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CONTINUOUS SYSTEMS WITH ODD NONLINEARITIES: A GENERAL SOLUTION PROCEDURE

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Abstract- A generalized equation of motion with odd nonlinearities is considered. The nonlinearities of cubic and fifth order are represented in the form of arbitrary operators. The equation of motion, in its general form, may model a class of partial differential equations encountered in vibrations of continuous systems. Approximate analytical solutions are sought using the method of multiple scales, a perturbation technique. Forced vibrations with viscous damping are considered. Frequency-response relation is derived in its most general form. Finally, an application to a specific problem is given.

1. INTRODUCTION

A new notation of expressing the nonlinearities in continuous systems has been first proposed by Pakdemirli [1]. Quadratic and cubic nonlinearities of a general system were expressed by arbitrary spatial operators. Free vibrations with damping were considered for single-mode approximations. The analysis was generalized to infinite modes by Pakdemirli and Boyacı [2]. Primary resonances with forced vibrations were considered in that analysis. Subharmonic, superharmonic and combination resonances were treated using the general model by the same authors [3]. Finally, the same notation was also used by Boyacı and Pakdemirli [4] for expressing the nonlinearities of quadratic and cubic type. General solutions were constructed using different versions of the method of multiple scales.

In this work, we treat a general continuous-system model of odd nonlinearities as follows

$$\begin{gathered}\ddot{w} + \hat{\mu}\,\dot{w} + \boldsymbol{L}(w) + \varepsilon\,\boldsymbol{C}(w,w,w) + \varepsilon^2 \boldsymbol{E}(w,w,w,w,w) = \hat{F}\cos\Omega t \\ \boldsymbol{B}_1(w) = 0 \quad at \quad x = 0, \qquad \boldsymbol{B}_2(w) = 0 \quad at \quad x = 1\end{gathered} \tag{1}$$

where $w(x,t)$ is the deflection, $\hat{\mu}$ is the viscous damping coefficient, $\hat{F}$ is the external excitation amplitude and Ω is the external excitation frequency. $\boldsymbol{L}$, $\boldsymbol{C}$ and $\boldsymbol{E}$ are the linear self-adjoint, nonlinear cubic and nonlinear fifth order operators respectively. $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are the linear operators for the boundary conditions. All operators are spatial differential and/or integral operators. x and t are the spatial and time variables respectively. The dot denotes differentiation with respect to time and the prime denotes differentiation with respect to the spatial variable x. The equations of motion are in dimensionless form. The nonlinear operators possess the property of being multilinear as explained in more detail in previous work [1-4].

For finding steady-state solutions of partial differential system (1), we use the version of the method of multiple scales first proposed by Rahman and Burton [5]. Damped vibrations with forcing are considered. Frequency-response relation is derived in its most general form. Finally, the solution algorithm is applied to a beam resting on a foundation with odd nonlinearities.

2. GENERAL SOLUTION PROCEDURE

In this section, we will solve system (1) in its most general form. We follow the analysis given by Boyacı and Pakdemirli [4] for partial differential equations, which is an adaptation of the method proposed by Rahman and Burton [5] for ordinary differential equations. Defining a new time variable

$$T = \Omega t \tag{2}$$

and substituting into (1), we have

$$\begin{gathered} \Omega^2 \ddot{w} + \bar{\mu}\dot{w} + L(w) + \varepsilon C(w,w,w) + \varepsilon^2 E(w,w,w,w,w) = \hat{F}\cos T \\ B_1(w) = 0 \ \text{ at } \ x = 0, \qquad B_2(w) = 0 \ \text{ at } \ x = 1 \end{gathered} \tag{3}$$

where the dot now denotes differentiation with respect to the new variable T and $\bar{\mu} = \hat{\mu}\Omega$. The response, damping coefficient, excitation amplitude and frequency are expanded in terms of the small parameter ε

$$w(x,T;\varepsilon) = w_0(x,T_0,T_1,T_2) + \varepsilon w_1(x,T_0,T_1,T_2) + \varepsilon^2 w_2(x,T_0,T_1,T_2) + \ldots \tag{4}$$

$$\bar{\mu} = \varepsilon\mu_1 + \varepsilon^2\mu_2, \qquad \hat{F} = \varepsilon F, \qquad \Omega^2 = \omega^2 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 \tag{5,6,7}$$

where $T_0 = T$ is the usual fast time scale and $T_1 = \varepsilon T$ and $T_2 = \varepsilon^2 T$ are the slow time scales. Derivatives with respect to time are defined in terms of the new variables

$$\begin{aligned} d/dT &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \\ d^2/dT^2 &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots \end{aligned} \tag{8}$$

Expansion of $\hat{F}$ in eq. (6) is kept up to $O(\varepsilon)$. An expansion to $O(\varepsilon^2)$ yields redundant terms and requires compatibility conditions [4]. In search of approximate solutions, we are directly attacking the partial differential system rather than discretizing the system first and then applying perturbations. The former method has advantages over the latter one [1,2,6-9].

Inserting eqs. (4)-(8) into eq. (3) and separating each order of ε yields the set of equations

$$\omega^2 D_0^2 w_0 + L(w_0) = 0 \tag{9}$$

$$\begin{aligned} \omega^2 D_0^2 w_1 + L(w_1) = {} & -2\omega^2 D_0 D_1 w_0 - \sigma_1 D_0^2 w_0 - \mu_1 D_0 w_0 \\ & - C(w_0, w_0, w_0) + F\cos T_0 \end{aligned} \tag{10}$$

$$\omega^2 D_0^2 w_2 + L(w_2) = -2\omega^2 D_0 D_1 w_1 - \omega^2 (D_1^2 + 2D_0 D_2) w_0 - \sigma_1 D_0^2 w_1 - 2\sigma_1 D_0 D_1 w_0$$
$$- \sigma_2 D_0^2 w_0 - \mu_1 D_0 w_1 - \mu_1 D_1 w_0 - \mu_2 D_0 w_0 - C(w_0, w_0, w_1) \tag{11}$$
$$- C(w_0, w_1, w_0) - C(w_1, w_0, w_0) - E(w_0, w_0, w_0, w_0, w_0)$$

Equation (9) possesses solutions of the form

$$w_0 = (A(T_1, T_2)e^{iT_0} + cc)Y(x) \tag{12}$$

where cc denotes complex conjugate of the preceding terms and $Y(x)$ satisfy

$$\begin{gathered} L(Y) - \omega^2 Y = 0 \\ B_1(Y) = 0 \quad \text{at } x = 0, \quad B_2(Y) = 0 \quad \text{at } x = 1 \end{gathered} \tag{13}$$

The above boundary value problem is an eigenvalue-eigenfunction problem with ω^2 (Square of the natural frequencies of the system) the eigenvalues and $Y(x)$ the corresponding eigenfunctions. For continuous systems, there are infinite number of eigenvalues and corresponding eigenfunctions.

Substituting eq. (12) into eq. (10) and finding the solvability condition at this order, (see Nayfeh [10] for details of finding solvability conditions) we obtain

$$2i\omega^2 D_1 A = (\sigma_1 - \mu_1 i)A - 3\alpha_1 A^2 \overline{A} + f/2 \tag{14}$$

where

$$\alpha_1 = \int_0^1 YC(Y, Y, Y)dx, \qquad \mathrm{f} = \int_0^1 YFdx \tag{15}$$

For steady-state solutions, requiring $D_1 A = 0$, writing the complex amplitude in its polar form $A = (1/2)ae^{i\beta}$, separating the real and imaginary parts, we have

$$\sigma_1 = \frac{3}{4}\alpha_1 a^2 \pm \sqrt{\frac{f^2}{a^2} - \mu_1^2} \tag{16}$$

A solution at this order free from secular and resonant terms is

$$w_1 = (A^3(T_2)e^{3iT_0} + cc)\phi(x) \tag{17}$$

where $\phi(x)$ satisfies

$$\begin{gathered} L(\phi) - 9\omega^2 \phi = -C(Y, Y, Y) \\ B_1(\phi) = 0 \quad \text{at } x = 0, \quad B_2(\phi) = 0 \quad \text{at } x = 1 \end{gathered} \tag{18}$$

At order ε^2, we substitute solutions (12) and (17) into eq. (11) and find the solvability condition

$$2i\omega^2 D_2 A = (\sigma_2 - \mu_2 i)A - \alpha_2 A^3 \overline{A}^2 \tag{19}$$

where

$$\alpha_2 = \int_0^1 Y[C(Y,Y,\phi)+C(Y,\phi,Y)+C(\phi,Y,Y)+10E(Y,Y,Y,Y,Y)]dx \tag{20}$$

In the steady-state $D_2 A = 0$. Writing $A = (1/2)ae^{i\beta}$, separating real and imaginary parts, we have

$$\sigma_2 = \frac{1}{16}\alpha_2 a^4, \qquad \mu_2 = 0 \tag{21}$$

Substituting σ_1 and σ_2 into equation (7), we obtain the frequency-response equation

$$\Omega^2 = \omega^2 + \varepsilon\left(\frac{3}{4}\alpha_1 a^2 \pm \sqrt{\frac{f^2}{a^2} - \mu_1^2}\right) + \varepsilon^2 \frac{1}{16}\alpha_2 a^4 \tag{22}$$

The coefficients α_1 and α_2 are defined in their most general form in terms of the arbitrary operators in eqs. (15) and (20). For the specific forms of the operators, these coefficients can be calculated with ease by evaluating the integrals. When ε is taken as zero, the nonlinear result reduces to that of linear one.

The approximate steady-state solution can now be written as

$$\begin{aligned} w &= a\cos\left[(\omega + \frac{1}{2}\varepsilon\sigma_1 + \frac{1}{2}\varepsilon^2\sigma_2)t + \beta\right]Y(x) \\ &+ \frac{1}{4}\varepsilon a^3 \cos\left[3(\omega + \frac{1}{2}\varepsilon\sigma_1 + \frac{1}{2}\varepsilon^2\sigma_2)t + 3\beta\right]\phi(x) + O(\varepsilon^2) \end{aligned} \tag{23}$$

To summarize the algorithm developed, we have to solve the boundary value problems appearing at each order of ε (i.e. eqs. (13) and (18)) and then evaluate the integrals to find the coefficients (eqs. (15) and (20)). The general solutions (22) and (23) can then be written for specific continuous systems.

3. AN EXAMPLE

In this section, we apply the general algorithm to a simply-supported beam resting on an elastic foundation with odd nonlinearities of cubic and fifth order. The equation of motion for the problem is

$$\begin{aligned} &\ddot{u} + \hat{\mu}\dot{u} + u^{iv} + k_1 u + k_2 u^3 + k_3 u^5 = \bar{F}\cos\Omega t \\ &u(0,t) = u''(0,t) = 0, \qquad u(1,t) = u''(1,t) = 0 \end{aligned} \tag{24}$$

where k_1 is the linear coefficient and k_2 and k_3 are the nonlinear coefficients of the elastic foundation. Assuming that the vibrations are small (a weakly nonlinear system with ε a small parameter), we make the transformations

$$u = \varepsilon^{1/2} w, \qquad T = \Omega t \tag{25}$$

and substitute into eq. (24)

$$\Omega^2\ddot{w}+\bar{\mu}\dot{w}+w^{iv}+k_1 w+\varepsilon k_2 w^3+\varepsilon^2 k_3 w^5=\hat{F}\cos T$$
$$w(0,T)=w''(0,T)=0, \qquad w(1,T)=w''(1,T)=0 \tag{26}$$

where $\hat{F}=\bar{F}/\varepsilon^{1/2}$. Comparing eq. (26) with eq. (3), we define the operators

$$\boldsymbol{L}(w)=w^{iv}+k_1 w\,, \quad \boldsymbol{C}(w,w,w)=k_2 w^3\,, \quad \boldsymbol{E}(w,w,w,w,w)=k_3 w^5 \tag{27}$$

We first solve the boundary value problem in eq. (13). With the above linear operator, equation (13) takes the form

$$Y^{iv}+(k_1-\omega^2)Y=0$$
$$Y(0)=Y''(0)=Y(1)=Y''(1)=0 \tag{28}$$

Solution of this eigenvalue-eigenfunction problem yields

$$Y(x)=\sqrt{2}\sin n\pi x, \qquad \omega=\sqrt{\mathrm{n}^4\pi^4+k_1}\,, \qquad \mathrm{n}=1,2,3,\ldots \tag{29}$$

Now, we can calculate the coefficient α_1 from eq. (15)

$$\alpha_1=\int_0^1 k_2 Y^4 dx=\frac{3}{2}k_2 \tag{30}$$

Next, we solve the boundary value problem given in eq. (18)

$$\phi^{iv}+(k_1-9\omega^2)\phi=-2\sqrt{2}k_2\sin^3 n\pi x$$
$$\phi(0)=\phi''(0)=\phi(1)=\phi''(1)=0 \tag{31}$$

The solution is

$$\phi(x)=\frac{3\sqrt{2}k_2}{16(n^4\pi^4+k_1)}\sin n\pi\, x+\frac{\sqrt{2}k_2}{16(9n^4\pi^4-k_1)}\sin 3n\pi\, x \tag{32}$$

Having determined $Y(x)$ and $\phi(x)$, we can calculate α_2

$$\alpha_2=\int_0^1 Y(3k_2Y^2\phi+10k_3Y^5)dx=\frac{3k_2^2}{32}\left(\frac{9}{n^4\pi^4+k_1}-\frac{1}{9n^4\pi^4-k_1}\right)+25k_3 \tag{33}$$

We now write the frequency-response relation for this case

$$\Omega^2=n^4\pi^4+k_1+\varepsilon\left(\frac{9}{8}k_2a^2\pm\sqrt{\frac{f^2}{a^2}-\mu_1^2}\right)$$
$$+\varepsilon^2\left[\frac{3k_2^2}{512}\left(\frac{9}{n^4\pi^4+k_1}-\frac{1}{9n^4\pi^4-k_1}\right)+\frac{25}{16}k_3\right]a^4 \tag{34}$$

The approximate solution may be written by substituting first eq. (33) into eq. (21), eq. (30) into eq. (16) and then the results together with eq. (29) and eq.(32) into eq. (23).

4. CONCLUDING REMARKS

We showed the essential steps of solving a general odd nonlinearity problem. Approximate solutions of the general equation has been found using a special notation developed previously. Cubic and fifth order nonlinearities are represented by arbitrary spatial operators. The method of multiple scales was used in the analysis. Frequency-response relations and approximate steady-state solutions were found in their most general form. A nonlinear beam problem was solved using the algorithm developed.

Only the primary resonances were considered in the analysis. Secondary resonances can be considered as an extension of the method.

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