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A GENERALIZED APPROACH TO COUPLED NONLINEAR VIBRATIONS OF CONTINUOUS SYSTEMS

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Abstract – A mathematical model covering many practical vibration problems of continuous systems has been proposed. The equations of motion consist of two nonlinearly coupled partial differential equations. The quadratic and cubic nonlinearities as well as the linear part of the equations are represented by arbitrary operators. A perturbation approach (method of multiple time scales) has been applied directly to the partial differential equations. The responses as well as the amplitude and phase modulation equations are found for the case of primary resonances of the external excitation and one-to-one internal resonances between the natural frequencies of the equations. The coefficients of the amplitude and phase modulation equations are calculated in their most general form. Results are then applied to a nonlinear cable vibration problem having small sag-to-span ratios.

1. INTRODUCTION

A generalized system of nonlinear partial differential equations is treated. This general model covers many practical vibration problems of continuous systems such as beam, string and cable vibrations. The quadratic and cubic nonlinearities, which occur frequently in vibrations of continuous systems, are inserted into the equations in the form of arbitrary spatial operators. Also the linear part of the equation is left arbitrary. With the aid of these arbitrary operators, the equations have the advantage of modeling a wide range of nonlinear problems. A perturbation method (method of multiple time scales) is applied directly to the partial differential equations. This directly applying perturbations to the partial differential system (direct perturbation method) has, at least, the advantage of yielding spatial corrections to the linear normal modes[1-7]. The responses and the amplitude and phase modulation equations are presented for this general system of equations. Because of the arbitrariness, the solution procedure yields an algorithm for solutions of many practical problems. Using this algorithm, solutions are presented for a specific problem of nonlinear cable vibrations.

The dimensionless equations of motion, considered, are written as follows

$$\ddot{w}_1 + L_1(w_1) + \hat{\mu}_1\dot{w}_1 + Q_1(w_1,w_1) + Q_2(w_2,w_2) + Q_3(w_1,w_2) + C_1(w_1,w_1,w_1) + C_2(w_1,w_1,w_2) + C_3(w_1,w_2,w_2) + \hat{F}\cos\Omega t = 0 \tag{1}$$

$$\ddot{w}_2 + L_2(w_2) + \hat{\mu}_2\dot{w}_2 + Q_4(w_1,w_1) + Q_5(w_2,w_2) + Q_6(w_1,w_2) + C_4(w_2,w_2,w_2) + C_5(w_2,w_2,w_1) + C_6(w_2,w_1,w_1) = 0 \tag{2}$$

where $(\dot{\ })$ denotes time differentiation. $\hat{\mu}_1$ and $\hat{\mu}_2$ are the damping coefficients, and $\hat{F}$ and Ω are the external excitation amplitude and frequency respectively. L_i are the linear spatial differential and/or integral operators. Q_i and C_i are the arbitrary spatial differential and/or integral operators producing quadratic and cubic nonlinearities respectively. The quadratic and cubic operators possess the property of being multilinear such that

$$Q(c_1w_1+c_2w_2\ ,\ c_3w_3+c_4w_4) \ = c_1\,c_3\ Q(w_1\,,w_3\ )+c_1\,c_4\ Q(w_1\,,w_4\ )+c_2\,c_3\ Q(w_2\,,w_3)+c_2\,c_4\ Q(w_2\,,w_4)$$

$$\{\ Q(w_1\,,w_2) \neq Q(w_2\,,w_1) \text{ in general } \} \tag{3}$$

where c_i are some arbitrary constants. Instead of $Q(w)$, a more common notation, the reason that we use $Q(w,w)$ is that it gives the flexibility to represent the contributions of nonlinearity at higher orders of approximation correctly.

The associated boundary conditions for Eqs. (1) and (2) are assumed to be linear and homogenous and free from time derivatives.

There have been attempts of generalizing the direct perturbation method for a wide range of nonlinear vibration problems very recently. For a single arbitrary nonlinear partial differential equation, Pakdemirli[3] presented results for primary resonance case. For finite mode truncations, he[3] also compared results of direct approach method with discretization methods. The analysis is generalized to infinite modes by Pakdemirli and Boyacı[4]. Using the same model of single differential equation, Pakdemirli and Boyacı[8] treated the subharmonic, superharmonic and combination resonance cases in detail. Nayfeh *et al.*[5] compared results of direct perturbation method with those of discretization methods by considering several examples from continuous systems. Solutions are also presented for an arbitrary cubic nonlinear spatial and temporal operator using the nonlinear normal mode concept. Two different versions of method of multiple scales are compared using general models, one with an arbitrary cubic operator and the other arbitrary quadratic and cubic operators[9]. Finally a general odd-nonlinearity model was treated by Pakdemirli, Boyacı and Yılmaz[10].

2. DIRECT PERTURBATION METHOD

In this section, we apply the method of multiple scales[11,12] directly to the partial differential equations (1) and (2). Contrary to the harmonic balance or discretization methods, this direct method requires no initial assumptions for the form of solutions. Due to cubic nonlinearities, we assume expansions up to third order for w_1 and w_2 of the form

$$w_1\,(x,t;\varepsilon) = \varepsilon\ w_{11}\,(x\,,T_0,T_2) + \varepsilon^2 w_{12}\,(x\,,T_0,T_2) + \varepsilon^3\, w_{13}(x\,,T_0,T_2) + \ldots \tag{4}$$

$$w_2\,(x,t;\varepsilon) = \varepsilon\ w_{21}\,(x\,,T_0,T_2) + \varepsilon^2 w_{22}\,(x\,,T_0,T_2) + \varepsilon^3\, w_{23}(x\,,T_0,T_2) + \ldots \tag{5}$$

where ε is a small dimensionless measure of the amplitude of w_1 and w_2, used as a book-keeping device, w_{1n} and w_{2n} are $O(1)$. $T_0 = t$ is a fast-time scale characterizing changes occuring at the external excitation and natural frequencies; $T_2 = \varepsilon^2 t$ is a slow-time scale. The later analysis shows that there is no $T_1 = \varepsilon t$ dependence, hence we omit it from the beginning. Assuming a weakly nonlinear system, damping and excitation coefficients are ordered such that their effects appear at the third order,

$$\hat{\mu}_{1,2} = \varepsilon^2 \mu_{1,2}\,, \qquad\qquad \hat{F} = \varepsilon^3 F \tag{6}$$

In terms of the T_n, the time derivatives become

$$(\dot{\ }) = D_0 + \varepsilon^2 D_2 + \ldots \tag{7}$$

$$(\ddot{\ }) = D_0 + 2\varepsilon^2 D_0 D_2 + \ldots \tag{8}$$

where $D_n = \dfrac{\partial}{\partial T_n}$. Substituting Eqs. (4)-(8) into Eqs. (1) and (2) and seperating the equations at each order of ε ; we finally obtain

Order ε

$$D_0^2 w_{11} + L_1(w_{11}) = 0 \tag{9}$$

$$D_0^2 w_{21} + L_2(w_{21}) = 0 \tag{10}$$

Order ε^2

$$D_0^2 w_{12} + L_1(w_{12}) = -Q_1(w_{11},w_{11}) - Q_2(w_{21},w_{21}) - Q_3(w_{11},w_{21}) \tag{11}$$

$$D_0^2 w_{22} + L_2(w_{22}) = -Q_4(w_{11},w_{11}) - Q_5(w_{21},w_{21}) - Q_6(w_{11},w_{21}) \tag{12}$$

Order ε^3

$$\begin{aligned} D_0^2 w_{13} + L_1(w_{13}) = & -2D_0D_2 w_{11} - \mu_1 D_0 w_{11} - Q_1(w_{11},w_{12}) - Q_1(w_{12},w_{11}) - Q_2(w_{21},w_{22}) \\ & - Q_2(w_{22},w_{21}) - Q_3(w_{11},w_{22}) - Q_3(w_{12},w_{21}) - C_1(w_{11},w_{11},w_{11}) \\ & - C_2(w_{11},w_{11},w_{21}) - C_3(w_{11},w_{21},w_{21}) - F\cos \Omega T_0 \end{aligned} \tag{13}$$

$$\begin{aligned} D_0^2 w_{23} + L_2(w_{23}) = & -2D_0D_2 w_{21} - \mu_2 D_0 w_{21} - Q_4(w_{11},w_{12}) - Q_4(w_{12},w_{11}) - Q_5(w_{21},w_{22}) \\ & - Q_5(w_{22},w_{21}) - Q_6(w_{11},w_{22}) - Q_6(w_{12},w_{21}) - C_4(w_{21},w_{21},w_{21}) \\ & - C_5(w_{21},w_{21},w_{11}) - C_6(w_{21},w_{11},w_{11}) \end{aligned} \tag{14}$$

3. SOLUTIONS

In this section, we search for general solutions to the above equations at each order of approximation. At order ε, the solution can be written as

$$w_{11} = (A(T_2)e^{i\omega_1 T_0} + cc)Y_1(x) \tag{15}$$

$$w_{21} = (B(T_2)e^{i\omega_2 T_0} + cc)Y_2(x) \tag{16}$$

where cc stands for the complex conjugate of the preceeding terms. $Y_1(x)$ and $Y_2(x)$ satisfy the following equations

$$L_1(Y_1) - \omega_1^2 Y_1 = 0 \tag{17}$$

$$L_2(Y_2) - \omega_2^2 Y_2 = 0 \tag{18}$$

where ω_1 and ω_2 are the eigenvalues and Y_1 and Y_2 are the corresponding eigenfunctions. Note that there are infinite number of eigenvalues for continuous systems. In the presence of damping, and for weakly nonlinear systems, the mode that is directly excited by an external excitation or indirectly excited by an internal resonance would survive and all other modes would decay with time [1]. In the following analysis, we will assume that ω_1 is directly excited by an external excitation and ω_2 is indirectly excited by a one-to-one internal resonance with ω_1. Note that the boundary conditions for Eqs. (17) and (18) are homogenous and linear, as assumed previously.

At order ε^2, the right hand sides of Eqs. (11) and (12) are known functions and we assume suitable solutions for w_{12} and w_{22} of the form

$$w_{12} = (A^2 e^{2i\omega_1 T_0} + cc)\xi_1(x) + 2A\bar{A}\,\xi_2(x) + (B^2 e^{2i\omega_2 T_0} + cc)\xi_3(x) + 2B\bar{B}\,\xi_4(x)$$

$$+ (ABe^{i(\omega_1+\omega_2)T_0} + cc)\xi_5(x) + (A\bar{B}\,e^{i(\omega_1-\omega_2)T_0} + cc)\xi_6(x) \tag{19}$$

$$w_{22} = (A^2 e^{2i\omega_1 T_0} + cc)\xi_7(x) + 2A\bar{A}\,\xi_8(x) + (B^2 e^{2i\omega_2 T_0} + cc)\xi_9(x) + 2B\bar{B}\,\xi_{10}(x)$$

$$+ (ABe^{i(\omega_1+\omega_2)T_0} + cc)\xi_{11}(x) + (A\bar{B}\,e^{i(\omega_1-\omega_2)T_0} + cc)\xi_{12}(x) \tag{20}$$

Substituting Eqs. (15), (16), (19) and (20) into Eqs. (11) and (12), we find the equations determining the unknown functions $\xi_i(x)$

$$L_1(\xi_1) - 4\omega_1^2\xi_1 = -Q_1(Y_1, Y_1) \tag{21}$$

$$L_1(\xi_2) = -Q_1(Y_1, Y_1) \tag{22}$$

$$L_1(\xi_3) - 4\omega_2^2\xi_3 = -Q_2(Y_2, Y_2) \tag{23}$$

$$L_1(\xi_4) = -Q_2(Y_2, Y_2) \tag{24}$$

$$L_1(\xi_5) - (\omega_1 + \omega_2)^2\xi_5 = -Q_3(Y_1, Y_2) \tag{25}$$

$$L_1(\xi_6) - (\omega_1 - \omega_2)^2\xi_6 = -Q_3(Y_1, Y_2) \tag{26}$$

$$L_2(\xi_7) - 4\omega_1^2\xi_7 = -Q_4(Y_1, Y_1) \tag{27}$$

$$L_2(\xi_8) = -Q_4(Y_1, Y_1) \tag{28}$$

$$L_2(\xi_9) - 4\omega_2^2\xi_9 = -Q_5(Y_2, Y_2) \tag{29}$$

$$L_2(\xi_{10}) = -Q_5(Y_2, Y_2) \tag{30}$$

$$L_2(\xi_{11}) - (\omega_1 + \omega_2)^2\xi_{11} = -Q_6(Y_1, Y_2) \tag{31}$$

$$L_2(\xi_{12}) - (\omega_1 - \omega_2)^2\xi_{12} = -Q_6(Y_1, Y_2) \tag{32}$$

At order ε^3, we assume that $\Omega \approx \omega_1$ and $\omega_2 \approx \omega_1$, that is

$$\Omega = \omega_1 + \varepsilon^2\sigma \tag{33}$$

$$\omega_1 = \omega_2 + \varepsilon^2\rho \tag{34}$$

where σ and ρ are detuning parameters of $O(1)$. Since the homogenous part of Eqs. (13) and (14) have a nontrivial solution, the inhomogenous Eqs. (13) and (14) have a solution, only if a solvability condition is satisfied[7]. To find this condition, we express their solution in the form

$$w_{13} = \psi_1(x,T_2)e^{i\omega_1 T_0} + cc + W_1(x,T_0,T_2) \tag{35}$$

$$w_{23} = \psi_2(x,T_2)e^{i\omega_2 T_0} + cc + W_2(x,T_0,T_2) \tag{36}$$

where W_1 and W_2 are governed by Eqs. (13) and (14) with the terms proportional to $exp(i\omega_1 T_0)$ and $exp(i\omega_2 T_0)$ being deleted. Hence W_1 and W_2 are unique and free of secular and small-divisor terms. Substituting Eqs. (35), (36), (19), (20), (15) and (16) into Eqs. (13) and (14), using Eqs. (33) and (34), and equating the coefficients of $exp(i\omega_1 T_0)$ and $exp(i\omega_2 T_0)$ on both sides of each equation results in

$$\begin{aligned}
L_1(\psi_1) - \omega_1{}^2\psi_1 = &- i\omega_1(2A' + \mu_1 A)Y_1 - A^2\overline{A}\{Q_1(Y_1,\xi_1) + Q_1(\xi_1,Y_1) + Q_3(Y_1,\xi_7) \\
&+ 2[Q_1(Y_1,\xi_2) + Q_1(\xi_2,Y_1) + Q_3(Y_1,\xi_8)] + 3C_1(Y_1,Y_1,Y_1)\} \\
&- AB\overline{B}\{2[Q_1(Y_1,\xi_4) + Q_1(\xi_4,Y_1) + Q_3(Y_1,\xi_{10})] + Q_2(Y_2,\xi_{11}) \\
&+ Q_2(\xi_{11},Y_2) + Q_3(\xi_5,Y_2) + Q_2(Y_2,\xi_{12}) + Q_2(\xi_{12},Y_2) + Q_3(\xi_6,Y_2) \\
&+ 2C_3(Y_1,Y_2,Y_2)\} - B^2\overline{A}e^{-2i\rho T_2}\{Q_1(Y_1,\xi_3) + Q_1(\xi_3,Y_1) + Q_3(Y_1,\xi_9) \\
&+ Q_2(Y_2,\xi_{12}) + Q_2(\xi_{12},Y_2) + Q_3(\xi_6,Y_2) + C_3(Y_1,Y_2,Y_2)\} \\
&- A\overline{A}B\,e^{-i\rho T_2}\{Q_1(Y_1,\xi_5) + Q_1(\xi_5,Y_1) + Q_3(Y_1,\xi_{11}) + Q_1(Y_1,\xi_6) \\
&+ Q_1(\xi_6,Y_1) + Q_3(Y_1,\xi_{12}) + 2[Q_2(Y_2,\xi_8) + Q_2(\xi_8,Y_2) + Q_3(\xi_2,Y_2) \\
&+ C_2(Y_1,Y_1,Y_2)]\} - A^2\overline{B}e^{i\rho T_2}\{Q_1(Y_1,\xi_6) + Q_1(\xi_6,Y_1) + Q_3(Y_1,\xi_{12}) \\
&+ Q_2(Y_2,\xi_7) + Q_2(\xi_7,Y_2) + Q_3(\xi_1,Y_2) + C_2(Y_1,Y_1,Y_2)\} \\
&- B^2\overline{B}e^{-i\rho T_2}\{Q_2(Y_2,\xi_9) + Q_2(\xi_9,Y_2) + Q_3(\xi_3,Y_2) + 2[Q_2(Y_2,\xi_{10}) \\
&+ Q_2(\xi_{10},Y_2) + Q_3(\xi_4,Y_2)]\} - \frac{1}{2}F e^{i\sigma T_2}
\end{aligned} \tag{37}$$

$$\begin{aligned}
L_2(\psi_2) - \omega_2{}^2\psi_2 = &- i\omega_2(2B' + \mu_2 B)Y_2 - B^2\overline{B}\{Q_5(Y_2,\xi_9) + Q_5(\xi_9,Y_2) + Q_6(\xi_3,Y_2) \\
&+ 2[Q_5(Y_2,\xi_{10}) + Q_5(\xi_{10},Y_2) + Q_6(\xi_4,Y_2)] + 3C_4(Y_2,Y_2,Y_2)\} \\
&- A\overline{A}B\{Q_4(Y_1,\xi_5) + Q_4(\xi_5,Y_1) + Q_6(Y_1,\xi_{11}) + Q_4(Y_1,\xi_6) \\
&+ Q_4(\xi_6,Y_1) + Q_6(Y_1,\xi_{12}) + 2[Q_5(Y_2,\xi_8) + Q_5(\xi_8,Y_2) + Q_6(\xi_2,Y_2) \\
&+ C_6(Y_2,Y_1,Y_1)]\} - A^2\overline{B}e^{2i\rho T_2}\{Q_4(Y_1,\xi_6) + Q_4(\xi_6,Y_1) + Q_6(Y_1,\xi_{12}) \\
&+ Q_5(Y_2,\xi_7) + Q_5(\xi_7,Y_2) + Q_6(\xi_1,Y_2) + C_6(Y_2,Y_1,Y_1)\} \\
&- AB\overline{B}e^{i\rho T_2}\{2[Q_4(Y_1,\xi_4) + Q_4(\xi_4,Y_1) + Q_6(Y_1,\xi_{10})] + Q_5(Y_2,\xi_{11}) \\
&+ Q_5(\xi_{11},Y_2) + Q_6(\xi_5,Y_2) + Q_5(Y_2,\xi_{12}) + Q_5(\xi_{12},Y_2) + Q_6(\xi_6,Y_2) \\
&+ 2C_5(Y_2,Y_2,Y_1)\} - B^2\overline{A}e^{-i\rho T_2}\{Q_4(Y_1,\xi_3) + Q_4(\xi_3,Y_1) + Q_6(Y_1,\xi_9) \\
&+ Q_5(Y_2,\xi_{12}) + Q_5(\xi_{12},Y_2) + Q_6(\xi_6,Y_2) + C_5(Y_2,Y_2,Y_1)\} \\
&- A^2\overline{A}e^{i\rho T_2}\{Q_4(Y_1,\xi_1) + Q_4(\xi_1,Y_1) + Q_6(Y_1,\xi_7) + 2[Q_4(Y_1,\xi_2) \\
&+ Q_4(\xi_2,Y_1) + Q_6(Y_1,\xi_8)]\}
\end{aligned} \tag{38}$$

We now assume that the linear operators L_i with the associated boundary conditions are self-adjoint. The solvability conditions for Eqs. (37) and (38) are

$$i\omega_1(2A' + \mu_1 A) + \alpha_1 A^2\overline{A} + \alpha_2 AB\overline{B} + \alpha_3 B^2\overline{A}e^{-2i\rho T_2} + \alpha_4 A\overline{A}B\,e^{-i\rho T_2} + \alpha_5 A^2\overline{B}e^{i\rho T_2}$$

$$+ \alpha_6 B^2 \overline{B}\, e^{-i\rho T_2} + \frac{1}{2} f e^{i\sigma T_2} = 0 \tag{39}$$

$$i\omega_2 (2B' + \mu_2 B) + \alpha_7 B^2 \overline{B} + \alpha_8 A \overline{A} B + \alpha_9 A^2 \overline{B}\, e^{2i\rho T_2} + \alpha_{10} AB \overline{B}\, e^{i\rho T_2} + \alpha_{11} B^2 \overline{A}\, e^{-i\rho T_2} + \alpha_{12} A^2 \overline{A}\, e^{i\rho T_2} = 0 \tag{40}$$

where coefficients α_i are defined as follows

$$\alpha_1 = \int_D Y_1 \{ Q_1 (Y_1 , \xi_1) + Q_1 (\xi_1 , Y_1) + Q_3 (Y_1 , \xi_7) + 2[Q_1 (Y_1 , \xi_2) + Q_1 (\xi_2 , Y_1) + Q_3 (Y_1 , \xi_8)] + 3\, C_1 (Y_1 , Y_1 , Y_1) \}\, dx \tag{41}$$

$$\alpha_2 = \int_D Y_1 \{2[Q_1 (Y_1 , \xi_4) + Q_1 (\xi_4 , Y_1) + Q_3 (Y_1 , \xi_{10})] + Q_2 (Y_2 , \xi_{11}) + Q_2 (\xi_{11} , Y_2) + Q_3 (\xi_5 , Y_2) + Q_2 (Y_2 , \xi_{12}) + Q_2 (\xi_{12} , Y_2) + Q_3 (\xi_6 , Y_2) + 2\, C_3 (Y_1 , Y_2 , Y_2) \}\, dx \tag{42}$$

$$\alpha_3 = \int_D Y_1 \{ Q_1 (Y_1 , \xi_3) + Q_1 (\xi_3 , Y_1) + Q_3 (Y_1 , \xi_9) + Q_2 (Y_2 , \xi_{12}) + Q_2 (\xi_{12} , Y_2) + Q_3 (\xi_6 , Y_2) + C_3 (Y_1 , Y_2 , Y_2) \}\, dx \tag{43}$$

$$\alpha_4 = \int_D Y_1 \{ Q_1 (Y_1 , \xi_5) + Q_1 (\xi_5 , Y_1) + Q_3 (Y_1 , \xi_{11}) + Q_1 (Y_1 , \xi_6) + Q_1 (\xi_6 , Y_1) + Q_3 (Y_1 , \xi_{12}) + 2[Q_2 (Y_2 , \xi_8) + Q_2 (\xi_8 , Y_2) + Q_3 (\xi_2 , Y_2) + C_2 (Y_1 , Y_1 , Y_2)] \}\, dx \tag{44}$$

$$\alpha_5 = \int_D Y_1 \{ Q_1 (Y_1 , \xi_6) + Q_1 (\xi_6 , Y_1) + Q_3 (Y_1 , \xi_{12}) + Q_2 (Y_2 , \xi_7) + Q_2 (\xi_7 , Y_2) + Q_3 (\xi_1 , Y_2) + C_2 (Y_1 , Y_1 , Y_2) \}\, dx \tag{45}$$

$$\alpha_6 = \int_D Y_1 \{Q_2 (Y_2 , \xi_9) + Q_2 (\xi_9 , Y_2) + Q_3 (\xi_3 , Y_2) + 2[Q_2 (Y_2 , \xi_{10}) + Q_2 (\xi_{10} , Y_2) + Q_3 (\xi_4 , Y_2)]\}\, dx \tag{46}$$

$$\alpha_7 = \int_D Y_2 \{ Q_5 (Y_2 , \xi_9) + Q_5 (\xi_9 , Y_2) + Q_6 (\xi_3 , Y_2) + 2\, [Q_5 (Y_2 , \xi_{10}) + Q_5 (\xi_{10} , Y_2) + Q_6 (\xi_4 , Y_2)] + 3\, C_4 (Y_2 , Y_2 , Y_2) \}\, dx \tag{47}$$

$$\alpha_8 = \int_D Y_2 \{ Q_4 (Y_1 , \xi_5) + Q_4 (\xi_5 , Y_1) + Q_6 (Y_1 , \xi_{11}) + Q_4 (Y_1 , \xi_6) + Q_4 (\xi_6 , Y_1) + Q_6 (Y_1 , \xi_{12}) + 2\, [Q_5 (Y_2 , \xi_8) + Q_5 (\xi_8 , Y_2) + Q_6 (\xi_2 , Y_2) + C_6 (Y_2 , Y_1 , Y_1)] \}\, dx \tag{48}$$

$$\alpha_9 = \int_D Y_2 \{ Q_4 (Y_1 , \xi_6) + Q_4 (\xi_6 , Y_1) + Q_6 (Y_1 , \xi_{12}) + Q_5 (Y_2 , \xi_7) + Q_5 (\xi_7 , Y_2) + Q_6 (\xi_1 , Y_2) + C_6 (Y_2 , Y_1 , Y_1) \}\, dx \tag{49}$$

$$\alpha_{10} = \int_D Y_2 \{2[Q_4(Y_1,\xi_4) + Q_4(\xi_4, Y_1) + Q_6(Y_1,\xi_{10})] + Q_5(Y_2,\xi_{11}) + Q_5(\xi_{11},Y_2) + Q_6(\xi_5,Y_2)$$
$$+ Q_5(Y_2,\xi_{12}) + Q_5(\xi_{12},Y_2) + Q_6(\xi_6,Y_2) + 2\,C_5(Y_2,Y_2,Y_1)\}\,dx \quad (50)$$

$$\alpha_{11} = \int_D Y_2 \{Q_4(Y_1,\xi_3) + Q_4(\xi_3,Y_1) + Q_6(Y_1,\xi_9) + Q_5(Y_2,\xi_{12}) + Q_5(\xi_{12},Y_2) + Q_6(\xi_6,Y_2)$$
$$+ C_5(Y_2,Y_2\,Y_1)\}\,dx \quad (51)$$

$$\alpha_{12} = \int_D Y_2 \{Q_4(Y_1,\xi_1) + Q_4(\xi_1,Y_1) + Q_6(Y_1,\xi_7) + 2[Q_4(Y_1,\xi_2) + Q_4(\xi_2,Y_1) + Q_6(Y_1,\xi_8)]\}dx \quad (52)$$

$$f = \int_D Y_1\,F\,dx \quad (53)$$

where D is the domain of integration. Representing the solutions in this form makes it convenient to see explicitly the contributions of each operator to the coefficients. Note that, an extension of the above formulas for two or more spatial variables is trivial. In arriving at Eqs. (39) and (40), we normalized the following integrals

$$\int_D Y_1^{\,2}\,dx = 1\,,\ \int_D Y_2^{\,2}\,dx = 1 \quad (54)$$

Eqs. (39) and (40) determine the complex functions $A(T_2)$ and $B(T_2)$. Note that, once the functions Y and ξ are known either in closed form or numerically, α_i coefficients can be calculated numerically for specific operators.

4. RESPONSES AND MODULATION EQUATIONS

To find the amplitude and phase modulation equations, we insert the following polar form into Eqs. (39) and (40)

$$A(T_2) = \frac{1}{2}a(T_2)e^{i\lambda_1(T_2)}\,, \qquad B(T_2) = \frac{1}{2}b(T_2)e^{i\lambda_2(T_2)} \quad (55)$$

and obtain the amplitude and phase modulation equations for the responses

$$a' = -\frac{1}{2}\mu_1 a - \frac{1}{8}\frac{\alpha_3}{\omega_1}ab^2\sin 2\gamma_1 + \frac{1}{8}\left(\frac{\alpha_5}{\omega_1} - \frac{\alpha_4}{\omega_1}\right)a^2 b\sin\gamma_1 - \frac{1}{8}\frac{\alpha_6}{\omega_1}b^3\sin\gamma_1 - \frac{1}{2}\frac{f}{\omega_1}\sin\gamma_2 \quad (56)$$

$$b' = -\frac{1}{2}\mu_2 b + \frac{1}{8}\frac{\alpha_9}{\omega_2}a^2 b\sin 2\gamma_1 + \frac{1}{8}\left(\frac{\alpha_{10}}{\omega_2} - \frac{\alpha_{11}}{\omega_2}\right)ab^2\sin\gamma_1 + \frac{1}{8}\frac{\alpha_{12}}{\omega_2}a^3\sin\gamma_1 \quad (57)$$

$$\gamma_1' = -\rho + \frac{1}{8}\left(\frac{\alpha_7}{\omega_2} - \frac{\alpha_2}{\omega_1}\right)b^2 + \frac{1}{8}\left(\frac{\alpha_8}{\omega_2} - \frac{\alpha_1}{\omega_1}\right)a^2 + \frac{1}{8}\left(\frac{\alpha_9}{\omega_2}a^2 - \frac{\alpha_3}{\omega_1}b^2\right)\cos 2\gamma_1$$
$$+ \frac{1}{8}\left(\frac{\alpha_{10}}{\omega_2} + \frac{\alpha_{11}}{\omega_2} - \frac{\alpha_4}{\omega_1} - \frac{\alpha_5}{\omega_1}\right)ab\cos\gamma_1 + \frac{1}{8}\left(\frac{\alpha_{12}}{\omega_2}\frac{a^3}{b} - \frac{\alpha_6}{\omega_1}\frac{b^3}{a}\right)\cos\gamma_1 - \frac{1}{2}\frac{f}{\omega_1 a}\cos\gamma_2 \quad (58)$$

$$\gamma_2' = \sigma - \frac{1}{8}\frac{\alpha_1}{\omega_1}a^2 - \frac{1}{8}\frac{\alpha_2}{\omega_1}b^2 - \frac{1}{8}\frac{\alpha_3}{\omega_1}b^2\cos 2\gamma_1 - \frac{1}{8}(\frac{\alpha_4}{\omega_1} + \frac{\alpha_5}{\omega_1})ab\cos\gamma_1 - \frac{1}{8}\frac{\alpha_6}{\omega_1}\frac{b^3}{a}\cos\gamma_1$$
$$- \frac{1}{2}\frac{f}{\omega_1 a}\cos\gamma_2 \tag{59}$$

where γ_1 and γ_2 are defined as follows

$$\gamma_1 = \lambda_2 - \lambda_1 - \rho T_2, \qquad \gamma_2 = \sigma T_2 - \lambda_1 \tag{60}$$

Once the numerical values of α coefficients are determined for specific problems, stability and bifurcation analysis of equations (56)-(59) can be made.

The responses are found by substituting Eqs. (55), (60), (33), (34), (19), (20), (15) and (16) into Eqs. (4) and (5). The final results are

$$w_1 = \varepsilon a\cos(\Omega t - \gamma_2)Y_1(x) + \frac{\varepsilon^2}{2}\{a^2\cos[2(\Omega t - \gamma_2)]\,\xi_1(x) + a^2\xi_2(x) + b^2\cos[2(\Omega t + \gamma_1 - \gamma_2)].$$
$$\xi_3(x) + b^2\xi_4(x) + ab\cos(2\Omega t + \gamma_1 - 2\gamma_2)\,\xi_5(x) + ab\cos\gamma_1\,\xi_6(x)\} + \dots \tag{61}$$

$$w_2 = \varepsilon b\cos(\Omega t + \gamma_1 - \gamma_2)Y_2(x) + \frac{\varepsilon^2}{2}\{a^2\cos[2(\Omega t - \gamma_2)]\,\xi_7(x) + a^2\xi_8(x)$$
$$+ b^2\cos[2(\Omega t + \gamma_1 - \gamma_2)]\xi_9(x) + b^2\xi_{10}(x) + ab\cos(2\Omega t + \gamma_1 - 2\gamma_2)\,\xi_{11}(x)$$
$$+ ab\cos\gamma_1\,\xi_{12}(x)\} + \dots \tag{62}$$

where $\xi_i(x)$ satisfy Eqs. (21)-(32). The amplitudes and phases appearing in Eqs. (61) and (62) are governed by Eqs. (56)-(59). Therefore Eqs. (61) and (62) together with Eqs. (56)-(59) constitute the final solutions to Eqs. (1) and (2). In the next section, a specific problem will be treated for illustration.

5. APPLICATION TO NONLINEAR CABLE VIBRATIONS

In this section, we will apply the formalism derived in the previous sections to differential equations modelling the nonlinear vibrations of cables with small sag-to-span ratios. Following the previous analysis, we will investigate primary resonances of the excitation and one-to-one internal resonances between the natural frequencies of the in-plane and out-of-plane vibrations. The equations of motion, first derived by Lee and Perkins are[13]

$$[v_t^2 + v_l^2 g(t)]\,w_1'' + \frac{v_l^2}{v_t^2}g(t) + \hat{F}(x)\cos\Omega t = \ddot{w}_1 + \hat{\mu}_1\dot{w}_1 \tag{63}$$

$$[v_t^2 + v_l^2 g(t)]\,w_2'' = \ddot{w}_2 + \hat{\mu}_2\dot{w}_2 \tag{64}$$

$$g(t) = \int_0^1 \{-\frac{1}{v_t^2}w_1 + \frac{1}{2}[w_1'^2 + w_2'^2]\}\,dx \tag{65}$$

$$w_{1,2}(0,t) = w_{1,2}(1,t) = 0 \tag{66}$$

where x is the dimensionless arclength coordinate, and $(\,)'$ denotes differentiation with respect to x. The constants v_t and v_l are the dimensionless propagation speeds of transverse and longitudinal waves respectively. w_1 is the in-plane and w_2 is the out-of-plane displacements.

The specific form of the operators in Eqs. (1) and (2) for this case can be defined as follows

$$L_1(w_1) = -v_t^2 w_1'' + \frac{v_l^2}{v_t^4}\int_0^1 w_1\,dx\,, \qquad L_2(w_2) = -v_t^2 w_2''$$

$$Q_1(w_1,w_1) = \frac{v_l^2}{v_t^2}\left(w_1''\int_0^1 w_1\,dx - \frac{1}{2}\int_0^1 w_1'^2\,dx\right), \qquad Q_2(w_2,w_2) = -\frac{v_l^2}{2v_t^2}\int_0^1 w_2'^2\,dx$$

$$Q_3(w_1,w_2) = Q_4(w_1,w_1) = Q_5(w_2,w_2) = 0\,, \qquad Q_6(w_1,w_2) = \frac{v_l^2}{v_t^2}w_2''\int_0^1 w_1\,dx \tag{67}$$

$$C_1(w_1,w_1,w_1) = -\frac{v_l^2}{2}w_1''\int_0^1 w_1'^2\,dx\,, \qquad C_3(w_1,w_2,w_2) = -\frac{v_l^2}{2}w_1''\int_0^1 w_2'^2\,dx$$

$$C_4(w_2,w_2,w_2) = -\frac{v_l^2}{2}w_2''\int_0^1 w_2'^2\,dx\,, \qquad C_6(w_2,w_1,w_1) = -\frac{v_l^2}{2}w_2''\int_0^1 w_1'^2\,dx$$

$$C_2(w_1,w_1,w_2) = C_5(w_2,w_2,w_1) = 0$$

Assuming expansions (4) and (5) for the displacements, we obtain solutions (15) and (16) for the linear problem where the eigenfunctions $Y_1(x)$ and $Y_2(x)$ satisfy Eqs. (17) and (18) which can be written for this case as follows

$$v_t^2 Y_1'' + \omega_1^2 Y_1 - \frac{v_l^2}{v_t^4}\int_0^1 Y_1\,dx = 0 \tag{68}$$

$$v_t^2 Y_2'' + \omega_2^2 Y_2 = 0 \tag{69}$$

$$Y_{1,2}(0) = Y_{1,2}(1) = 0 \tag{70}$$

Equation (68) with the boundary conditions (70) possess two types of solutions, namely the symmetric and the anti-symmetric in-plane solutions with respect to the mid-span of the cable. The symmetric in-plane solution is

$$Y_1(x) = C\left\{1 - \tan\left(\frac{\omega_1}{2v_t}\right)\sin\left(\frac{\omega_1}{v_t}x\right) - \cos\left(\frac{\omega_1}{v_t}x\right)\right\} \tag{71}$$

where ω_1 satisfies the equation

$$\frac{\omega_1^3 v_t^3}{v_l^2} - \frac{\omega_1}{v_t} + 2\tan\left(\frac{\omega_1}{2v_t}\right) = 0 \tag{72}$$

C should be chosen such that $\int_0^1 Y_1{}^2\, dx = 1$. The anti-symmetric in-plane solution is

$$Y_1(x) = \sqrt{2}\, sin(\frac{\omega_1}{v_t}x), \qquad \frac{\omega_1}{v_t} = 2n\pi, \qquad n = 1,2,3,... \tag{73}$$

The symmetric in-plane solution can account for stretching of the cable whereas the anti-symmetric solution corresponds to zero stretching case[13]. We will consider only the symmetric solution in the following analysis.

The out-of-plane solution is

$$Y_2(x) = \sqrt{2}\, sin(\frac{\omega_2}{v_t}x), \qquad \frac{\omega_2}{v_t} = n\pi, \qquad n = 1,2,3,... \tag{74}$$

The next step is to find the solutions at order ε^2. The form of the solutions are Eqs. (19) and (20) where ξ_i satisfy Eqs. (21)-(32). Substituting the specific form of the operators from Eq. (67) into Eqs. (21)-(32), we obtain the following set of equations

$$v_t^2\xi_1'' + 4\omega_1{}^2\xi_1 - \frac{v_l^2}{v_t^4}\int_0^1 \xi_1\, dx = \frac{v_l^2}{v_t^2}(Y_1''\int_0^1 Y_1\, dx - \frac{1}{2}\int_0^1 Y_1'^2\, dx) \tag{75}$$

$$v_t^2\xi_2'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_2\, dx = \frac{v_l^2}{v_t^2}(Y_1''\int_0^1 Y_1\, dx - \frac{1}{2}\int_0^1 Y_1'^2\, dx) \tag{76}$$

$$v_t^2\xi_3'' + 4\omega_2{}^2\xi_3 - \frac{v_l^2}{v_t^4}\int_0^1 \xi_3\, dx = -\frac{v_l^2}{2v_t^2}\int_0^1 Y_2'^2\, dx \tag{77}$$

$$v_t^2\xi_4'' - \frac{v_l^2}{v_t^4}\int_0^1 \xi_4\, dx = -\frac{v_l^2}{2v_t^2}\int_0^1 Y_2'^2\, dx \tag{78}$$

$$v_t^2\xi_5'' + (\omega_1 + \omega_2)^2\xi_5 - \frac{v_l^2}{v_t^4}\int_0^1 \xi_5\, dx = 0 \tag{79}$$

$$v_t^2\xi_6'' + (\omega_1 - \omega_2)^2\xi_6 - \frac{v_l^2}{v_t^4}\int_0^1 \xi_6\, dx = 0 \tag{80}$$

$$v_t^2\xi_7'' + 4\omega_1{}^2\xi_7 = 0 \tag{81}$$

$$v_t^2\xi_8'' = 0 \tag{82}$$

$$v_t^2\xi_9'' + 4\omega_2{}^2\xi_9 = 0 \tag{83}$$

$$v_t^2\xi_{10}'' = 0 \tag{84}$$

$$v_t^2\xi_{11}'' + (\omega_1 + \omega_2)^2\xi_{11} = \frac{v_l^2}{v_t^2}Y_2''\int_0^1 Y_1\, dx \tag{85}$$

$$v_t^2 \xi_{12}'' + (\omega_1 - \omega_2)^2 \xi_{12} = \frac{v_l^2}{v_t^2} Y_2'' \int_0^1 Y_1 \, dx \tag{86}$$

$$\xi_i(0) = \xi_i(1) = 0, \qquad i = 1, 2, \ldots, 12 \tag{87}$$

Substituting the linear solutions (71) and (74) into Eqs. (75)-(86), using Eq. (72) when necessary, we finally obtain the solutions for ξ_i

$$\xi_1(x) = \left(\frac{v_l^2}{2v_t^2} \int_0^1 Y_1'^2 \, dx - C^2 \omega_1^4\right) \left\{ \frac{v_l^2}{v_t^4}\left(1 - \frac{v_t}{\omega_1} \tan\left(\frac{\omega_1}{v_t}\right)\right) - 4\omega_1^2 \right\}^{-1} \cdot$$
$$\left\{ 1 - \tan\left(\frac{\omega_1}{v_t}\right) \sin\left(\frac{2\omega_1}{v_t} x\right) - \cos\left(\frac{2\omega_1}{v_t} x\right) \right\} - \frac{1}{3} C \omega_1^2 Y_1(x) \tag{88}$$

$$\xi_2(x) = \frac{1}{12 v_t^6 + v_l^2} \left(6 C^2 \omega_1^4 v_t^4 - 3 v_t^2 v_l^2 \int_0^1 Y_1'^2 \, dx \right) (x^2 - x) + C \omega_1^2 Y_1(x) \tag{89}$$

$$\xi_3(x) = \frac{\omega_2^2 v_l^2}{2(4\omega_2^2 v_t^4 - v_l^2)} \left\{ \cos\left(\frac{2\omega_2}{v_t} x\right) - 1 \right\} \tag{90}$$

$$\xi_4(x) = -\frac{3\omega_2^2 v_l^2}{12 v_t^6 + v_l^2} (x^2 - x) \tag{91}$$

$$\xi_{11}(x) = \frac{\sqrt{2} C \omega_1^2 \omega_2^2}{\omega_2^2 - (\omega_1 + \omega_2)^2} \sin\left(\frac{\omega_2}{v_t} x\right) \tag{92}$$

$$\xi_{12}(x) = \frac{\sqrt{2} C \omega_1^2 \omega_2^2}{\omega_2^2 - (\omega_1 - \omega_2)^2} \sin\left(\frac{\omega_2}{v_t} x\right) \tag{93}$$

$$\xi_5 = \xi_6 = \xi_7 = \xi_8 = \xi_9 = \xi_{10} = 0 \tag{94}$$

At order ε^3, we obtain the solvability conditions (39) and (40) where the coefficients α_i are defined in Eqs. (41)-(52). Substituting the specific forms of the operators defined in Eqs. (67) into Eqs. (41)-(52), we obtain the coefficients of the modulation equations

$$\alpha_1 = -\left\{ \frac{v_l^2}{v_t^2} (b_2 b_6 + 2b_1 b_8 + 2b_3 b_6 + 4b_1 b_9) - \frac{2}{3} v_l^2 b_6^2 \right\} \tag{95}$$

$$\alpha_2 = -\left\{ \frac{v_l^2}{v_t^2} (2b_5 b_6 + 4b_1 b_{11} + b_1 b_{12} + b_1 b_{13}) - v_l^2 b_6 b_7 \right\} \tag{96}$$

$$\alpha_3 = -\left\{ \frac{v_l^2}{v_t^2} (2b_1 b_{10} + b_1 b_{13} + b_4 b_6) - \frac{1}{2} v_l^2 b_6 b_7 \right\} \tag{97}$$

$$\alpha_7 = -\{ \frac{v_l^2}{v_t^2}(b_4 + 2b_5)\, b_7 - \frac{3}{2} v_l^2 b_7^2 \} \tag{98}$$

$$\alpha_8 = -\{ \frac{v_l^2}{v_t^2}(b_1 b_{12} + b_1 b_{13} + 2b_3 b_7) - v_l^2 b_6 b_7 \} \tag{99}$$

$$\alpha_9 = -\{ \frac{v_l^2}{v_t^2}(b_1 b_{13} + b_2 b_7) - \frac{1}{2} v_l^2 b_6 b_7 \} \tag{100}$$

$$\alpha_4 = \alpha_5 = \alpha_6 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0 \tag{101}$$

where

$$b_1 = \int_0^1 Y_1\, dx, \quad b_2 = \int_0^1 \xi_1\, dx, \quad b_3 = \int_0^1 \xi_2\, dx, \quad b_4 = \int_0^1 \xi_3\, dx,$$
$$b_5 = \int_0^1 \xi_4\, dx, \quad b_6 = \int_0^1 Y_1'^2\, dx, \quad b_7 = \int_0^1 Y_2'^2\, dx, \quad b_8 = \int_0^1 Y_1'\xi_1' dx,$$
$$b_9 = \int_0^1 Y_1'\xi_2' dx, \quad b_{10} = \int_0^1 Y_1'\xi_3' dx, \quad b_{11} = \int_0^1 Y_1'\xi_4' dx, \quad b_{12} = \int_0^1 Y_2'\xi_{11}' dx,$$
$$b_{13} = \int_0^1 Y_2'\xi_{12}' dx \tag{102}$$

The amplitude and phase modulation equations can now be written from eqs. (56)-(59)

$$a' = -\frac{1}{2}\mu_1 a - \frac{1}{8}\frac{\alpha_3}{\omega_1} a b^2 \sin 2\gamma_1 - \frac{1}{2}\frac{f}{\omega_1}\sin\gamma_2 \tag{103}$$

$$b' = -\frac{1}{2}\mu_2 b + \frac{1}{8}\frac{\alpha_9}{\omega_2} a^2 b \sin 2\gamma_1 \tag{104}$$

$$\gamma_1' = -\rho + \frac{1}{8}(\frac{\alpha_7}{\omega_2} - \frac{\alpha_2}{\omega_1}) b^2 + \frac{1}{8}(\frac{\alpha_8}{\omega_2} - \frac{\alpha_1}{\omega_1}) a^2 + \frac{1}{8}(\frac{\alpha_9}{\omega_2} a^2 - \frac{\alpha_3}{\omega_1} b^2) \cos 2\gamma_1$$
$$- \frac{1}{2}\frac{f}{\omega_1 a}\cos\gamma_2 \tag{105}$$

$$\gamma_2' = \sigma - \frac{1}{8}\frac{\alpha_1}{\omega_1} a^2 - \frac{1}{8}\frac{\alpha_2}{\omega_1} b^2 - \frac{1}{8}\frac{\alpha_3}{\omega_1} b^2 \cos 2\gamma_1 - \frac{1}{2}\frac{f}{\omega_1 a}\cos\gamma_2 \tag{106}$$

where γ_1 and γ_2 are defined in Eqs. (60). The responses are found from Eqs. (61) and (62)

$$w_1 = \varepsilon a \cos(\Omega t - \gamma_2) Y_1(x) + \frac{\varepsilon^2}{2}\{ a^2 \cos[2(\Omega t - \gamma_2)]\, \xi_1(x) + a^2 \xi_2(x) + b^2 \cos[2(\Omega t + \gamma_1 - \gamma_2)] \cdot$$
$$\xi_3(x) + b^2 \xi_4(x) \} + \ldots \tag{107}$$

$$w_2 = \varepsilon b\, cos(\Omega t + \gamma_1 - \gamma_2) Y_2(x) + \frac{\varepsilon^2}{2} ab\{ cos(2\Omega t + \gamma_1 - 2\gamma_2)\, \xi_{11}(x)$$

$$+ ab\, cos\gamma_1\, \xi_{12}(x) \} + \dots \qquad (108)$$

All results presented here are in full agreement with those given in references [2] and [4] provided that the following notation is adapted

$$x \equiv s\,,\quad Y_1(x) \equiv \phi(s)\,,\quad Y_2(x) \equiv \psi(s)$$

$$\xi_1(x) \equiv \Phi_1(s)\,, \xi_2(x) \equiv \Phi_2(s)\,, \xi_3(x) \equiv \Phi_3(s)\,, \xi_4(x) \equiv \Phi_4(s)\,,\ \xi_{11}(x) \equiv \Phi_5(s)\,,\ \xi_{12}(x) \equiv \Phi_6(s)$$

$$\alpha_1 \equiv -\Gamma_1\,, \alpha_2 \equiv -\Gamma_3\,, \alpha_3 \equiv -\Gamma_2\,, \alpha_7 \equiv -\Gamma_6\,, \alpha_8 \equiv -\Gamma_4\,, \alpha_9 \equiv -\Gamma_5\,,\quad f \equiv b_{14}$$

$$\gamma_1 \equiv -\gamma_1/2\,,\ \gamma_2 \equiv \gamma_2\,,\ \rho \equiv \sigma_1\,,\ \sigma \equiv \sigma_2\,,\ \lambda_1 \equiv \beta_1\,,\ \lambda_2 \equiv \beta_2\,,\ \omega_1 \equiv \omega_3\,,\ \omega_2 \equiv \lambda_3 \qquad (109)$$

Reference[2] includes stability and bifurcation analysis of the modulation equations.

The formalism developed in this paper can be cast into a purely numerical algorithm for more involved problems where it becomes hard to find the functions explicitly.

6. CONCLUDING REMARKS

A general nonlinear vibration problem has been proposed. The problem generalizes many vibration problems of continuous systems such as beam, string and cable vibrations. Solutions of this problem are presented in their most general form so that an algorithm can be constructed for solutions of a wide range of problems. As an illustration, the algorithm is used to solve a nonlinear cable vibration problem.

In this study, as an initial step, only the arbitrary linear and homogenous boundary conditions are considered. Nonlinear boundary conditions can be added as a next step. Subharmonic and superharmonic resonances of the external excitation as well as different internal resonances can be considered in a similar way. Numerical solutions can be searched when it is hard to find explicitly the functions appearing at the first and second orders of approximation. This will not involve any problem at the last level of approximation, since the coefficients of the modulation equations are presented in a suitable way to enable further numerical calculations.

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