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THE DARBOUX TRIHEDRONS OF REGULAR CURVES ON A REGULAR TIME-LIKE SURFACE

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Abstract- In this work, we study Differential Geometry in the Minkowski 3-space R_1^3 of the curves on a time-like regular surface by using parameter curves which are not perpendicular to each other. The aim of this study is to investigate the formulae between the Darboux Vectors of the time-like curve (c), the time-like parameter curve (c_1) and the space-like parameter curve (c_2) which are not intersecting perpendicularly.

Key words- Time-like and space-like curves, time-like surface , minkowski 3-space.

1. INTRODUCTION

Classical differential geometry of the curves may be surrounded by the topics which are general helices, involute-evolute curve couples, spherical curves and Bertrand curves. Such special curves are investigated and used in some of real world problems like mechanical design or robotics by well-known Frenet-Serret equations. At the beginning of the twentieth century, Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold. In recent years, the theory of degenerate submanifolds has been treated by researchers and some of the classical differential geometry topics have been extended to Lorentzian manifolds. Some authors have aimed to determine Frenet–Serret invariants in higher dimensions. There exists a vast literature on this subject, for instance [1-4,6,7]. In the light of the available literature, in [4] the author extended spherical images of curves to a four-dimensional Lorentzian space and studied such curves in the case where the base curve is a space-like curve according to the signature (+,+,+,-).

By using the Darboux vector, various well-known formulas of differential geometry had been produced by [5]. Then, in [1], authors had been given these formulae in Minkowski 3-space R_1^3.

In this work, we investigate the formulae between the Darboux Vectors of the curve (c) , the parameter curves (c_1) and (c_2) which are not intersecting perpendicularly. Thus, we will find an opportunity to investigate regular time-like surface by taking the parameter curves which are intersecting under the angle θ.

2. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space R_1^3 are briefly presented (A more complete elementary treatment can be found in [1]). The Minkowski 3-space E_1^3 provided with the standard flat metric is given by

$$\langle \, , \, \rangle = dx_1^2 + dx_2^2 - dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of R_1^3. Recall that, the norm of an arbitrary vector $a \in R_1^3$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$. Let $\phi = \phi(s)$ be a regular curve in R_1^3. ϕ is called a unit speed curve if the velocity vector v of ϕ satisfies $\|v\| = 1$. For the vectors $u, w \in R_1^3$ they are said to be orthogonal if and only if $\langle u, v \rangle = 0$.

On the other hand , if we consider any orthogonal trihedron as $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ we can write their derivative formulaes as follows:

$$\frac{d\vec{e}_i}{dt} = \vec{w} \wedge \vec{e}_i \qquad i = 1, 2, 3 \tag{2.2}$$

where $\vec{e}_1$ is time-like vector, $\vec{e}_2$ and $\vec{e}_3$ are the space-like vectors and Darboux instantaneous rotation vector is

$$\vec{w} = a\vec{e}_1 + b\vec{e}_2 - c\vec{e}_3$$

Moreover , $\wedge$ is Lorentzian vectoral product,[1].

Let us take a time-like surface as $y = y(u, v)$. Denote by $\{\vec{t}, \vec{n}, \vec{b}\}$ the moving Frenet-Serret frame along the time-like curve (c) on $y = y(u, v)$. Another orthogonal frame on $y = y(u, v)$ is the Darboux trihedron as $\{\vec{t}, \vec{g}, \vec{N}\}$. For an arbitrary time-like curve (c) on time-like surface, the orientation of the Darboux trihedron is written as

$$\vec{N} \wedge \vec{t} = -\vec{g} \quad , \vec{t} \wedge \vec{g} = -\vec{N} \quad , \ \vec{g} \wedge \vec{N} = \vec{t} \tag{2.3}$$

and the Darboux derivative formulae can be written as follows:

$$\frac{d\vec{t}}{ds} = \vec{w} \wedge \vec{t} \quad , \ \frac{d\vec{g}}{ds} = \vec{w} \wedge \vec{g} \ , \ \ \frac{d\vec{N}}{ds} = \vec{w} \wedge \vec{N} \tag{2.4}$$

And also the Darboux vector of this trihedron is written as

$$\vec{w} = \frac{\vec{t}}{T_g} + \frac{\vec{g}}{R_n} - \frac{\vec{N}}{R_g} \tag{2.5}$$

where $\langle \vec{t}, \vec{t} \rangle = -1$, $\langle \vec{g}, \vec{g} \rangle = 1$, $\langle \vec{N}, \vec{N} \rangle = 1$ and $\frac{1}{T_g}$, $\frac{1}{R_n}$ and $\frac{1}{R_g}$ are geodesic torsion, normal curvature and geodesic curvature, respectively,[1].

3. THE DARBOUX VECTOR FOR THE DARBOUX TRIHEDRON OF A TIME-LIKE CURVE

Let us express the parameter curves u=const. as (c_1) and v=const. as (c_2) which are on a time-like surface $y = y(u,v)$. But, these curves intersect under the angle θ (not perpendicular). Let any time-like curve that is passing through a point P on the surface be (c). Let us take time-like and space-like parameter curves which are passing through the same point P as (c_1) and (c_2). Let the unit tangent vectors of curves (c), (c_1) and (c_2) at the point P be $\vec{t}, \vec{t}_1$ and $\vec{t}_2$, respectively. From $[1]$, the edges of the Darboux trihedrons of parameter curves are

$$\vec{N} \wedge \vec{t}_1 = -\vec{g}_1 \quad , \quad \vec{t}_1 \wedge \vec{g}_1 = -\vec{N} \quad , \quad \vec{g}_1 \wedge \vec{N} = \vec{t}_1 \tag{3.1}$$

Here, three Darboux trihedrons are written as below:

$$\left[\vec{t}, \vec{g}, \vec{N}\right] , \quad \left[\vec{t}, \vec{g}_1, \vec{N}\right] \quad , \quad \left[\vec{t}, \vec{g}_2, \vec{N}\right]$$

Let s, s_1 and s_2 be the arc-elements of the curves (c), (c_1) and (c_2), respectively. Thus, we can write

$$\begin{aligned} \vec{t}_1 &= \frac{\vec{r}_u}{\|\vec{r}_u\|} = \frac{\vec{r}_u}{\sqrt{E}} \\ \vec{t}_2 &= \frac{\vec{r}_v}{\|\vec{r}_v\|} = \frac{\vec{r}_v}{\sqrt{G}} \\ \vec{t} &= \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds} \end{aligned} \tag{3.2}$$

where $E = \left|\vec{r}_u \vec{r}_u\right|$, $G = \left|\vec{r}_v \vec{r}_v\right|$.

Moreover, because of the parameter curves intersect under the angle θ we have

$$\vec{t}_1 \vec{t}_2 = \left|sh\theta\right|$$

(3.3),[8].

In the paper the sign will be taken as positive i.e., it will be assumed that $\vec{t}_1 \vec{t}_2 = sh\theta$.

Then, the normal vector of time-like surface is

$$\vec{N} = \frac{\vec{t}_1 \wedge \vec{t}_2}{\left\|\vec{t}_1 \wedge \vec{t}_2\right\|} = \frac{\vec{t}_1 \wedge \vec{t}_2}{ch\theta} \tag{3.4}$$

Other then, considering the first two formulae of (3.2) in the third term ,

$$\vec{t} = \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds} = \vec{t}_1 \sqrt{E} \frac{du}{ds} + \vec{t}_2 \sqrt{G} \frac{dv}{ds} \tag{3.5}$$

is written,[1].

On the other hand, let us consider the hyperbolic angle between $\vec{t}$ and $\vec{t}_1$ as α, and if we take inner product both sides of (3.5) with $\vec{t}_1$ and $\vec{t}_2$ then

$$\vec{t}.\vec{t}_1 = -ch\alpha = -\sqrt{E}\frac{du}{ds} + sh\theta\sqrt{G}\frac{dv}{ds} \tag{3.6}$$

$$\vec{t}.\vec{t}_2 = sh(\theta-\alpha) = sh\theta\sqrt{E}\frac{du}{ds} + \sqrt{G}\frac{dv}{ds} \tag{3.7}$$

are obtained. Thus , from (3.6) and (3.7)

$$\begin{aligned} \frac{ch(\theta-\alpha)}{ch\theta} &= \sqrt{E}\frac{du}{ds} \\ -\frac{sh\alpha}{ch\theta} &= \sqrt{G}\frac{dv}{ds} \end{aligned} \tag{3.8}$$

are written. Finally, if we put (3.8) into (3.5), we have the following equation between the tangent vectors of the curves (c), (c_1) and (c_2)

$$\vec{t} = \frac{ch(\theta-\alpha)}{ch\theta}\vec{t}_1 - \frac{sh\alpha}{ch\theta}\vec{t}_2 \tag{3.9}$$

Here, we shall denote the arc elements ds, ds_1 and ds_2 of the parameter curves which are belongs to time-like surface $y = y(u,v)$, and then we express as follows:

$$\begin{aligned} ds^2 &= Edu^2 + 2Fdudv + Gdv^2 \\ ds_1^2 &= Edu^2 \\ ds_2^2 &= Gdv^2 \end{aligned} \tag{3.10}$$

Thus, considering (3.8) and (3.10), we have

$$\begin{aligned} \frac{ch(\theta-\alpha)}{ch\theta} &= \sqrt{E}\frac{du}{ds} = \frac{ds_1}{ds} \\ -\frac{sh\alpha}{ch\theta} &= \sqrt{G}\frac{dv}{ds} = \frac{ds_2}{ds} \end{aligned} \tag{3.11}$$

Corollary 3.1 : The third elements $\vec{g}$, $\vec{g}_1$ and $\vec{g}_2$ of the Darboux trihedrons $\left[\vec{t},\vec{g},\vec{N}\right]$ $\left[\vec{t}_1,\vec{g}_1,\vec{N}\right]$, $\left[\vec{t}_2,\vec{g}_2,\vec{N}\right]$ are linearly dependent :

Proof: If we substitute the equation (3.9) in the first equality of (2.3) and consider the Darboux trihedrons of (c_1) and (c_2) we have

$$\vec{g} = \frac{ch(\theta-\alpha)}{ch\theta}\vec{g}_1 + \frac{sh\alpha}{ch\theta}\vec{g}_2 \tag{3.12}$$

Thus, we get the expression .

Teorem 3.1: The Darboux trihedrons $\left[\vec{t}_1, \vec{g}_1, \vec{N}\right]$ and $\left[\vec{t}_2, \vec{g}_2, \vec{N}\right]$ of the parameters curves (c_1) and (c_2) of the time-like surface are written by Darboux instantaneous vectors as follows:

$$\frac{\partial \vec{t}_i}{\partial s_j} = \vec{w}_i \wedge \vec{t}_i \quad , \quad \frac{\partial \vec{g}_i}{\partial s_j} = \vec{w}_i \wedge \vec{g}_i \quad , \quad \frac{\partial \vec{N}}{\partial s_j} = \vec{w}_i \wedge \vec{N} \qquad (i, j = 1, 2) \tag{3.13}$$

Proof: If we consider the Darboux trihedrons $\left[\vec{t}_1, \vec{g}_1, \vec{N}\right]$ and $\left[\vec{t}_2, \vec{g}_2, \vec{N}\right]$ of parameter curves, we see that the normal vector N is coincide . Then, considering (3.4)

$$\vec{g}_1 = \vec{t}_1 \wedge \vec{N} = t_1 \wedge \left(\frac{\vec{t}_1 \wedge \vec{t}_2}{ch\theta}\right) = \left[\frac{\vec{t}_2(\vec{t}_1\vec{t}_1) - \vec{t}_1(\vec{t}_1\vec{t}_2)}{ch\theta}\right] = \left[\frac{-\vec{t}_1 sh\theta - \vec{t}_2}{ch\theta}\right] \tag{3.14}$$

$$\vec{g}_2 = \vec{N} \wedge \vec{t}_2 = \left(\frac{\vec{t}_1 \wedge \vec{t}_2}{ch\theta}\right) \wedge t_2 = \left[\frac{-\vec{t}_2(\vec{t}_1\vec{t}_2) + \vec{t}_1(\vec{t}_2\vec{t}_2)}{ch\theta}\right] = \left[\frac{-sh\theta\vec{t}_2 + \vec{t}_1}{ch\theta}\right] \tag{3.15}$$

are obtained. From (2.2), we write

$$\frac{\partial \vec{t}_1}{\partial s_1} = \vec{w}_1 \wedge \vec{t}_1 \quad , \quad \frac{\partial \vec{N}}{\partial s_1} = \vec{w}_1 \wedge \vec{N}_1 \quad , \quad \frac{\partial \vec{g}_1}{\partial s_1} = \vec{w}_1 \wedge \vec{g}_1 \tag{3.16}$$

$$\frac{\partial \vec{t}_2}{\partial s_1} = \vec{w}_2 \wedge \vec{t}_2 \quad , \quad \frac{\partial \vec{N}}{\partial s_2} = \vec{w}_2 \wedge \vec{N} \quad , \quad \frac{\partial \vec{g}_2}{\partial s_1} = \vec{w}_2 \wedge \vec{g}_2 \tag{3.17}$$

If (3.14) is substituted in the third equality of (3.16), we get

$$\frac{\partial \vec{g}_1}{\partial s_1} = \frac{\partial\left[\frac{1}{ch\theta}\left(-sh\theta\vec{t}_1 - \vec{t}_2\right)\right]}{\partial s_1} = \frac{1}{ch\theta}\left(-\frac{\partial \vec{t}_1}{\partial s_1}sh\theta - \frac{\partial \vec{t}_2}{\partial s_1}\right) = \frac{1}{ch\theta}\left(-(w_1 \wedge t_1)sh\theta - \frac{\partial \vec{t}_2}{\partial s_1}\right) \tag{3.18}$$

$$\frac{\partial \vec{g}_1}{\partial s_1} = w_1 \wedge g_1 = w_1 \wedge \left[\frac{1}{ch\theta}\left(-sh\theta\vec{t}_1 - \vec{t}_2\right)\right] = \frac{1}{ch\theta}\left[-(w_1 \wedge t_1)sh\theta - (w_1 \wedge t_2)\right] \tag{3.19}$$

Then, from (3.18) and (3.19), we have

$$\frac{\partial \vec{t}_2}{\partial s_1} = \vec{w}_1 \wedge \vec{t}_2 \tag{3.20}$$

Thus, the derivative of $\vec{t}_2$ with respect to s_1 is written by the Lorentzian cross product of $\vec{w}_1$ and $\vec{t}_2$.Similarly, it is easy to see that the other vectors can be written by the same method.

Corollary 3.2: By using the vectors $\vec{t}_1, \vec{t}_2$ and $\vec{N}$, **w**e can express $\vec{w}, \vec{w}_1$ and $\vec{w}_2$ as follows:

(3.21)

$$\vec{w}_1 = \vec{t}_1\left[\frac{1}{\left(T_g\right)_1} - \frac{sh\theta}{ch\theta\left(R_n\right)_1}\right] + \frac{\vec{t}_2}{ch\theta\left(R_n\right)_1} - \frac{\vec{N}}{\left(R_g\right)_1} \tag{3.22}$$

$$\vec{w}_2 = \left[-\frac{1}{ch\theta\left(T_g\right)_2}\vec{t}_1 + \left(-\frac{1}{\left(T_g\right)_2} + \frac{sh\theta}{ch\theta\left(R_n\right)_2}\right)\vec{t}_2\right] + \frac{\vec{N}}{\left(R_g\right)_2} \tag{3.23}$$

Proof: From (2.5), we can write the darboux vectors of the $\left[\vec{t}, \vec{g}, \vec{N}\right]$, $\left[\vec{t}_1, \vec{g}_1, \vec{N}\right]$ and $\left[\vec{t}_2, \vec{g}_2, \vec{N}\right]$ as

$$\begin{cases} \vec{w} = \dfrac{\vec{t}}{T_g} + \dfrac{\vec{g}}{R_n} - \dfrac{\vec{N}}{R_g} \\ \vec{w}_1 = \dfrac{\vec{t}_1}{\left(T_g\right)_1} + \dfrac{\vec{g}_1}{\left(R_n\right)_1} - \dfrac{\vec{N}}{\left(R_g\right)_1} \\ \vec{w}_2 = -\dfrac{\vec{t}_2}{\left(T_g\right)_2} - \dfrac{\vec{g}_2}{\left(R_n\right)_2} + \dfrac{\vec{N}}{\left(R_g\right)_2} \end{cases} \tag{3.24}$$

Then, if we consider the equations (3.9) , (3.14) and (3.15) according to the vectors t_1 and t_2 and substitute in (3.24) , we get (3.21),(3.22) and (3.23).

Teorem 3.2: If we consider the tangent vectors t_1 and t_2 of the parameter curves (c_1) and (c_2) on the time-like surface , then we obtain the following relations:

$$i) \quad \vec{t}_1 \frac{\partial \vec{t}_2}{\partial s_1} = -\vec{t}_2 \frac{\partial \vec{t}_1}{\partial s_1} = \frac{\left(\sqrt{E}\right)_v - sh\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}} \tag{3.25}$$

$$ii)\ \vec{t}_2 \frac{\partial \vec{t}_1}{\partial s_2} = -\vec{t}_1 \frac{\partial \vec{t}_2}{\partial s_2} = \frac{\left(\sqrt{G}\right)_u - sh\theta\left(\sqrt{E}\right)_v}{\sqrt{EG}} \tag{3.26}$$

Proof: i) From (3.2) , $\vec{t}_1 = \dfrac{\vec{r}_u}{\sqrt{E}}$ and $\vec{t}_2 = \dfrac{\vec{r}_v}{\sqrt{G}}$ are written . And also, we know that

$$\vec{t}_1\vec{t}_2 = sh\theta \quad \Rightarrow \quad \vec{r}_u\vec{r}_v = sh\theta\sqrt{E}\sqrt{G} \tag{3.27}$$

$$E = \left(\sqrt{E}\right)^2 = \vec{r}_u^{\,2} \quad \Rightarrow \quad \sqrt{E}\left(\sqrt{E}\right)_v = \vec{r}_{uv}\vec{r}_u \tag{3.28}$$

$$G = \left(\sqrt{G}\right)^2 = \vec{r}_v^{\,2} \quad \Rightarrow \quad \sqrt{G}\left(\sqrt{G}\right)_u = \vec{r}_{vu}\vec{r}_v \tag{3.29}$$

By taking differential from $\vec{t}_1 = \dfrac{\vec{r}_u}{\sqrt{E}}$ and $\vec{t}_2 = \dfrac{\vec{r}_v}{\sqrt{G}}$, we obtain

$$\frac{\partial \vec{t}_1}{\partial v} = \frac{\vec{r}_{uv}\left(\sqrt{E}\right) - \left(\sqrt{E}\right)_v \vec{r}_u}{E}$$

$$\frac{\partial \vec{t}_2}{\partial u} = \frac{\vec{r}_{uv}\left(\sqrt{G}\right) - \left(\sqrt{G}\right)_u \vec{r}_v}{G}$$

Thus, we write

$$\vec{t}_2 \frac{\partial \vec{t}_1}{\partial v} = \frac{\vec{r}_v}{\sqrt{G}}\left(\frac{\vec{r}_{uv}\left(\sqrt{E}\right) - \left(\sqrt{E}\right)_v \vec{r}_u}{E}\right) = \frac{\left(\sqrt{G}\right)_u - sh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}} \tag{3.30}$$

$$\vec{t}_1 \frac{\partial \vec{t}_2}{\partial u} = \frac{\vec{r}_u}{\sqrt{E}}\left(\frac{\vec{r}_{uv}\left(\sqrt{G}\right) - \left(\sqrt{G}\right)_u \vec{r}_v}{G}\right) = \frac{\left(\sqrt{E}\right)_v - sh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}} \tag{3.31}$$

On the other hand, we have

$$\frac{\partial \vec{t}_1}{\partial s_2} = \frac{\partial \vec{t}_1}{\partial v} . \frac{\partial v}{\partial s_2} = \frac{1}{\sqrt{G}}\frac{\partial \vec{t}_1}{\partial v} \quad (ds_2 = \sqrt{G}dv) \Rightarrow \frac{\partial v}{\partial s_2} = \frac{1}{\sqrt{G}} \tag{3.32}$$

$$\frac{\partial \vec{t}_2}{\partial s_2} = \frac{\partial \vec{t}_2}{\partial u} . \frac{\partial u}{\partial s_1} = \frac{1}{\sqrt{E}}\frac{\partial \vec{t}_2}{\partial u} \quad (ds_1 = \sqrt{E}dv) \Rightarrow \frac{\partial u}{\partial s_1} = \frac{1}{\sqrt{E}} \tag{3.33}$$

Thus, taking inner product of (3.32) and (3.33) by the vector $\vec{t}_2$ and the vector $\vec{t}_1$, and considering (3.30) and (3.31) , we have (3.25) and (3.26).
The other cases can be seen easily.

Result 3.1: If we take differential from $\vec{t}_1\vec{t}_2 = sh\theta$ with respect to u and v , we get

$$\vec{t}_1\frac{\partial \vec{t}_2}{\partial v}=-\vec{t}_2\frac{\partial \vec{t}_1}{\partial v}=-\left(\frac{\left(\sqrt{G}\right)_u-sh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right) \tag{3.34}$$

$$\vec{t}_1\frac{\partial \vec{t}_2}{\partial u}=\frac{\left(\sqrt{E}\right)_v-sh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}} \tag{3.35}$$

Thus, we have

$$\begin{aligned}\frac{\partial}{\partial v}\left(\vec{t}_1\frac{\partial \vec{t}_2}{\partial u}\right)-\frac{\partial}{\partial u}\left(\vec{t}_1\frac{\partial \vec{t}_2}{\partial v}\right)=\frac{\partial}{\partial v}&\left(\frac{\left(\sqrt{E}\right)_v-sh\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right)\\ &+\frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_v-sh\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right)\end{aligned} \tag{3.36}$$

Theorem 3.3: The following geodesic curvature equalities are satisfied for the parameter curves (c_1) and (c_2)

$$\frac{1}{(R_g)_1}=\frac{1}{ch\theta\sqrt{EG}}\left(-sh\theta\left(\sqrt{G}\right)_u+\left(\sqrt{E}\right)_v\right) \tag{3.37}$$

$$\frac{1}{(R_g)_2}=\frac{1}{ch\theta\sqrt{EG}}\left(\left(\sqrt{G}\right)_u-sh\theta\left(\sqrt{E}\right)_v\right) \tag{3.38}$$

Proof: i) From (3.25) and (3.20), we write

$$\vec{t}_1\frac{\partial \vec{t}_2}{\partial s_1}=\left(\frac{\left(\sqrt{E}\right)_v-sh\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}}\right)$$

$$\frac{\partial \vec{t}_2}{\partial s_1}=w_1\wedge t_2$$

From here, we have

$$\frac{\left(\sqrt{E}\right)_v-sh\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}}=\vec{t}_1.\left(\vec{w}_1\wedge\vec{t}_2\right)=w_1.\left(\vec{t}_2\wedge\vec{t}_1\right)=-ch\theta\vec{N}.\vec{w}_1$$

Then , from (3.24), if we take inner product both of side $\vec{w}_1$ with $-ch\theta\vec{N}$ we obtain

$$-ch\theta\vec{N}\,\vec{w}_1 = \frac{ch\theta\vec{N}^2}{(R_g)_1} \Rightarrow \vec{N}\,\vec{w}_1 = \frac{-1}{(R_g)_1}$$

Thus,

$$\frac{1}{(R_g)_1} = \frac{1}{ch\theta\sqrt{EG}}.\left(-sh\theta(\sqrt{G})_u + (\sqrt{E})_v\right)$$

is obtained. Similarly, (ii) can be proofed.

Theorem 3.4: Let us consider any time-like curve (c) on the time-like surface $y = y(u,v)$ and the arc elements of curves (c), (c_1) and (c_2) as s, s_1 and s_2, respectively. Let the Darboux instantaneous rotation vectors of (c_1) and (c_2) be $\vec{w}_1$, $\vec{w}_2$, and if the hyperbolic angle between the tangent $\vec{t}$ of curve (c) and $\vec{t}_1$ is α, then

$$\left(\frac{ch(\theta-\alpha)}{ch\theta}\vec{w}_1 - \frac{sh\alpha}{ch\theta}\vec{w}_2\right)\wedge\vec{t}_1 = \vec{A}\wedge\vec{t}_1 \tag{3.39}$$

and

$$\frac{d\vec{t}_1}{ds} = \vec{A}\wedge\vec{t}_1 \quad , \quad \frac{d\vec{t}_2}{ds} = \vec{A}\wedge\vec{t}_2 \quad , \quad \frac{d\vec{N}}{ds} = \vec{A}\wedge\vec{N} \tag{3.40}$$

are satisfied.

Proof: If we consider (3.11) and (3.13), then

$$\frac{d\vec{t}_1}{ds} = \frac{\partial\vec{t}_1}{\partial s_1}\frac{ds_1}{ds} + \frac{\partial\vec{t}_1}{\partial s_2}\frac{ds_2}{ds} = \frac{ch(\theta-\alpha)}{ch\theta}\left(\vec{w}_1\wedge\vec{t}_1\right) + \frac{sh\alpha}{ch\theta}\left(\vec{w}_2\wedge\vec{t}_1\right)$$

$$= \left(\frac{ch(\theta-\alpha)}{ch\theta}\vec{w}_1 - \frac{sh\alpha}{ch\theta}\vec{w}_2\right)\wedge\vec{t}_1 = \vec{A}\wedge\vec{t}_1$$

is obtained. Similarly, the others are satisfied.

Result 3.2: The following equality

$$\vec{t}_2\frac{d\vec{t}_1}{ds} = -\vec{t}_1\frac{d\vec{t}_2}{ds} = ch\theta\,\vec{A}.\vec{N} \tag{3.41}$$

is valid.

Theorem 3.5: Let us consider the curves (c) , (c_1) and (c_2) which intersect a point P on time-like surface. Let the Darboux instantaneous rotation vectors of these curves at the point P be $\vec{w}$, $\vec{w}_1$ and $\vec{w}_2$, respectively. we obtain following equality between the darboux vectors $\vec{w}$, $\vec{w}_1$ and $\vec{w}_2$.

$$\vec{w}=\frac{ch(\theta-\alpha)}{ch\theta}\vec{w}_1-\frac{sh\alpha}{ch\theta}\vec{w}_2+\vec{N}\frac{d\alpha}{ds} \tag{3.42}$$

Proof: From (3.9)

$$\vec{t}=\frac{ch(\theta-\alpha)}{ch\theta}\vec{t}_1-\frac{sh\alpha}{ch\theta}\vec{t}_2 \tag{3.43}$$

can be written. Then, by taking derivatives with respect to s from equation (3.43), we obtain

$$\frac{d\vec{t}}{ds}=\frac{ch(\theta-\alpha)}{ch\theta}\frac{d\vec{t}_1}{ds}-\frac{sh\alpha}{ch\theta}\frac{d\vec{t}_2}{ds}-\left[\frac{sh(\theta-\alpha)}{ch\theta}\vec{t}_1+\frac{ch\alpha}{ch\theta}\vec{t}_2\right]\frac{d\alpha}{ds} \tag{3.44}$$

On the other hand, considering the Darboux trihedrons $\left[\vec{t}_1,\vec{g}_1,\vec{N}\right]$ and $\left[\vec{t}_2,\vec{g}_2,\vec{N}\right]$**,** we write

$$\vec{t}_1=\vec{g}_1\wedge\vec{N} \quad , \quad \vec{t}_2=-\vec{g}_2\wedge\vec{N} \tag{3.45}$$

From (3.14) and (3.15) , if $\vec{g}_1$ and $\vec{g}_2$ are substituted in (3.45) we obtain

$$\vec{t}_1=\frac{1}{ch\theta}\left(-sh\theta\,\vec{t}_1-\vec{t}_2\right)\wedge N \qquad \vec{t}_2=\frac{1}{ch\theta}\left(sh\theta\,\vec{t}_2-\vec{t}_1\right)\wedge N \tag{3.46}$$

And then, substituting the equations (3.46) in (3.44), we have

$$\frac{d\vec{t}}{ds}=\frac{ch(\theta-\alpha)}{ch\theta}\frac{d\vec{t}_1}{ds}-\frac{sh\alpha}{ch\theta}\frac{d\vec{t}_2}{ds}-\left[\frac{sh(\theta-\alpha)}{ch^2\theta}\left(-sh\theta\,\vec{t}_1-\vec{t}_2\right)\wedge N+\frac{ch\alpha}{ch^2\theta}\left(-\vec{t}_1+\vec{t}_2\,sh\theta\right)\wedge N\right]\frac{d\alpha}{ds}$$

According to the theorem 3.4 ,

$$\frac{d\vec{t}_1}{ds}=\vec{A}\wedge\vec{t}_1 \quad , \quad \frac{d\vec{t}_2}{ds}=\vec{A}\wedge\vec{t}_2 \quad , \quad \vec{A}=\frac{ch(\theta-\alpha)}{ch\theta}\vec{w}_1-\frac{sh\alpha}{ch\theta}\vec{w}_2$$

are known. And, by using the trigonometric expression, we find

$$\frac{d\vec{t}}{ds}=\frac{ch(\theta-\alpha)}{ch\theta}\vec{A}\wedge\vec{t}_1-\frac{sh\alpha}{ch\theta}\vec{A}\wedge\vec{t}_2+\frac{d\alpha}{ds}\left[\frac{ch(\theta-\alpha)}{ch\theta}\vec{t}_1-\frac{sh\alpha}{ch\theta}\vec{t}_2\right]\wedge N$$
$$=\vec{A}\wedge\left(\frac{ch(\theta-\alpha)}{ch\theta}\vec{t}_1-\frac{sh\alpha}{ch\theta}\vec{t}_2\right)+\frac{d\alpha}{ds}\left[\frac{ch(\theta-\alpha)}{ch\theta}\vec{t}_1-\frac{sh\alpha}{ch\theta}\vec{t}_2\right]\wedge N$$
$$=\vec{A}\wedge\vec{t}-(\vec{N}\wedge\vec{t}\,)\frac{d\alpha}{ds}$$
$$=\left[\vec{A}-\vec{N}\frac{d\alpha}{ds}\right]\wedge\vec{t}$$

Thus,

$$\frac{d\vec{t}}{ds}=\vec{b}\wedge\vec{t} \tag{3.47}$$
$$\vec{b}=\left[\vec{A}-\vec{N}\frac{d\alpha}{ds}\right] \tag{3.48}$$

are written. After that,

$$\vec{A}=\vec{b}+\vec{N}\frac{d\alpha}{ds} \tag{3.49}$$

is obtained. By writing (3.49) in the third expression of (3.40) we obtain

$$\frac{d\vec{N}}{ds}=\vec{A}\wedge\vec{N}=\left(\vec{b}+\frac{d\alpha}{ds}\vec{N}\right)\wedge N=\vec{b}\wedge\vec{N} \tag{3.50}$$

Since $\vec{w}$ is Darboux vector, we have

$$\frac{d\vec{t}}{ds}=\vec{w}\wedge t \qquad , \qquad \frac{d\vec{N}}{ds}=\vec{w}\wedge\vec{N} \tag{3.51}$$

Then, considering (2.4) , (3.47) , (3.50) and (3.51)

$$\frac{d\vec{t}}{ds}=\vec{w}\wedge\vec{t}=\vec{b}\wedge\vec{t}\Rightarrow 0=\vec{b}\wedge\vec{t}-\vec{w}\wedge\vec{t}=\left(\vec{b}-\vec{w}\right)\wedge\vec{t}$$
$$\vec{b}-\vec{w}=\lambda\vec{t} \tag{3.52}$$

$$\frac{d\vec{N}}{ds}=\vec{w}\wedge\vec{N}=\vec{b}\wedge\vec{N}\Rightarrow 0=\vec{b}\wedge\vec{N}-\vec{w}\wedge\vec{N}=\left(\vec{b}-\vec{w}\right)\wedge\vec{N}$$

$$\vec{b} - \vec{w} = \mu \vec{N} \tag{3.53}$$

are written. At the end, if we make equal (3.52) to (3.53), we have

$$\vec{b} - \vec{w} = \lambda t = \mu \vec{N} \quad \Rightarrow \quad \lambda = \mu = 0$$

Finally, $\vec{b} - \vec{w} = \vec{0}$ can be written. Thus, we get the theorem.

4. REFERENCES

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