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FIBONACCI COLLOCATION METHOD FOR SOLVING LINEAR DIFFERENTIAL - DIFFERENCE EQUATIONS

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Abstract- This study presents a new method for the solution of *mth*-order linear differential-difference equations with variable coefficients under the mixed conditions. We introduce a Fibonacci collocation method based on the Fibonacci polynomials for the approximate solution. Numerical examples are included to demonstrate the applicability of the technique. The obtained results are compared by the known results.

Key Words- Fibonacci polynomials, Fibonacci collocation method, differential-difference equations, collocation points.

1. INTRODUCTION

The study of the differential-difference equations developed very rapidly in recent years [1-3]. These equations play an important role in various branches of science such as engineering, mechanics, physics, biology, control theory etc. Differential-difference equations occur also frequently as a mathematical model for problems [3-4].

Since some equations are hard to solve analytically, they are solved by using the approximate methods by many authors [5-7]. Approximate solutions of linear differential, difference, differential-difference, integral and integro-differential-difference, pantograph equations have been found using the Taylor collocation method and Chebyshev polynomial method by Sezer et. al. [8-14]. Also, the Fibonacci matrix method has been used to find the approximate solutions of differential and integro-differential equations [15].

In this paper, we consider the approximate solution of the *mth*-order linear differential-difference equation with variable coefficients,

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x)+\sum_{r=0}^{s} Q_r(x)y^{(r)}(\mu_r x+\tau_r)=g(x),\ s\le m,\ \tau_r \text{ are the integer} \tag{1}$$

with the conditions

$$\sum_{k=0}^{m-1}[a_{jk}y^{(k)}(a)+b_{jk}y^{(k)}(b)]=\lambda_j,\quad j=1,2,3,...,m,\quad a\le x\le b \tag{2}$$

where $P_k(x)$, $Q_r(x)$ and $g(x)$ are functions defined on $a\le x\le b$; a_{jk}, b_{jk} and λ_j are suitable constants.

Our aim is to find an approximate solution of (1) expressed in the truncated Fibonacci series form

$$y(x)=\sum_{n=1}^{N} a_n F_n(x) \tag{3}$$

where a_n, $n=1,2,3,...N$ are the unknown Fibonacci coefficients. Here n is chosen as any positive integer such that $n \geq 1$, and $F_n(x)$, $n=1,2,3,...N$ are the Fibonacci polynomials defined by

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}, \quad [(n-1)/2] = \begin{cases} (n-2)/2, & n \text{ even} \\ (n-1)/2, & n \text{ odd}. \end{cases}$$

2. FUNDAMENTAL MATRIX RELATIONS

We can write the Fibonacci polynomials $F_n(x)$ in the matrix form as follows

$$\boldsymbol{F}^T(x) = \boldsymbol{C}\boldsymbol{X}^T(x) \Leftrightarrow \boldsymbol{F}(x) = \boldsymbol{X}(x)\boldsymbol{C}^T \tag{4}$$

where

$$\boldsymbol{F}(x) = [F_1(x) \ \mathrm{F}_2(x) \ \mathrm{F}_3(x) \ ... \ \mathrm{F}_N(x)], \ \boldsymbol{X}(x) = [1 \ x \ x^2 \ \ldots \ x^{N-1}]$$

and if n is even,

$$\boldsymbol{C} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0 \\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{(n-2)/2}{(n-2)/2} & 0 & \binom{n/2}{(n-4)/2} & 0 & \cdots & 0 \\ 0 & \binom{n/2}{(n-2)/2} & 0 & \binom{(n+2)/2}{(n-4)/2} & \cdots & \binom{n-1}{0} \end{bmatrix}_{NxN}$$

if n is odd,

$$\boldsymbol{C} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0 \\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \binom{(n-1)/2}{(n-3)/2} & 0 & \binom{(n+1)/2}{(n-5)/2} & \cdots & 0 \\ \binom{(n-1)/2}{(n-1)/2} & 0 & \binom{(n+1)/2}{(n-3)/2} & \cdots & \cdots & \binom{n-1}{0} \end{bmatrix}_{NxN}.$$

Let us show Eq(1) in the form

$$P(x) + Q(x) = g(x) \tag{5}$$

where $P(x)=\sum_{k=0}^{m}P_k(x)y^{(k)}(x)$, $Q(x)=\sum_{r=0}^{s}Q_r(x)y^{(r)}(\mu_r x+\tau_r)$.

2.1. Matrix relations for the differential part P(x)

We write the solution $y(x)$ and its kth derivate $y^{(k)}(x)$ in the matrix forms, respectively,

$$\boldsymbol{y}(x)=\boldsymbol{F}(x)\boldsymbol{A},\quad \boldsymbol{A}=\begin{bmatrix}a_1 & a_2 & \cdots & a_N\end{bmatrix}^T \tag{6}$$

and

$$\boldsymbol{y}^{(k)}(x)=\boldsymbol{F}^{(k)}(x)\boldsymbol{A}. \tag{7}$$

Then, from relations (4) and (6), we can obtain matrix form

$$\boldsymbol{y}(x)=\boldsymbol{X}(x)\boldsymbol{C}^T\boldsymbol{A} \tag{8}$$

Similar to Eq. (8), from relations (4), (6) and (7), we can obtain $\boldsymbol{y}^{(k)}(x)$ matrix form as

$$\boldsymbol{y}^{(k)}(x)=\boldsymbol{X}^{(k)}(x)\boldsymbol{C}^T\boldsymbol{A} \tag{9}$$

To find the matrix $\boldsymbol{X}^{(k)}(x)$ in terms of the matrix $\boldsymbol{X}(x)$, we can use the following relation

$$\boldsymbol{X}^{(1)}(x)=\boldsymbol{X}(x)\boldsymbol{T}^T$$

$$\boldsymbol{X}^{(2)}(x)=\boldsymbol{X}^{(1)}(x)\boldsymbol{T}^T=(\boldsymbol{X}(x)\boldsymbol{T}^T)\boldsymbol{T}^T=\boldsymbol{X}(x)(\boldsymbol{T}^T)^2$$

$$\vdots$$

$$\boldsymbol{X}^{(k)}(x)=\boldsymbol{X}(x)(\boldsymbol{T}^T)^k \tag{10}$$

where

$$\boldsymbol{T}^T=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 2 & \cdots & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & N-1\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}.$$

Consequently, by substituting the matrix form (10) into Eq.(9), we obtain the matrix relations

$$\boldsymbol{y}^{(k)}(x)=\boldsymbol{X}(x)(\boldsymbol{T}^T)^k\boldsymbol{C}^T\boldsymbol{A}. \tag{11}$$

2.2. Matrix relations for the difference part Q(x)

If we put $x\to\mu_r x+\tau_r$ in the relation (6), we have the matrix form

$$\boldsymbol{y}(\mu_r x+\tau_r)=\boldsymbol{F}(\mu_r x+\tau_r)\boldsymbol{A}. \tag{12}$$

Also, it is seen that the relation between the matrices $\boldsymbol{X}(x)$ and $\boldsymbol{X}(\mu_r x+\tau_r)$ is

$$\boldsymbol{X}(\mu_r x+\tau_r)=\boldsymbol{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r) \tag{13}$$

where

$$\boldsymbol{\beta}(\mu_r,\tau_r)=\begin{bmatrix} \binom{0}{0}\mu_r^{\,0}\tau_r^{\,0} & \binom{1}{0}\mu_r^{\,0}\tau_r^{\,1} & \cdots & \binom{N-1}{0}\mu_r^{\,0}\tau_r^{\,N-1} \\ 0 & \binom{1}{1}\mu_r^{\,1}\tau_r^{\,0} & \cdots & \binom{N-1}{1}\mu_r^{\,1}\tau_r^{\,N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N-1}{N-1}\mu_r^{\,N-1}\tau_r^{\,0} \end{bmatrix}.$$

By using the relations (10) and (13), we can get

$$\boldsymbol{X}^{(k)}(\mu_r x+\tau_r)=\boldsymbol{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^k. \tag{14}$$

Thus from (9) and (14), we can find

$$\boldsymbol{y}^{(k)}(\mu_r x+\tau_r)=\boldsymbol{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^k\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}. \tag{15}$$

By using the expressions (9) and (15), we obtain the matrix forms

$$P(x)=\sum_{k=0}^{m}P_k(x)\boldsymbol{X}(x)(\boldsymbol{T}^{\boldsymbol{T}})^k\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A} \tag{16}$$

$$Q(x)=\sum_{r=0}^{s}Q_r(x)\boldsymbol{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^r\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}\;. \tag{17}$$

2.3. Matrix relations for the conditions

By means of (11), the corresponding matrix forms for the conditions (2) can be shown as

$$\sum_{k=0}^{m-1}[a_{jk}\boldsymbol{X}(a)+b_{jk}\boldsymbol{X}(b)](\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{k}}\boldsymbol{C}^T\boldsymbol{A}=\lambda_j,\;\; j=1,2,3,...,m. \tag{18}$$

3. METHOD OF SOLUTION

In this part, we construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, we substitute the matrix relations (16) and (17) into Eq. (5). So we obtain the matrix equation

$$\sum_{k=0}^{m}P_k(x)\boldsymbol{X}(x)\boldsymbol{C}^{\boldsymbol{T}}(\boldsymbol{T}^{\boldsymbol{T}})^k\boldsymbol{A}+\sum_{r=0}^{s}Q_r(x)\boldsymbol{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^r\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}=\left[g(x)\right]. \tag{19}$$

By using in Eq. (19) the collocation points x_i defined by,

$$x_i=a+\left(\frac{b-a}{N-1}\right)(i-1),\quad i=1,2,\ldots,N, \tag{20}$$

the system of the matrix equations is obtained as

$$\sum_{k=0}^{m}P_k(x_i)\boldsymbol{X}(x_i)\boldsymbol{C}^{\boldsymbol{T}}(\boldsymbol{T}^{\boldsymbol{T}})^k\boldsymbol{A}+\sum_{r=0}^{s}Q_r(x_i)\boldsymbol{X}(x_i)\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^r\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{A}=\left[g(x_i)\right]$$

or shortly the fundamental matrix equation becomes

$$\left\{\sum_{k=0}^{m}\boldsymbol{P_k}\boldsymbol{X}\boldsymbol{C}^{\boldsymbol{T}}(\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{k}}+\sum_{r=0}^{s}\boldsymbol{Q_r}\boldsymbol{X}\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{r}}\boldsymbol{C}^{\boldsymbol{T}}\right\}\boldsymbol{A}=\boldsymbol{G} \tag{21}$$

where

$$\boldsymbol{P_k}=\begin{bmatrix} P_k(x_1) & 0 & \cdots & 0 \\ 0 & P_k(x_2) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix} \quad \boldsymbol{Q_r}=\begin{bmatrix} Q_r(x_1) & 0 & \cdots & 0 \\ 0 & Q_r(x_2) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & Q_r(x_N) \end{bmatrix}$$

$$\boldsymbol{X(x)}=\begin{bmatrix} X(x_1) \\ X(x_2) \\ \vdots \\ X(x_N) \end{bmatrix}=\begin{bmatrix} 1 & x_1 & \cdots & x_1^{N-1} \\ 1 & x_2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^{N-1} \end{bmatrix} \quad \boldsymbol{G}=\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}.$$

Therefore, the fundamental matrix equation (21) corresponding to Eq. (1) can be written in the augmented form

$$\boldsymbol{WA=G} \text{ or } [\boldsymbol{W;G}] \tag{22}$$

where

$$\boldsymbol{W}=\sum_{k=0}^{m}\boldsymbol{P_k X}(\boldsymbol{T}^{\boldsymbol{T}})^k\boldsymbol{C}^{\boldsymbol{T}}+\sum_{r=0}^{s}\boldsymbol{Q_r X}\boldsymbol{\beta}(\mu_r,\tau_r)(\boldsymbol{T}^{\boldsymbol{T}})^r\boldsymbol{C}^{\boldsymbol{T}}.$$

Eq. (22) corresponds to a system of N linear algebraic equations with unknown Fibonacci coefficients a_1,a_2 ,..., a_N. Further, we can express the matrix form (18) for conditions as

$$\boldsymbol{U_j A}=[\lambda_j] \text{ or } [\boldsymbol{U_j};\lambda_j],\ j=1,2,3,...,m \tag{23}$$

where

$$\boldsymbol{U_j}=\sum_{k=0}^{m-1}[a_{jk}\boldsymbol{X}(a)+b_{jk}\boldsymbol{X}(b)](\boldsymbol{T}^{\boldsymbol{T}})^{\boldsymbol{k}}C^T=\begin{bmatrix} u_{j1} & u_{j2} & u_{j3} & \cdots & u_{jN} \end{bmatrix}$$

To obtain the solution of Eq. (1) under the conditions (2), by replacing the row matrices (23) by the last m rows of the matrices (22), we have the new augmented matrix

$$\boldsymbol{\tilde{W}A}=\boldsymbol{\tilde{G}}.$$

If the last m rows of the (22) are replaced, the augmented matrix of the above system is obtained as follows

$$[\boldsymbol{\tilde{W}};\boldsymbol{\tilde{G}}]=\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ w_{21} & w_{22} & \cdots & w_{2N} & ; & g(x_2) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{21} & u_{22} & \cdots & u_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mN} & ; & \lambda_m \end{bmatrix}. \tag{24}$$

If rank $\boldsymbol{\tilde{W}}=rank\ [\boldsymbol{\tilde{W};\tilde{G}}]=N$, then we can write

$$\boldsymbol{A}=(\boldsymbol{\tilde{W}})^{-1}\boldsymbol{\tilde{G}}. \tag{25}$$

Hence, the matrix $\boldsymbol{A}$ (thereby the coefficients $a_1,a_2,...,a_N$) is uniquely determined. Further the Eq. (1) with conditions (2) has a unique solution. This solution is given by the truncated Fibonacci series (3).

4. ACCURACY OF SOLUTION

We can check the accuracy of the method. The truncated Fibonacci series in (3) have to be approximately satisfying Eq. (1). For each $x = x_i \in [a,b],\ i = 1,2,3,...$

$$E(x_i) = \left|P(x_i) - Q(x_i) - g(x_i)\right| \cong 0$$

or

$$E(x_i) \le 10^{-k_i} \qquad (k_i \text{ is any positive integer})$$

If $\max(10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

5. NUMERICAL EXAMPLES

In this section, several numerical examples are given to show the accuracy and the efficiency of this method.

Example 5.1. [6] Let us first consider the linear differential-difference equation given by

$$y''(x) - xy'(x-1) + y(x-2) = -x^2 - 2x + 5,\ -2 \le x \le 0 \tag{26}$$

with the conditions

$$y(0) = -1,\ y'(-1) = -2$$

and the approximate solution $y(x)$ by the truncated Fibonacci series

$$y(x) = \sum_{n=1}^{4} a_n F_n(x).$$

So that $P_2(x) = 1,\ Q_0(x) = 1, Q_1(x) = -x,\ g(x) = -x^2 - 2x + 5$

$$\mu_0 = 1,\ \tau_0 = 2,\ \mu_1 = 1,\ \tau_1 = 1.$$

From Eq. (20), the collocation points (20) for $N = 4$ is computed

$$\left\{x_1 = -2,\ x_2 = -\frac{4}{3},\ x_3 = -\frac{2}{3},\ x_4 = 0\right\}$$

and from Eq. (21), the fundamental matrix equation of the problem becomes

$$\left\{\boldsymbol{P_2 X(T^T)^2 C^T + Q_0 X\beta(1,\text{-}2)(T^T)^0 C^T + Q_1 X\beta(1,\text{-}1)(T^T)^0 C^T}\right\}\boldsymbol{A} = \boldsymbol{G}$$

where

$$\boldsymbol{P_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Q}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Q}_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{T}^T=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&0\\0&0&0&3\end{bmatrix}\quad \boldsymbol{C}^T=\begin{bmatrix}1&0&1&0\\0&1&0&2\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad \boldsymbol{G}=\begin{bmatrix}5\\53/9\\53/9\\5\end{bmatrix},$$

$$\boldsymbol{\beta}(1,\text{-}2)=\begin{bmatrix}1&-2&4&-8\\0&1&-4&12\\0&0&1&-6\\0&0&0&1\end{bmatrix}\quad \boldsymbol{\beta}(1,\text{-}1)=\begin{bmatrix}1&-1&1&-1\\0&1&-2&3\\0&0&1&-3\\0&0&0&1\end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated

$$[\boldsymbol{W};\boldsymbol{G}]=\begin{bmatrix}1&-2&7&-26&;&5\\1&-2&71/9&-736/27&;&53/9\\1&-2&71/9&-578/27&;&53/9\\1&-2&7&-12&;&5\end{bmatrix}.$$

From Eq. (23), the matrix forms for the conditions are

$$[\boldsymbol{U_0};\lambda_0]=[1\quad 0\quad 1\quad 0\quad ;\quad -1],$$
$$[\boldsymbol{U_1};\lambda_1]=[0\quad 1\quad -2\quad 5\quad ;\quad -2].$$

The new augmented matrix based on conditions can be calculated as

$$[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{G}}]=\begin{bmatrix}1&-2&7&-26&;&5\\1&-2&71/9&-736/27&;&53/9\\1&0&1&0&;&-1\\0&1&-2&5&;&-2\end{bmatrix}.$$

Solving this system, Fibonacci coefficients are obtained as

$$\boldsymbol{A}=[-2\quad 0\quad 1\quad 0]^T.$$

Hence, by substituting the Fibonacci coefficients matrix into Eq. (6),

$$y(x)=\sum_{n=1}^{4}a_nF_n(x)=a_1F_1(x)+a_2F_2(x)+a_3F_3(x)+a_4F_4(x)$$
$$=\text{-}2.1+0.x+1.(x^2+1)+0.(x^3+2x)$$
$$=x^2-1.$$

We obtain the solution $y(x)=x^2-1$, which is the exact solution.

Example 5.2. [5] Consider the following linear differential-difference equation given by

$$y''(x)+xy'(x)+xy(x)+y'(x-1)+y(x-1)=e^{-x},\ -2\le x\le 0 \tag{27}$$

with the conditions

$$y(0)=1,\ y'(0)=-1$$

So that $P_2(x)=1,\ P_1(x)=x,\ P_0(x)=x,\ Q_0(x)=1,\ Q_1(x)=1,\ g(x)=e^{-x}$

$$\mu_0=1,\ \tau_0=-1,\ \mu_1=1,\ \tau_1=-1.$$

The solutions obtained for $N = 10,11,12$ are compared with the exact solution is e^{-x}, which are given in Fig 1. We compare the numerical solution and absolute errors for $N = 10,11,12$ in Table 1.

Table 1. Comparison of the absolute errors of Example 5.2

		Present method					
		$N = 10$		$N = 11$		$N = 12$	
x_i	Exact Solution	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors
-2	7.389056	7.37577	0.01328584	7.382423	0.006632786	7.387371	0.001684938
-1,8	6.049648	6.040341	0.009306033	6.045003	0.004644495	6.048077	0.001570507
-1.6	4.953032	4.946573	0.006459052	4.94981	0.003222258	4.951569	0.001463605
-1.4	4.055200	4.050788	0.004412089	4.05300	0.002199794	4.05384	0.001359659
-1.2	3.320117	3.317182	0.002935219	3.318655	0.001462234	3.318866	0.001251266
-1	2.718282	2.716411	0.001870547	2.717351	0.0009307117	2.717153	0.001128723
-0.8	2.225541	2.22443	0.001110914	2.224989	0.0005517244	2.22456	0.000981122
-0.6	1.822119	1.821534	0.000584719	1.821829	0.0002895305	1.821321	0.000798193
-0.4	1.491825	1.49158	0.000244463	1.491704	0.0001204192	1.491252	0.000572870
-0.2	1.221403	1.221345	0.000057666	1.221375	0.0000281227	1.221099	0.000304212
0,0	1.0	1.0	0.0	1.0	0.0	1.0	0.0

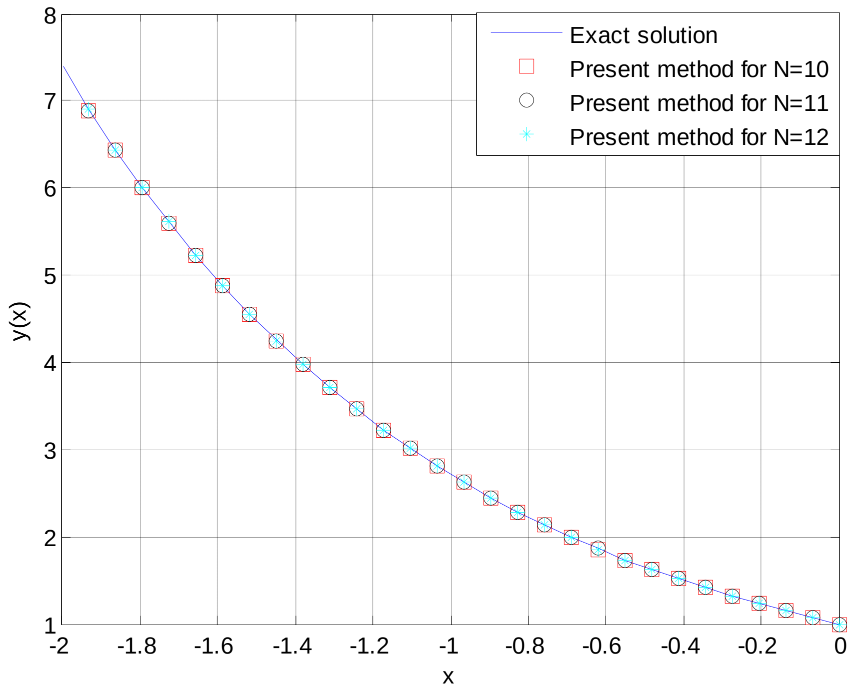

Figure 1. Numerical and exact solution of Example 5.2 for $N = 10,11,12$

Example 5.3. [6] We now consider the problem

$$y'''(x)-\cos(x)y'(x)-\sin(x)y'\left(x-\frac{\pi}{2}\right)+\sqrt{2}y\left(x-\frac{\pi}{4}\right)=\sin(x)-2\cos(x)-1 \quad (28)$$

with the conditions $y(0)=0,\ y'(0)=1,\ y''(0)=0,\ -\frac{\pi}{2}\leq x\leq 0.$

So that, $P_3(x)=1,\ P_1(x)=-\cos(x),\ \ Q_0(x)=\sqrt{2},\ Q_1(x)=-\sin(x),$

$$g(x)=\sin(x)\text{-}2\cos(x)\text{-}1,\ \mu_0=1,\ \ \tau_0=-\pi/4,\ \ \mu_1=1,\ \ \tau_1=-\pi/2.$$

The solutions obtained for $N=6,9,11$ are compared with the exact solution is $\sin(x)$ which are given in Fig 2. Also, we compare the numerical solution and absolute errors for $N=6,9,11$ in Table 2.

Table 2. Comparison of the absolute errors of Example 5.3

		Present method					
		$N=6$		$N=9$		$N=11$	
x_i	Exact Solution	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors	$y(x_i)$	Absolute Errors
-π/2	-1.0	-0.8533821	0.1466179	-0.9998634	1.366421E-4	-1.000001	1.201886E-6
-5π/12	-0.9659258	-0.870813	0.09511279	-0.9658354	9.047086E-5	-0.9659267	8.489779E-7
-π/3	-0.8660254	-0.8110107	0.05501466	-0.8659708	5.465341E-5	-0.8660259	5.132985E-7
-π/4	-0.7071068	-0.6807406	0.02636616	-0.7070786	2.819873E-5	-0.707107	2.575395E-7
-π/6	-0.5	-0.4910961	0.008903864	-0.4999893	1.070918E-5	-0.5000001	9.64397E-8
-π/12	-0.258819	-0.2575491	0.001269951	-0.2588172	1.82441E-6	-0.2588191	1.713066E-8
0	0	4.16334E-17	4.16334E-17	9.49409E-16	9.49409E-16	1.77720E-15	1.77720E-15

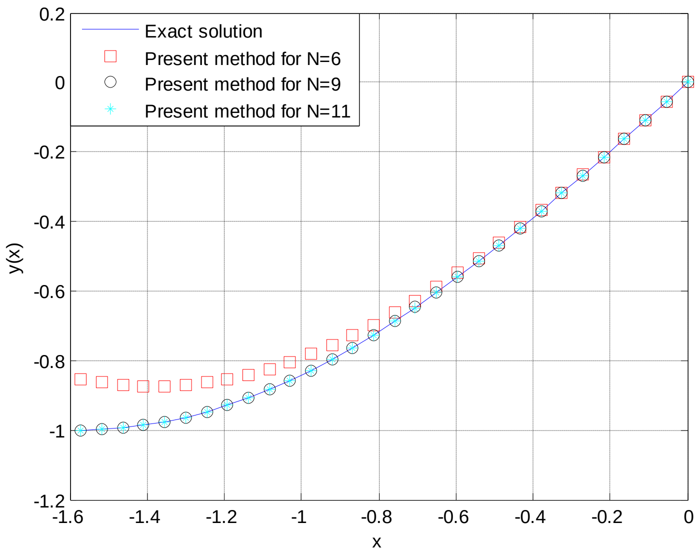

Figure 2. Numerical and exact solution of Example 5.3 for $\mathrm{N}=6,9,11$

6. CONCLUSIONS

Differential-difference equations with variable coefficients are usually difficult to solve analytically; therefore, approximate solutions are required. To have the best approximate solution for the equation, we take more terms from the Fibonacci expansion of functions, that is, the accuracy improves when N is increased. A considerable advantage of this method is that Fibonacci coefficients of the solution are obtained very easily by using the computer programs. We use the MATLAB program to obtain the solution of equations.

In this study, examples, tables and figures indicate that the present method is convenient, reliable and effective. So, we can say that the Fibonacci collocation method can be a suitable method for solving analytic solutions to linear differential-difference equations.

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