>0$. If $h+k$ is odd, then*

$$HS_1(h,k)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\tan\left(\frac{\pi h(2n-1)}{2k}\right) \tag{13}$$

and if h and k are odd, then

$$HS_2(h,k)=\frac{2}{\pi}\sum_{\substack{n=1\\ 2n-1\not\equiv 0 (mod\ k)}}^{\infty}\frac{1}{2n-1}\tan\left(\frac{\pi h(2n-1)}{2k}\right) \tag{14}$$

Now we will give the finite series representation of the Hardy-Berndt sums:

Theorem 7. *[9] Let h and k be coprime integers with $k>0$. If $h+k$ is odd, then*

$$HS_1(h,k)=\frac{1}{k}\sum_{j=1}^{k}\tan\left(\frac{\pi h(2j-1)}{2k}\right)\cot\left(\frac{\pi(2j-1)}{2k}\right) \tag{15}$$

and if h and k are odd, then

$$HS_2(h,k)=\frac{1}{2k}\sum_{\substack{j=1\\ j\neq\frac{k+1}{2}}}^{k}\tan\left(\frac{\pi h(2j-1)}{2k}\right)\cot\left(\frac{\pi(2j-1)}{2k}\right) \tag{16}$$

In [16], Simsek gave the following new sums: Let h is an integer and k is a positive integer with $(h,k)=1$. Then

$$Y(h,k)=4k\sum_{j\bmod k}(-1)^{j+[\frac{hj}{k}]}\left(\left(\frac{j}{k}\right)\right)$$

We observe that $Y(h,k)$ sums are also related to the Hardy sums $HS_2(h,k)$. That is

$$Y(h,k)=4kHS_2(h,k) \tag{17}$$

Reciprocity law for this sum was given by Simsek in [16] (p. 5, Theorem 4) as below:

$$hY(h,k)+kY(k,h)=2hk-2 \tag{18}$$

Simsek gave two different proofs of this reciprocity law. Another proof was also given in [17]. $Y(h,k)$ sums are also related to the three term polynomial relations, [17–20]

In this paper we study the Hardy sums, the Simsek sums $Y(h,k)$ and the Dedekind sums $DS(h,k)$ which are related to the symmetric pairs [21], and the Fibonacci numbers. Before starting our results, we need some properties of the Fibonacci numbers which are given as follows: The Fibonacci numbers are defined by means of the following generating function [22]:

$$F(x) = \frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n \tag{19}$$

One can easily derive the following recurrence relation from (19):

$$F_{n+1} = F_n + F_{n-1}$$

From (19), we also easily compute the first few Fibonacci numbers as follows: $0,1,1,2,3,5,8,13,21,\cdots$ In [21], Meyer studied a special case of the Dedekind sums. In that paper, Meyer investigated the pairs of integers $\{h,k\}$ for which $DS(h,k) = DS(k,h)$. Meyer defined that $\{h,k\}$ is a symmetric pair if this property holds and he showed that $\{h,k\}$ is a symmetric pair if and only if $h = F_{2n+1}$ and $k = F_{2n+3}$ for $n \in \mathbb{N}$ where F_m is the $m-$th Fibonacci number. In [21], Meyer proved the following theorem:

Theorem 8. *If $(h,k) = 1$ and $\{h,k\}$ is a symmetric pair, then $DS(h,k) = 0$.*

In [17], Cetin et al. defined the sum $C_1(h,k)$ as follows:

$$C_1(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{hj}{k} \right) \right) (-1)^{j+\left[\frac{hj}{k}\right]} \tag{20}$$

where h,k are positive integers with $(h,k) = 1$.

For the odd values of k, the below theorem is given in [23]:

Theorem 9. *If $(h,k) = 1$, h and k are odd integers with $k > 0$, then we have*

$$C_1(h,k) = \frac{1}{2} - \frac{1}{2k} \tag{21}$$

In [17], Cetin et al. also defined the sum $Y_{n-1}(a_1, a_1, \cdots; a_n)$ as follows:

$$Y_{n-1}(a_1, a_1, \cdots; a_n) = \sum_{j=1}^{a_n - 1} (2j-1)(-1)^{j+\left[\frac{a_1 j}{a_n}\right]+\cdots+\left[\frac{a_{n-1} j}{a_n}\right]} \left[\frac{a_1 j}{a_n}\right] \cdots \left[\frac{a_{n-1} j}{a_n}\right]$$

where $a_1, a_2, \cdots, a_n$ are pairwise positive integers.

Two-term polynomial relation has an important role in the next section. So we need to remind it in the following theorem:

Theorem 10. *If a, b, and c are pairwise coprime positive integers, then*

$$(u-1)\sum_{x=1}^{a-1} u^{x-1} v^{\left[\frac{bx}{a}\right]} + (v-1)\sum_{y=1}^{b-1} v^{y-1} u^{\left[\frac{ay}{b}\right]} = u^{a-1}v^{b-1} - 1 \tag{22}$$

Equation (22) is originally due to Berndt and Dieter [24].

Next corollary, which was given in [23] (Corollary 7), will be useful in the next section.

Corollary 11. *Let h and k be positive integers and $\{h,k\}$ is a symmetric pair. If $(h,k)=1$, $h=F_{6n-1}$ and $k=F_{6n+1}$ with n is a natural number, where F_m is the $m-$th Fibonacci number, then*

$$HS_2(h,k)+HS_2(k,h)=\frac{1}{2}\left(\frac{h}{k}+\frac{k}{h}-2\right) \tag{23}$$

and

$$hY(h,k)+kY(k,h)=2h^2+2k^2-4hk$$

2. The Sum $B_1(h,k)$ and Its Properties

In [17] we defined a new sum as follows:

$$B_1(h,k)=\sum_{j=1}^{k-1}(-1)^{j+\left[\frac{hj}{k}\right]}\left[\frac{hj}{k}\right]$$

which $(h,k)=1$ and $k>0$. The sum $B_1(h,k)$ has the following arithmetic property:

$$B_1(-h,-k)=B_1(h,k) \tag{24}$$

To show that the last equality holds true, we use the definition of the $[.]$ function, and the fact that $((-x))=-((x))$. If we also consider the equation

$$(-1)^{[x]}=2((x))-4\left(\left(\frac{x}{2}\right)\right) \tag{25}$$

when x is not an integer, then we get the Equation (24). The Equation (25) is originally reduced from [15].

Now we will give a relation between the sums $B_1(h,k)$ and $HS_1(h,k)$.

Theorem 12. *If $h+k$ is odd, $k>0$, and $(h,k)=1$, then*

$$B_1(h,k)=\frac{1}{2}(1-h)HS_1(h,k) \tag{26}$$

Proof. We consider the two-term relation which is given in Equation (22). If we take the partial derivative of Equation (22) with respect to u, and substitute $u=v=-1$, then we have

$$\sum_{x=1}^{h-1}(-1)^{x+\left[\frac{kx}{h}\right]}-2\sum_{x=1}^{h-1}x(-1)^{x+\left[\frac{kx}{h}\right]}-2\sum_{y=1}^{k-1}\left[\frac{hy}{k}\right](-1)^{y+\left[\frac{hy}{k}\right]}=(h-1)(-1)^{h+k-1}$$

After some elementary calculations and by using Equation (5), we get

$$-HS_1(h,k)-2h\left(HS_2(k,h)-\frac{1}{2}HS_1(k,h)\right)-2B_1(h,k)=h-1$$

We know from Equation (7) that $HS_2(h,k)=HS_2(k,h)=0$. If we use this fact, then we have

$$-2B_1(h,k)=(h-1)(1-HS_1(k,h)) \tag{27}$$

From Equation (8) we can write

$$HS_1(h,k)=1-HS_1(k,h) \tag{28}$$

If we put Equation (28) into Equation (27), then we have the desired result. □

In the next theorem, we will give the relation between the sums $B_1(h,k)$ and the Hardy-Berndt sums $HS_2(h,k)$:

Theorem 13. *If h and k are relatively prime odd numbers with $k > 0$, then*

$$B_1(h,k) = hHS_2(h,k) + \frac{1}{2k} - \frac{1}{2} \tag{29}$$

Proof. From the definition of the sum $B_1(h,k)$ after basic calculations, we get

$$B_1(h,k) = hHS_2(h,k) - C_1(h,k) + \frac{1}{2}HS_1(h,k)$$

From [23], we know that Equation (21) holds true. If we also use Equation (9), then we get the desired result. □

In the next theorem we will give the relation between the sums $B_1(h,k)$ and the sums $Y(h,k)$:

Theorem 14. *If h and k are relatively prime odd numbers with $k > 0$, then*

$$B_1(h,k) = \frac{h}{4k}Y(h,k) + \frac{1}{2k} - \frac{1}{2}$$

Proof. It can be directly obtained from Theorem 13 and Equation (17). □

Now, we will give a relation for the sums $B_1(h,k)$ as follows:

Theorem 15. *If $h+k$ is an odd positive integer and $(h,k) = 1$, then*

$$(k-1)B_1(h,k) + (h-1)B_1(k,h) = -\frac{1}{2}(k-1)(h-1) \tag{30}$$

Proof. From Theorem 12, we showed that Equation (26) holds. Similarly, when $h+k$ is an odd positive integer with $h > 0$, we can also write

$$B_1(k,h) = \frac{1}{2}(1-k)HS_1(h,k) \tag{31}$$

So first, if we multiply Equation (26) by k and Equation (31) by h respectively, then if we sum the two equations side by side and use (8), we get the following identity:

$$kB_1(h,k) + hB_1(k,h) = \frac{1}{2}\left[kHS_1(h,k) + hHS_1(k,h) - hk\right] \tag{32}$$

Now we will consider the sum $B_1(h,k) + B_1(k,h)$. From (26) and (31), we can see that

$$B_1(h,k) + B_1(k,h) = \frac{1}{2}\left[1 - (hHS_1(h,k) + kHS_1(k,h))\right]$$

So from this last equation, we can write

$$hHS_1(h,k) + kHS_1(k,h) = 1 - 2B_1(h,k) - 2B_1(k,h) \tag{33}$$

Now we will use the Equation (8). First, we multiply Equation (8) by h, and we multiply Equation (8) by k. Then if add these two equations side by side and if we use the Equation (32), we get the desired result. □

In the below theorem, we will give the reciprocity theorem for the sums $B_1(h,k)$:

Theorem 16. *If h and k are odd positive integers with $(h,k)=1$, then*

$$kB_1(h,k)+hB_1(k,h)=\frac{1}{2}(h-1)(k-1)$$

Proof. From Theorem 13, we know that Equation (29) holds. Similarly, we can also write

$$B_1(k,h)=kHS_2(k,h)+\frac{1}{2h}-\frac{1}{2} \tag{34}$$

If we multiply Equation (29) by k, and Equation (34) by h respectively, and add these equations side by side, we get

$$kB_1(h,k)+hB_1(k,h)=hk\left(HS_2(h,k)+HS_2(k,h)\right)+1-\frac{k}{2}-\frac{h}{2} \tag{35}$$

In this last equation if we use Equation (6), then we get the desired result. □

Theorem 17. *If $h+k$ is an odd positive integer and $(h,k)=1$, then*

$$Y_1(h,k)+Y_1(k,h)=2(k-1)B_1(h,k)+2(h-1)B_1(k,h)$$

Proof. In [17], if we take $n=2$ in Theorem 4, and use it with Equation (30) we have desired result. □

Theorem 18. *If $h+k$ is an odd positive integer and $(h,k)=1$, then*

$$Y_1(h,k)+Y_1(k,h)=2kB_1(h,k)+2hB_1(k,h) \tag{36}$$

Proof. In [17], if we take $n=2$ in Theorem 4, and use it with Equation (36) we have desired result. □

In the following three theorems, we will give the relations between the sums $B_1(h,k)$ and the Dedekind sums $DS(h,k)$:

Theorem 19. *Let $h+k$ is odd, $(h,k)=1$ with $k>0$. Then*

$$B_1(h,k)=(1-h)\left(4DS(h,2k)+4DS(2h,k)-10DS(h,k)\right)$$

Proof. It can be directly obtained from Theorem 12 and Equation (10). □

Theorem 20. *Let $h+k$ is odd, $(h,k)=1$ with $k>0$. Then*

$$B_1(h,k)=2(1-h)\left(DS(h,k)-2DS(h+k,2k)\right).$$

Proof. It can be directly obtained from Theorem 12 and Equation (12). □

Theorem 21. *If h and k are relatively prime odd numbers with $k>0$, then*

$$B_1(h,k)=-10hDS(h,k)+4hDS(2h,k)+4hDS(h,2k)+\frac{1}{2k}-\frac{1}{2}$$

Proof. It can be directly obtained from Equation (29) and Equation (11). □

Now we will give two different infinite series representations of the sums $B_1(h,k)$:

Theorem 22. *Let h and k denote relatively prime integers with $k > 0$. If $h + k$ is odd, then*

$$B_1(h,k) = \frac{2(1-h)}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \tan\left(\frac{\pi h(2n-1)}{2k}\right)$$

Proof. It can be directly obtained from Theorem 12 and Equation (13). □

Theorem 23. *Let h and k denote relatively prime integers with $k > 0$. If h and k are odd, then*

$$B_1(h,k) = \frac{2h}{\pi} \sum_{\substack{n=1 \\ 2n-1 \not\equiv mod k}}^{\infty} \frac{1}{2n-1} \tan\left(\frac{\pi h(2n-1)}{2k}\right) + \frac{1}{2k} - \frac{1}{2}$$

Proof. It can be directly obtained from Theorem 13 and Equation (14). □

Similarly, we give two different finite series representations of the sums $B_1(h,k)$ below:

Theorem 24. *Let h and k denote relatively prime integers with $k > 0$. If $h + k$ is odd, then*

$$B_1(h,k) = \frac{1-h}{2k} \sum_{j=1}^{k} \tan\left(\frac{\pi h(2j-1)}{2k}\right) \cot\left(\frac{\pi(2j-1)}{2k}\right)$$

Proof. It can be directly obtained from Theorem 12 and Equation (15). □

Theorem 25. *Let h and k be coprime integers with $k > 0$. If h and k are odd, then*

$$B_1(h,k) = \frac{h}{2k} \sum_{\substack{j=1 \\ j \neq \frac{k+1}{2}}}^{k} \tan\left(\frac{\pi h(2j-1)}{2k}\right) \cot\left(\frac{\pi(2j-1)}{2k}\right) + \frac{1}{2k} - \frac{1}{2}$$

Proof. It can be directly obtained from Theorem 13 and Equation (16). □

Now, we will give the relation between the sums $B_1(h,k)$ and the Fibonacci numbers.

Theorem 26. *Let h and k be positive integers and $\{h,k\}$ is a symmetric pair. If $(h,k) = 1$, $h = F_{6n-1}$ and $k = F_{6n+1}$ with n is a natural number, where F_m is the $m-$th Fibonacci number, then*

$$kB_1(h,k) + hB_1(k,h) = \frac{h^2 - h - k + k^2}{2} - hk + 1$$

Proof. It can be obtained similarly with Theorem 16's proof. From Theorem 16's proof, we know that Equation (35) holds. If we also use the Equation (23) into Equation (35), then we get desired result. □

3. Conclusions

In this paper, we gave some properties of the sum $B_1(h,k)$, with the help of the Fibonacci numbers. We thereby built a connection between analysis and number theory. The Dedekind sums were studied around the 1900s and the Dedekind sums for higher-dimensions were studied around the 1950s. These sums have been studied prominently in so many different areas such as enumerating lattice points in polytopes and special values of L-functions, Modular forms, arithmetical functions, the three-term polynomial relations, Theta functions and some other special numbers and polynomials. Dedekind sums and Hardy sums have many useful applications in mathematics. Therefore, the connections between the sum $B_1(h,k)$ and the other well-known sums, can be useful in many areas like physics and engineering.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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