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APPROXIMATE SOLUTIONS OF LINEAR FREDHOLM INTEGRAL EQUATIONS SYSTEM WITH VARIABLE COEFFICIENTS

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Abstract – In this paper, a new approximate method has been presented to solve the linear Fredholm integral equations system (FIEs). The technique is based on, first, differentiating both sides of integral equations *n* times and then substituting the Taylor series the unknown functions in the resulting equation and later, transforming to a matrix equation. By merging these results, a new system which corresponds to a system of linear algebraic equations is obtained. The solution of this system yields the Taylor coefficients of the solution function. Also, this method gives the analytic solution when the exact solutions are polynomials. So as to Show this capability and robustness, some systems of FIEs are solved by the presented method in order to obtain their approximate solutions.

Key Words - Taylor polynomials and series, System of integral equations, Fredholm systems.

1. INTRODUCTION

The solutions of integral equations have a major role in the fields of science and engineering. A physical event can be modeled by the differential equation or an integro-differential equation or a system of these. Since few of these equations can be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [1,2]. Furthermore, there are also expansion methods for integral equations such as El-gendi's, Wolfe's and Galerkin methods [3]. Conversely, the solution of integral equations system which occur in physics [4], biology [5] and engineering [6,7] is based on numerical integration methods such as Euler- Chebyshev [8] and Runge-Kutta [9] methods, and also in a recent research, the first-order linear Fredholm integral equations system is solved by using rationalized Haar functions method [10] and by Galerkin methods with hybrid functions [11].

Besides, a Taylor method for solving Fredholm integral equations has been presented by Kanwal and Liu [12] and then this method has been extended by Sezer to Volterra integral equations [13] and to differential equations [14]. Similar approach has been used to solve linear Volterra-Fredholm integro-differential equations has been applied by Yalçınbaş and Sezer [15], nonlinear Volterra-Fredholm integral equations by Yalçınbaş [16], high-order linear differential equation system by [17,18] and linear Volterra integral equation systems [19]. Thus, the presented method which is an expansion method has been proposed to obtain approximate solution and also analytical solution of systems of higher-order linear integral equations.

In this paper, the basic ideas of the above studies are developed and applied to the systems of s linear Fredholm type integral equations of the second kind (FIE) in the general form

$$\sum_{j=1}^{s} a_{mj}(x) y_j(x) = f_m(x) + \sum_{j=1}^{s} \int_a^b K_{mj}(x,t) y_j(t) dt \quad , \quad m = 1,2,\ldots,s \ , \ a \le x,t \le b \quad \text{(1a)}$$

where $a_{mj}(x)(m, j = 1,2,\ldots,s)$, $f_m(x)$, $K_{mj}(x,t)$ are functions having nth derivatives on an interval $a \le x,t \le b$, and a, b are appropriate constants; and the solutions is expresed in the form

$$y_m(x) = \sum_{n=0}^{N} \frac{y_m^{(n)}(c)}{n!}(x-c)^n \ , \quad m = 1,2,\ldots,s \quad , \quad a \le x,c \le b \qquad (2)$$

which is a Taylor polynomial of degree N at $x = c$, where $y_m^{(n)}(c)$, $n = 0,1,\ldots,N$ are the coefficients to be determined.

2. FUNDAMENTAL RELATIONS AND SOLUTION METHOD

Let us first consider the systems of s linear integral equations of Fredholm type (FIEs) that is given by (1a) in the form of

$$E_m(x) = f_m(x) + F_m(x) \quad \text{or} \quad E_m(x) = I_m(x) \ , \ m = 1,2,\ldots,s \qquad \text{(1b)}$$

where

$$E_m(x) = \sum_{j=1}^{s} a_{mj}(x) y_j(x) \quad , \quad m = 1,2,\ldots,s$$

$$I_m(x) = f_m(x) + \sum_{j=1}^{s} \int_a^b K_{mj}(x,t) y_j(t) dt \quad , \quad m = 1,2,\ldots,s.$$

Here the expression $E_m(x)$ and $I_m(x)$, respectively, are called as the first part and second part (or integral part) of the Eq. (1b). To obtain the solution of the given problem in the form of expression (2) we first differentiate Eq. (1a) n times with respect to x to obtain

$$E_m^{(n)}(x) = f_m^{(n)}(x) + F_m^{(n)}(x) \quad \text{or} \quad E_m^{(n)}(x) = I_m^{(n)}(x) \quad , \quad m = 1,2,\ldots,s \qquad (3)$$

and then analyse the expressions $E_m(x)$ and $I_m(x)$ as follows:

2.1. Matrix Representation for the First Part

The expression $E_m^{(n)}(x)$ can be more clearly written as

$$E_m^{(n)}(x) = \left[\sum_{j=1}^{s} a_{mj}(x) y_j(x)\right]^{(n)} \quad , \quad m = 1,2,\ldots,s \quad ; \quad n = 0,1,2,\ldots,N \ . \qquad (4)$$

Using the Leibnitz's rule which is dealing with differentation of product of functions

$$\left[P(x).y(x)\right]^{(n)}\Big|_{x=c}=\sum_{i=0}^{n}\binom{n}{i}P^{(n-i)}(c)y^{(i)}(c)$$

and simplifying $x=c$ into the resulting relation (4), we have

$$E_m^{(n)}(x)=\left[\sum_{j=1}^{s}a_{mj}(x)y_j(x)\right]_{x=c}^{(n)}=\sum_{j=1}^{s}\sum_{i=1}^{n}\binom{n}{i}a_{mj}^{(n-i)}(c)y_j^{(i)}(c), \tag{5}$$

$$m=1,2,...,s \quad ; \quad n=0,1,2,...,N$$

where the $N+1$ unknown coefficients $y_j^{(0)}(c), y_j^{(1)}(c),..., y_j^{(N)}(c)$; $(j=1,2,...,s)$ are Taylor coefficients to be determined and $a_{mj}^{(i)}(c)$; $(m, j=1,2,...,s)$, respectively, denote the value of the *i*th derivative of the function $a_{mj}^{(i)}(x)$ at $x=c$.

We now write the matrix form of expression (5) as

$$\mathbf{E}=\mathbf{W.Y} \tag{6}$$

where

$$\mathbf{Y}=\begin{bmatrix}y_1^{(0)} & y_1^{(1)} & \cdots & y_1^{(N)} & y_2^{(0)} & y_2^{(1)} & \cdots & y_2^{(N)} & \cdots & y_s^{(0)} & y_s^{(1)} & \cdots & y_s^{(N)}\end{bmatrix}^{\mathrm{T}}$$
$$=\begin{bmatrix}\mathbf{Y_1} & \mathbf{Y_2} & \cdots & \mathbf{Y_s}\end{bmatrix}^{\mathrm{T}}$$

and

$$\mathbf{W}=\begin{bmatrix}\mathbf{W_{11}} & \mathbf{W_{12}} & \cdots & \mathbf{W_{1s}}\\ \mathbf{W_{21}} & \mathbf{W_{22}} & \cdots & \mathbf{W_{2s}}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{W_{s1}} & \mathbf{W_{s2}} & \cdots & \mathbf{W_{ss}}\end{bmatrix}$$

the elements of which are defined by

$$\mathbf{W_{11}}=\begin{bmatrix}(W_{11})_{00} & (W_{11})_{01} & \cdots & (W_{11})_{0s}\\ (W_{11})_{10} & (W_{11})_{11} & \cdots & (W_{11})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{11})_{s0} & (W_{11})_{s1} & \cdots & (W_{11})_{ss}\end{bmatrix}, \cdots\cdots\cdots, \mathbf{W_{1s}}=\begin{bmatrix}(W_{1s})_{00} & (W_{1s})_{01} & \cdots & (W_{1s})_{0s}\\ (W_{1s})_{10} & (W_{1s})_{11} & \cdots & (W_{1s})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{1s})_{s0} & (W_{1s})_{s1} & \cdots & (W_{1s})_{ss}\end{bmatrix}$$

$$\mathbf{W_{21}}=\begin{bmatrix}(W_{21})_{00} & (W_{21})_{01} & \cdots & (W_{21})_{0s}\\ (W_{21})_{10} & (W_{21})_{11} & \cdots & (W_{21})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{21})_{s0} & (W_{21})_{s1} & \cdots & (W_{21})_{ss}\end{bmatrix}, \cdots\cdots\cdots, \mathbf{W_{2s}}=\begin{bmatrix}(W_{2s})_{00} & (W_{2s})_{01} & \cdots & (W_{2s})_{0s}\\ (W_{2s})_{10} & (W_{2s})_{11} & \cdots & (W_{2s})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{2s})_{s0} & (W_{2s})_{s1} & \cdots & (W_{2s})_{ss}\end{bmatrix} \tag{7}$$

$$\vdots \qquad \ddots \qquad \vdots$$

$$\mathbf{W_{s1}}=\begin{bmatrix}(W_{s1})_{00} & (W_{s1})_{01} & \cdots & (W_{s1})_{0s}\\ (W_{s1})_{10} & (W_{s1})_{11} & \cdots & (W_{s1})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{s1})_{s0} & (W_{s1})_{s1} & \cdots & (W_{s1})_{ss}\end{bmatrix}, \cdots\cdots\cdots, \mathbf{W_{ss}}=\begin{bmatrix}(W_{ss})_{00} & (W_{ss})_{01} & \cdots & (W_{ss})_{0s}\\ (W_{ss})_{10} & (W_{ss})_{11} & \cdots & (W_{ss})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (W_{ss})_{s0} & (W_{ss})_{s1} & \cdots & (W_{ss})_{ss}\end{bmatrix}$$

The values $(W_{nm})_{ij}$, $(n,m=1,2,...,s;\ i,j=0,1,2,...,N)$ are defined by

$$
\begin{array}{llll}
(W_{11})_{ij}=\binom{i}{j}a_{11}^{(i-j)}(c), & (W_{12})_{ij}=\binom{i}{j}a_{12}^{(i-j)}(c), & \cdots, & (W_{1s})_{ij}=\binom{i}{j}a_{1s}^{(i-j)}(c) \\
(W_{21})_{ij}=\binom{i}{j}a_{21}^{(i-j)}(c), & (W_{22})_{ij}=\binom{i}{j}a_{22}^{(i-j)}(c), & \cdots, & (W_{2s})_{ij}=\binom{i}{j}a_{21s}^{(i-j)}(c) \\
\vdots & \vdots & \ddots & \vdots \\
(W_{s1})_{ij}=\binom{i}{j}a_{s1}^{(i-j)}(c), & (W_{s2})_{ij}=\binom{i}{j}a_{s2}^{(i-j)}(c), & \cdots, & (W_{ss})_{ij}=\binom{i}{j}a_{ss}^{(i-j)}(c).
\end{array}
\tag{8}
$$

Notice in the relation (8) clearly that $a_{ij}^{(l)}(c)=0$ for $l<0$ and $i,j=0,1,2,...,s$, and for $j<0$ and $j>i$, $\binom{i}{j}=0$, where i,j and l are integers. In this case, in Eq. (8), for $n=0,1,2,...,N-1; m=n+1,n+2,...,N$ leads to

$$(W_{nm})_{ij}=0.$$

Hence, the matrix $\mathbf{W}$ becomes, clearly,

$$
\mathbf{W}=\begin{bmatrix}
(W_{11})_{00} & (W_{11})_{01} & \cdots & (W_{11})_{0s} & \cdots & (W_{1s})_{00} & (W_{1s})_{01} & \cdots & (W_{1s})_{0s} \\
(W_{11})_{10} & (W_{11})_{11} & \cdots & (W_{11})_{1s} & \cdots & (W_{1s})_{10} & (W_{1s})_{11} & \cdots & (W_{1s})_{1s} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(W_{11})_{s0} & (W_{11})_{s1} & \cdots & (W_{11})_{ss} & \cdots & (W_{1s})_{s0} & (W_{1s})_{s1} & \cdots & (W_{1s})_{ss} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
(W_{s1})_{00} & (W_{s1})_{01} & \cdots & (W_{s1})_{0s} & \cdots & (W_{ss})_{00} & (W_{ss})_{01} & \cdots & (W_{ss})_{0s} \\
(W_{s1})_{10} & (W_{s1})_{11} & \cdots & (W_{s1})_{1s} & \cdots & (W_{ss})_{10} & (W_{ss})_{11} & \cdots & (W_{ss})_{1s} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(W_{s1})_{s0} & (W_{s1})_{s1} & \cdots & (W_{s1})_{ss} & \cdots & (W_{ss})_{s0} & (W_{ss})_{s1} & \cdots & (W_{ss})_{ss}
\end{bmatrix}. \tag{9}
$$

2.2. Matrix Representation for the Integral Part

The expression $I_m^{(n)}(x)$ can be more clearly written as

$$I_m^{(n)}(x)=f_m^{(n)}(x)+F_m^{(n)}(x) \tag{10}$$

or

$$I_m^{(n)}(x)=f_m^{(n)}(x)+\sum_{j=1}^{s}\int_a^b \frac{\partial^n K_{mj}(x,t)}{\partial x^n}y_j(t)dt \quad , \quad m=1,2,...,s.$$

First, we put $x=c$ in relation (10), thereby in expression (11), become

$$I_m^{(n)}(c) = f_m^{(n)}(c) + \sum_{j=1}^{s} \int_a^b \left. \frac{\partial^n K_{mj}(x,t)}{\partial x^n} \right|_{x=c} y_j(t)dt \quad , \quad m = 1,2,...,s \ . \tag{11}$$

Thereby in expression (11) and then substitute the Taylor expansion of $y_1(t), y_2(t),..., y_s(t)$ at $t = c$, i.e.

$$y_i(t) = \sum_{k=0}^{\infty} \frac{1}{k!} y_i^{(k)}(c)(t-c)^k \quad , \quad i = 1,2,...,s \tag{12}$$

in the resulting relation. Thus, expression (10) become

$$\begin{aligned}
I_1^{(n)}(c) &= f_1^{(n)}(c) + \sum_{k=0}^{\infty} {}^{11}T_{nk} y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{12}T_{nk} y_2^{(k)}(c) + \ldots + \sum_{k=0}^{\infty} {}^{1s}T_{nk} y_s^{(k)}(c) \\
I_2^{(n)}(c) &= f_2^{(n)}(c) + \sum_{k=0}^{\infty} {}^{21}T_{nk} y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{22}T_{nk} y_2^{(k)}(c) + \ldots + \sum_{k=0}^{\infty} {}^{2s}T_{nk} y_s^{(k)}(c) \\
\vdots \quad & \quad\ \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
I_s^{(n)}(c) &= f_s^{(n)}(c) + \sum_{k=0}^{\infty} {}^{s1}T_{nk} y_1^{(k)}(c) + \sum_{k=0}^{\infty} {}^{s2}T_{nk} y_2^{(k)}(c) + \ldots + \sum_{k=0}^{\infty} {}^{ss}T_{nk} y_s^{(k)}(c)
\end{aligned} \tag{13}$$

where

$${}^{mj}T_{nk} = \frac{1}{k!} \int_a^b \left. \frac{\partial^{(n)} K_{mj}(x,t)}{\partial x^n} \right|_{x=c} (t-c)^k \, dt \ .$$

The relation (13) gives us infinite linear equations. If we take $n = k = 0,1,2,,\ldots,N$ then relation (13) reduces to a system of $(N+1) \times s$ linear equations for the $(N+1) \times s$ unknown coefficients $y_i^{(0)}(c), y_i^{(1)}(c), \cdots, y_i^{(N)}(c), \quad i = 1,2,...,s$. This system can be put in a matrix form as

$$\mathbf{I} = \mathbf{F} + \mathbf{TY} \tag{14}$$

where the matrices $\mathbf{Y}$, $\mathbf{F}$ and $\mathbf{T}$ are defined by

$$\mathbf{Y} = \begin{bmatrix} y_1^{(0)} & y_1^{(1)} & \cdots & y_1^{(N)} & y_2^{(0)} & y_2^{(1)} & \cdots & y_2^{(N)} & \cdots & y_s^{(0)} & y_s^{(1)} & \cdots & y_s^{(N)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{Y_1} \\ \mathbf{Y_2} \\ \vdots \\ \mathbf{Y_s} \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} f_1^{(0)}(c) & f_1^{(1)}(c) & \cdots & f_1^{(N)}(c) & f_2^{(0)}(c) & f_2^{(1)}(c) & \cdots & f_2^{(N)}(c) & \cdots & f_s^{(0)}(c) & f_s^{(1)}(c) & \cdots & f_s^{(N)}(c) \end{bmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix} \mathbf{F_1} & \mathbf{F_2} & \cdots & \mathbf{F_s} \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T_{11}} & \mathbf{T_{12}} & \cdots & \mathbf{T_{1s}} \\ \mathbf{T_{21}} & \mathbf{T_{22}} & \cdots & \mathbf{T_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{T_{s1}} & \mathbf{T_{s2}} & \cdots & \mathbf{T_{ss}} \end{bmatrix}.$$

The matrices $\mathbf{T_{nk}}$, $(n,k = 1,2,...,s)$ are defined by

$$\mathbf{T_{11}}=\begin{bmatrix} {}^{11}T_{00} & {}^{11}T_{01} & \cdots & {}^{11}T_{0N} \\ {}^{11}T_{10} & {}^{11}T_{11} & \cdots & {}^{11}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{11}T_{N0} & {}^{11}T_{N1} & \cdots & {}^{11}T_{NN} \end{bmatrix}, \mathbf{T_{12}}=\begin{bmatrix} {}^{12}T_{00} & {}^{12}T_{01} & \cdots & {}^{12}T_{0N} \\ {}^{12}T_{10} & {}^{12}T_{11} & \cdots & {}^{12}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{12}T_{N0} & {}^{12}T_{N1} & \cdots & {}^{12}T_{NN} \end{bmatrix}, \cdots, \mathbf{T_{1s}}=\begin{bmatrix} {}^{1s}T_{00} & {}^{1s}T_{01} & \cdots & {}^{1s}T_{0N} \\ {}^{1s}T_{10} & {}^{1s}T_{11} & \cdots & {}^{1s}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{1s}T_{N0} & {}^{1s}T_{N1} & \cdots & {}^{1s}T_{NN} \end{bmatrix}$$

$$\mathbf{T_{21}}=\begin{bmatrix} {}^{21}T_{00} & {}^{21}T_{01} & \cdots & {}^{21}T_{0N} \\ {}^{21}T_{10} & {}^{21}T_{11} & \cdots & {}^{21}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{21}T_{N0} & {}^{21}T_{N1} & \cdots & {}^{21}T_{NN} \end{bmatrix}, \mathbf{T_{22}}=\begin{bmatrix} {}^{22}T_{00} & {}^{22}T_{01} & \cdots & {}^{22}T_{0N} \\ {}^{22}T_{10} & {}^{22}T_{11} & \cdots & {}^{22}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{22}T_{N0} & {}^{22}T_{N1} & \cdots & {}^{22}T_{NN} \end{bmatrix}, \cdots, \mathbf{T_{2s}}=\begin{bmatrix} {}^{2s}T_{00} & {}^{2s}T_{01} & \cdots & {}^{2s}T_{0N} \\ {}^{2s}T_{10} & {}^{2s}T_{11} & \cdots & {}^{2s}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{2s}T_{N0} & {}^{2s}T_{N1} & \cdots & {}^{2s}T_{NN} \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \qquad \ddots \qquad \vdots$$

$$\mathbf{T_{s1}}=\begin{bmatrix} {}^{s1}T_{00} & {}^{s1}T_{01} & \cdots & {}^{s1}T_{0N} \\ {}^{s1}T_{10} & {}^{s1}T_{11} & \cdots & {}^{s1}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{s1}T_{N0} & {}^{s1}T_{N1} & \cdots & {}^{s1}T_{NN} \end{bmatrix}, \mathbf{T_{s2}}=\begin{bmatrix} {}^{s2}T_{00} & {}^{s2}T_{01} & \cdots & {}^{s2}T_{0N} \\ {}^{s2}T_{10} & {}^{s2}T_{11} & \cdots & {}^{s2}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{s2}T_{N0} & {}^{s2}T_{N1} & \cdots & {}^{s2}T_{NN} \end{bmatrix}, \cdots, \mathbf{T_{ss}}=\begin{bmatrix} {}^{ss}T_{00} & {}^{ss}T_{01} & \cdots & {}^{ss}T_{0N} \\ {}^{ss}T_{10} & {}^{ss}T_{11} & \cdots & {}^{ss}T_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{ss}T_{N0} & {}^{ss}T_{N1} & \cdots & {}^{ss}T_{NN} \end{bmatrix}$$

2.3. Fundamental Matrix Equatios

Substituting the matrix forms (6) and (14) in expression of Eq. (3) at the point $x=c$, we get the matrix form of Eq. (3) as

$$\mathbf{WY}=\mathbf{F}+\mathbf{TY} \tag{15}$$

or

$$(\mathbf{W}-\mathbf{T})\mathbf{Y}=\mathbf{F},$$

which is a fundemental equation for the ingeral equations system (1a) . If we take $\mathbf{M}=\mathbf{W}-\mathbf{T}$, then we have

$$\mathbf{MY}=\mathbf{F} \tag{16}$$

where

$$\mathbf{M}=\begin{bmatrix}
(W_{11})_{00}-{}^{11}T_{00} & (W_{11})_{01}-{}^{11}T_{01} & \cdots & (W_{11})_{0s}-{}^{11}T_{0s} & & (W_{1s})_{00}-{}^{1s}T_{00} & (W_{1s})_{01}-{}^{1s}T_{01} & \cdots & (W_{1s})_{0s}-{}^{1s}T_{0s} \\
(W_{11})_{10}-{}^{11}T_{10} & (W_{11})_{11}-{}^{11}T_{11} & \cdots & (W_{11})_{1s}-{}^{11}T_{1s} & \cdots & (W_{1s})_{10}-{}^{1s}T_{10} & (W_{1s})_{11}-{}^{1s}T_{11} & \cdots & (W_{1s})_{1s}-{}^{1s}T_{1s} \\
\vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
(W_{11})_{s0}-{}^{11}T_{s0} & (W_{11})_{s1}-{}^{11}T_{s1} & \cdots & (W_{11})_{ss}-{}^{11}T_{ss} & & (W_{1s})_{s0}-{}^{1s}T_{s0} & (W_{1s})_{s1}-{}^{1s}T_{s1} & \cdots & (W_{1s})_{ss}-{}^{1s}T_{ss} \\
 & \vdots & & & \ddots & & \vdots & & \\
(W_{s1})_{00}-{}^{s1}T_{00} & (W_{s1})_{01}-{}^{s1}T_{01} & \cdots & (W_{s1})_{0s}-{}^{s1}T_{0s} & & (W_{ss})_{00}-{}^{ss}T_{00} & (W_{ss})_{01}-{}^{ss}T_{01} & \cdots & (W_{ss})_{0s}-{}^{ss}T_{0s} \\
(W_{s1})_{10}-{}^{s1}T_{10} & (W_{s1})_{11}-{}^{s1}T_{11} & \cdots & (W_{s1})_{1s}-{}^{s1}T_{1s} & \cdots & (W_{ss})_{10}-{}^{ss}T_{10} & (W_{ss})_{11}-{}^{ss}T_{11} & \cdots & (W_{ss})_{1s}-{}^{ss}T_{1s} \\
\vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
(W_{s1})_{s0}-{}^{s1}T_{s0} & (W_{s1})_{s1}-{}^{s1}T_{s1} & \cdots & (W_{s1})_{ss}-{}^{s1}T_{ss} & & (W_{ss})_{s0}-{}^{ss}T_{s0} & (W_{ss})_{s1}-{}^{ss}T_{s1} & \cdots & (W_{ss})_{ss}-{}^{ss}T_{ss}
\end{bmatrix}$$

where $\mathbf{Y}$ and $\mathbf{F}$ are defined in Eq. (14).

If $|\mathbf{M}|\neq 0$, then we can write

$$\mathbf{Y}=\mathbf{M}^{-1}\mathbf{F}\ . \tag{17}$$

Thus, the coefficients $y_m^{(n)}(c)$, $(m=1,2,...,s;\ n=0,1,2,...,N)$, are uniquely determined by Eq. (17). Also, by means of system (16) we may obtain some particular solutions. This solutions is given by the Taylor polynomial

$$y_m(x)\cong\sum_{n=0}^{N}\frac{y_m^{(n)}(c)}{n!}(x-c)^n\ ,\quad m=1,2,\ldots,s. \tag{18}$$

3. ACCURACY OF SOLUTION

We can easily check the accuracy of the solution obtained in the fom (18) as follws. Since the truncated Taylor series (18) or the corresponding polynomial expansion is an approximate solution of Eqs. (1a) and (1b), when the solution $y_m(x)$ are substituted Eqs. (1a) and (1b), resulting equation must be satisfied approximately; that is, for $x = x_r \in [a,b]$, $r = 0,1,2,\ldots$.

$$D(x_r) = \left|E_m(x_r) - f_m(x_r) - F_m(x_r)\right| \cong 0$$

or

$$D(x_r) \le 10^{-k_r} \qquad (\, k_r \text{ is any positive integer}\,).$$

If $\max\left(10^{-k_{ri}}\right) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $\left|D(x_r)\right|$ at each of the points x_r becomes smaller than the prescribed 10^{-k}.

On the other hand, the error function can be estimated by

$$D_N(x) = \sum_{j=1}^{s} a_{mj}(x) y_j(x) - f_m(x) - \int_a^b K_{mj}(x,t) y_j(t)dt$$

If $D_N(x) \to 0$ when N is sufficiently large enough as $N \to \infty$, then the error asymptotically vanishes.

4. NUMERICAL ILLUSTRATIONS

In this section we consider two examples of systems of Fredholm type to illustrate the use of presented method.

Example 1. Let us first consider the system of Fredholm with two unknown

$$\left.\begin{aligned} y_1(x) - 2xy_2(x) &= 22x + 3 + 3\int_{-1}^{1}(x+t)y_1(t)dt + 3\int_{-1}^{1}(x-t)y_2(t)dt \\ 5y_1(x) + y_2(x) &= -x + 9 + 3\int_{-1}^{1}x^2 y_1(t)dt + 3\int_{-1}^{1}\left(xt - t^2\right)y_2(t)dt \end{aligned}\right\} \tag{19}$$

and approximate the solution $y_m(x)$ by the Taylor polynomial

$$y_m(x) = \sum_{n=0}^{3} \frac{1}{n!} y_m^{(n)}(0)(x)^n \quad , \quad (m = 1,2\,)$$

where $a = -1, b = 1, c = 0, N = 3$.

Using the matrices $\mathbf{W}$ and $\mathbf{T}$, we find the matrix $\mathbf{M}$ in (16). We find the unknown coefficients $y_m^{(n)}(0)$ are uniquely determined as

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 & 4 & 0 & -4 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

By substituting the obtained coefficients in (18) the solution of (19) becomes

$$y_1(x) = \frac{(1)x^0}{0!} + \frac{(0)x^1}{1!} + \frac{(4)x^2}{2!} + \frac{(0)x^3}{3!} = 1 + 2x^2 \ , \ y_2(x) = \frac{(-4)x^0}{0!} + \frac{(1)x^1}{1!} + \frac{(0)x^2}{2!} + \frac{(0)x^3}{3!} = -4 + x$$

which are the exact solutions.

Example 2. Consider the system of Fredholm integral equations:

$$\left.\begin{aligned} y_1(x) - e^{-x} y_2(x) = e^x - \int_0^1 e^{x-2t} y_2(t)\,dt \\ y_2(x) = e^{2x} + \int_0^1 y_1(t)\,dt - \int_0^1 e^{-t} y_2(t)\,dt \end{aligned}\right\} \qquad (20)$$

Following the previous procedures, we find the unknown coefficients $y_m^{(n)}(0)$ as

$$\mathbf{Y} = \begin{bmatrix} 1.0068 & 0.99322 & 1.0068 & 0.99322 & 1.0068 & 0.99322 & 1.0068 & 2 & 4 & 8 & 16 & 32 \end{bmatrix}^{\mathrm{T}}.$$

We get the aproximate solution of problem (20) for $c = 0, N = 5$ as

$$y_1(x) = \frac{(1.0068)}{0!}x^0 + \frac{(0.99322)}{1!}x^1 + \frac{(1.0068)}{2!}x^2 + \frac{(0.99322)}{3!}x^3 + \frac{(1.0068)}{4!}x^4 + \frac{(0.99322)}{5!}x^5$$

$$y_2(x) = \frac{(1.0068)}{0!}x^0 + \frac{(2)}{1!}x^1 + \frac{(4)}{2!}x^2 + \frac{(8)}{3!}x^3 + \frac{(16)}{4!}x^4 + \frac{(32)}{5!}x^5.$$

Now let us find the solution of problem (20) taking $c = 0$; $N = 7,9,11$. The comparison of the solutions given above with exact solutions $y_1(x) = e^x$, $y_2(x) = e^{2x}$ of the problem is given below in Tables 1-4.

Table 1. Comparing the solutions of y_1 which has been found for $N = 5, 7, 9, 11$ at Example 2.

	Present Method				
x_i	Exact Solution $y_1(x_i) = e^x$	$N = 5$, $c = 0$ $y_1(x_i)$	$N = 7$, $c = 0$ $y_1(x_i)$	$N = 9$, $c = 0$ $y_1(x_i)$	$N = 11$, $c = 0$ $y_1(x_i)$
0	1	1.0068	1	1	1
0.1	1.105170918	1.111325814	1.105170918	1.105170918	1.105170918
0.2	1.221402758	1.226974062	1.221402758	1.221402758	1.221402758
0.3	1.349858808	1.354901398	1.349858806	1.349858808	1.349858808
0.4	1.491824698	1.496385021	1.491824681	1.491824698	1.491824698
0.5	1.648721271	1.652832609	1.648721169	1.64872127	1.648721271
0.6	1.8221188	1.825792247	1.822118355	1.822118799	1.8221188
0.7	2.013752707	2.016962359	2.013751160	2.013752699	2.013752707
0.8	2.225540928	2.228201646	2.225536368	2.225540897	2.225540928
0.9	2.459603111	2.461539012	2.459591265	2.459603007	2.459603111
1	2.718281828	2.719183500	2.718253972	2.718281526	2.718281826

Table 2. Comparing the solutions of y_2 which has been found for $N = 5,7,9,11$ at Example 2.

		Present Method			
x_i	Exact Solution $y_2(x_i) = e^{2x}$	$N = 5$, $c = 0$ $y_2(x_i)$	$N = 7$, $c = 0$ $y_2(x_i)$	$N = 9$, $c = 0$ $y_2(x_i)$	$N = 11$, $c = 0$ $y_2(x_i)$
0	1	1.0068	1.00000001	1	1
0.1	1.221402758	1.228202667	1.221402768	1.221402758	1.221402758
0.2	1.491824698	1.498618667	1.491824691	1.491824698	1.491824698
0.3	1. 8221188	1.828848	1.822118364	1.822118799	1.8221188
0.4	2.225540928	2.231930667	2.225536376	2.225540897	2.225540928
0.5	2.718281828	2.723466667	2.718253978	2.718281526	2.718281826
0.6	3.320116923	3.321936	3.319994159	3.320115010	3.320116902
0.7	4.055199967	4.049018667	4.054767904	4.055190849	4.055199834
0.8	4.953032424	4.929914667	4.951742466	4.952997048	4.953031755
0.9	6.049647464	5.993664	6.046250433	6.049530173	6.049644666
1	7.389056099	7.273466667	7.380952391	7.388712522	7.389046016

Table 3. Comparison of the error analysis of y_1 which has been found for $N = 5,7,9,11$ at Example 2.

x_i	$N = 5$, $c = 0$ $D_1(x_i)$	$N = 7$, $c = 0$ $D_1(x_i)$	$N = 9$, $c = 0$ $D_1(x_i)$	$N = 11$, $c = 0$ $D_1(x_i)$
0	6.80000E-03	0.00000E+00	0.00000E+00	0.00000E+00
0.1	6.15490E-03	0.00000E+00	0.00000E+00	0.00000E+00
0.2	5.57130E-03	0.00000E+00	0.00000E+00	0.00000E+00
0.3	5.04259E-03	2.00000E-09	0.00000E+00	0.00000E+00
0.4	4.56032E-03	1.70000E-08	0.00000E+00	0.00000E+00
0.5	4.11134E-03	1.02000E-07	1.00000E-09	0.00000E+00
0.6	3.67345E-03	4.45000E-07	1.00000E-09	0.00000E+00
0.7	3.20965E-03	1.54700E-06	8.00000E-09	0.00000E+00
0.8	2.66072E-03	4.56000E-06	3.10000E-08	0.00000E+00
0.9	1.93590E-03	1.18460E-05	1.04000E-07	0.00000E+00
1	9.01672E-04	2.78560E-05	3.02000E-07	2.00000E-09

Table 4. Comparison of the error analysis of y_2 which has been found for $N = 5,7,9,11$ at Example 2.

x_i	$N = 5$, $c = 0$ $D_2(x_i)$	$N = 7$, $c = 0$ $D_2(x_i)$	$N = 9$, $c = 0$ $D_2(x_i)$	$N = 11$, $c = 0$ $D_2(x_i)$
0	6.80000E-03	1.00000E-08	0.00000E+00	0.00000E+00
0.1	6.79991E-03	1.00000E-08	0.00000E+00	0.00000E+00
0.2	6.79397E-03	7.00000E-09	0.00000E+00	0.00000E+00
0.3	6.72920E-03	4.36000E-07	1.00000E-09	0.00000E+00
0.4	6.38974E-03	4.55200E-06	3.10000E-08	0.00000E+00
0.5	5.18484E-03	2.78500E-05	3.02000E-07	2.00000E-09
0.6	1.81908E-03	1.22764E-04	1.91300E-06	2.10000E-08
0.7	6.18130E-03	4.32063E-04	9.11800E-06	1.33000E-07
0.8	2.31178E-02	1.28996E-03	3.53760E-05	6.69000E-07
0.9	5.59835E-02	3.39703E-03	1.17291E-04	2.79800E-06
1	1.15589E-01	8.10371E-03	3.43577E-04	1.00830E-05

5. CONCLUSIONS

Linear Fredholm integral equations system with variable coefficients are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. A considerable advantage of the method is that it allows us to make use of the computer because this Taylor method transforms the problem to matrix equation, which is a linear algebraic system. Therefore, Taylor coefficients of the solution are found very easily by using the computer programs. Furthermore, after calculation of the series coefficients, the solutions $y(x)$ can be easily evaluated for arbitrary values of x at low computation effort.

If the functions $a_{mj}(x)(m, j = 1,2,\ldots,s),\ f_m(x),\ K_{mj}(x,t)$ are functions having nth derivatives on an interval $a \le x,t \le b$, then we can approach the solutions $y_m(x)$ by the Taylor polynomial

$$y_m(x) = \sum_{n=0}^{N} \frac{y_m^{(n)}(c)}{n!}(x-c)^n \ , \quad m = 1,2,\ldots,s \quad , \quad a \le x,c \le b$$

about $x = c$; otherwise, the method can not be used. On the other hand, it is observed that this method shows the best advantage when the known functions in equation can be expanded to Taylor series about $x = c$ with converge rapidly.

An interesting feature of this method is that when linear Fredholm integral equations system have linearly independent polynomial solution of degree N or less than N, the method can be used for finding the analytical solution. Besides, it is seen that when the truncation limit N is increased, there exists a solution, which is closer to the exact solution.

The method can be developed and applied to another high-order linear and nonlinear integro-differential equation systems with variable coefficients.

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