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BIQUATERNIONIC DESCRIPTION OF THE SCHRÖDINGER EQUATION

Murat TANIŞLI and Süleyman DEMİR

Anadolu University, Science Faculty, Department of Physics, 26470 Eskişehir, Turkey
mtanisli@anadolu.edu.tr, sudemir@anadolu.edu.tr

Abstract- This paper begins with the introduction of biquaternion algebra and its properties in compact, profound and comprehensible approach. The Schrödinger equation including both the scalar Aharonov-Bohm(sAB) and Aharonov-Casher(AC) effects, that are currently popular in particle physics and condensed-matter physics, has been re-written by biquaternions. The newly biquaternionic Schrödinger equation, which is defined with various assumptions and approaches, also includes the biquaternionic generalized momentum in Hamiltonian.

Key Words- Quaternion, Biquaternion, Schrödinger equation, Scalar Aharonov-Bohm (sAB) effect, Aharonov-Casher(AC) effect.

1. INTRODUCTION

In view of mathematicians, Dirac matrices are the generators of Clifford algebra. Quaternion [1, 2] and Dirac algebras are considered as the special cases of Clifford algebra [3-5]. Since a spinor with two elements is seen as an element of quaternion algebra, the Dirac spinor with four elements can also be considered as an element of Clifford algebra. Nowadays, the abstract Clifford algebras play a significant role in the investigation, quantization, improvement of supersymmetry concepts of particles with two valued spin such as fermions fitting the Pauli principle and quantum gravity [6].

In recent years, quaternions have been increasingly used in all fields. Accordingly to this, the quaternion algebra has been developing parallel to increasing usage fields of quaternions. These structures with applying of quaternions and octonions to exceptional groups and geometry have placed in physics. Complex quaternions (also named biquaternions) composed of an association of complex number and quaternion have been frequently used in general and special relativity, quantum mechanics, particle mechanics, acoustic and electromagnetism. Besides biquaternions belonging to Clifford algebra have a non-commutative algebraically structure in eight-dimensions, and matrix representations. These special features appear as an alternative way in the formulation of physical events [7-12].

It is a phenomenon that electromagnetic fields produce observable effects on quantum mechanical attitudes of moving particles, which exist out of the electromagnetic fields and its influence on non-commutative quantum mechanics is effective nearly in all fields in physics. The main problem here is how to define the biquaternionic generalized momentum in Hamiltonian. Actually, starting with Dirac equation [13-18] both sAB and AC effects have been investigated separately. After writing Dirac Lagrangian [19-26] and Hamiltonian [27-31] the above-mentioned Schrödinger equation can be expressed. In nonrelativistic limit, an effective Lagrangian can describe the interaction between charged particle and a magnetic moment. These effects are called as scalar Aharonov-Bohm (sAB) and Aharonov-Casher (AC) effects [19-21, 32]. Aharonov and Casher investigated a situation where electrically neutral particle exhibits a magnetic AB type effect in the presence of a fixed distribution of charge.

In the recent work, these effects have been investigated in an arbitrary direction of magnetic dipole [20]. These effects have already been investigated in a non-commutative space. Mirza and Zarei [21] developed a novel method to study the topological phase of the AC effect while Chaichaian *et al.* [22] and Krey [23] described AB effect. The AC effect for quantum motion of a neutral magnetized particle in the electric field is believed to be a topological effect closely related to the known AB effect [21].

In this work, we propose to investigate with a new method for both sAB and AC effects. In this formulation, the Schrödinger equation governing the sAB and AC effects can be expressed in compact, simple and elegant approach with biquaternions. The organization of the paper is presented as follows: In section 2, biquaternion algebra and its properties are introduced. Biquaternionic forms of Hamiltonian and Schrödinger equation responsible for sAB and AC effects have been expressed in section 3. Finally, some suggestions and discussions have been given in the last section.

2. BIQUATERNION ALGEBRA

A biquaternion as an eight dimensional hypercomplex number can be defined as

$$\begin{aligned}\mathbf{Q} &= \mathbf{q} + i\mathbf{q}' = (q_0\mathbf{e}_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3) + i(q'_0\mathbf{e}_0 + q'_1\mathbf{e}_1 + q'_2\mathbf{e}_2 + q'_3\mathbf{e}_3) \\ &= [q_0, q_1, q_2, q_3; iq'_0, iq'_1, iq'_2, iq'_3]\end{aligned} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are two quaternions and i is ordinary complex number ($i^2 = -1$). q_0, q_1, q_2, q_3 and q'_0, q'_1, q'_2, q'_3 all are real numbers. $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the quaternion basis elements that obey the non-commutative multiplication rules as following equation:

$$\mathbf{e}_j\mathbf{e}_k = -\delta_{jk}\mathbf{e}_0 + \varepsilon_{jkl}\,\mathbf{e}_l, \qquad j, k, l \in \{1, 2, 3\} \tag{2}$$

where δ_{jk} and ε_{jkl} are the Kronecker delta and three-index Levi-Civita symbol respectively.

Equation (1) can be written as

$$\begin{aligned}\mathbf{Q} &= (q_0 + iq'_0)\mathbf{e}_0 + (q_1 + iq'_1)\mathbf{e}_1 + (q_2 + iq'_2)\mathbf{e}_2 + (q_3 + iq'_3)\mathbf{e}_3 \\ &= Q_0\mathbf{e}_0 + Q_1\mathbf{e}_1 + Q_2\mathbf{e}_2 + Q_3\mathbf{e}_3\end{aligned} \tag{3}$$

where Q_0, Q_1, Q_2 and Q_3 are complex numbers. A biquaternion can be represented in terms of its scalar and vector parts as

$$\mathbf{P} = P_0 + \boldsymbol{P} \tag{4}$$

where the scalar and vector parts of biquaternion, $\mathbf{P}$, are

$$\langle\mathbf{P}\rangle_S = P_0, \tag{5}$$

$$\langle\mathbf{P}\rangle_V = P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3. \tag{6}$$

The product of two biquaternions as $\mathbf{P}$ and $\mathbf{Q}$ is

$$\mathbf{PQ} = (P_0 + \boldsymbol{P})(Q_0 + \boldsymbol{Q}) = P_0Q_0 + P_0\boldsymbol{Q} + Q_0\boldsymbol{P} - \boldsymbol{P}.\boldsymbol{Q} + \boldsymbol{P}\times\boldsymbol{Q} \tag{7}$$

where the dot and cross indicate the usual three-dimensional scalar and vector products respectively. The biquaternionic product is associative but not commutative. As in the familiar case of complex numbers, the biquaternionic conjugation is accomplished by changing the sign of the components of the imaginary basis elements,

$$\overline{\mathbf{Q}} = [q_0, -q_1, -q_2, -q_3; -iq_0', -iq_1', -iq_2', -iq_3']. \tag{8}$$

Similarly, the complex conjugate of $\mathbf{Q}$ is also defined as:

$$\mathbf{Q}^* = [q_0, q_1, q_2, q_3; -iq_0', -iq_1', -iq_2', -iq_3']. \tag{9}$$

If P_0 and Q_0 are zero in eq.(7), the products $\boldsymbol{P}.\boldsymbol{Q}$ and $\boldsymbol{P}\times\boldsymbol{Q}$ can be transcribed by the following equations [10]:

$$\boldsymbol{P}.\boldsymbol{Q} = -\frac{1}{2}(\mathbf{PQ} + \overline{\mathbf{PQ}}) \tag{10}$$

and

$$\boldsymbol{P}\times\boldsymbol{Q} = \frac{1}{2}(\mathbf{PQ} - \overline{\mathbf{PQ}}). \tag{11}$$

In general, the norm of a biquaternion $\mathbf{Q}$ is a complex number and is provided by

$$N(\mathbf{Q}) = \mathbf{Q}\overline{\mathbf{Q}} = \overline{\mathbf{Q}}\mathbf{Q} = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2, \tag{12}$$

however, the norm for biquaternions may be zero. Therefore, biquaternions do not form a division algebra. Biquaternions with unit norm are termed the unit biquaternions. The inverse of a biquaternion $\mathbf{Q}$, whose norm is non-zero, is given by

$$\mathbf{Q}^{-1} = \frac{\overline{\mathbf{Q}}}{N(\mathbf{Q})}. \tag{13}$$

3. BIQUATERNIONIC SCHRÖDINGER EQUATION FOR SAB AND AC EFFECTS

In this section, we aim to express the Schrödinger equation that contains sAB and AC effects in terms of biquaternions.

In former studies, there is a definition of Dirac Lagrangian with quaternions [25]. The traditional form for the Dirac Lagrangian density is

$$L = \frac{i}{2}[\overline{\psi}\,\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\overline{\psi})\gamma^\mu\psi] - m\overline{\psi}\,\psi. \tag{14}$$

Also, as an alternative method, the novel Dirac equation and electron theory have been investigated by the above-mentioned algebras for different states and conditions, as mentioned in [7, 9, 10, 18]. Dirac equation, which obeys an electrically neutral spin half particle of rest mass m carrying a magnetic moment μ and moving in an electromagnetic field, is

$$i\hbar\frac{\partial}{\partial t}\psi=(c\vec{\alpha}.\vec{p}+\beta mc^2-\beta\mu\vec{\sigma}.\vec{B}+i\frac{\mu}{c}\vec{\gamma}.\vec{E})\psi \tag{15}$$

where $\vec{\gamma}$ is the Dirac matrices, $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ and σ_k' s are the usual Pauli matrices, $\vec{E}$ and $\vec{B}$ define the electric and the magnetic fields. The four spinor ψ can be expressed in terms of two-spinors φ and χ for the nonrelativistic limit [20]:

$$\psi=e^{-i\left(\frac{mc^2}{\hbar}\right)t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{16}$$

The problem is to write the two-spinor equation as the biquaternionic Schrödinger equation in the nonrelativistic situation.

Using the vector identities, various assumptions and approaches, the Schrödinger equation governing the scalar Aharanov-Bohm and Aharanov-Casher effects has been suggested by Hyllus and Sjöqvist [20]. If we think of an electrically neutral spin $\frac{1}{2}$ particle of rest mass m carrying a magnetic moment $\mu=e\hbar/2mc$ and moving in an electromagnetic field, the two-spinor equation as the Schrödinger equation can be transcribed both vectors and matrices in different forms.

We introduce the biquaternionic generalized momentum Π as:

$$\Pi=[\,0,P_x,P_y,P_z\,;0,-i\mu E_x,-i\mu E_y,-i\mu E_z\,] \tag{17}$$

the biquaternionic time operator, magnetic dipole and magnetic field, respectively

$$\Box=[\,0;i\frac{\partial}{\partial t},0,0,0\,]\ , \tag{18}$$

$$\mu=[\,0,\mu_x,\mu_y,\mu_z\,;0\,], \tag{19}$$

and

$$\mathbf{B}=[\,0,B_x,B_y,B_z\,;0\,]. \tag{20}$$

The new biquaternionic Schrödinger equation responsible for the sAB and AC effects can be transcribed as the following equation:

$$\begin{aligned}\Box\,\varphi&=\left(\frac{1}{2m}\Pi\,\overline{\Pi}^{*}-\mu.\mathbf{B}\right)\varphi\\
&=\Big(\frac{1}{2m}[\,0,P_x,P_y,P_z\,;0,-i\mu E_x,-i\mu E_y,-i\mu E_z\,]\\
&\qquad[\,0,-P_x,-P_y,-P_z\,;0,-i\mu E_x,-i\mu E_y,-i\mu E_z\,]-\mu.\mathbf{B}\Big)\,\varphi\\
&=\Big(\frac{1}{2m}(P_x^2-i\mu E_y P_x\,\mathbf{e}_3+i\mu E_z P_x\,\mathbf{e}_2+P_y^2+i\mu E_x P_y\,\mathbf{e}_3-i\mu E_z P_y\,\mathbf{e}_1+P_z^2-i\mu E_x P_z\,\mathbf{e}_2\\
&\qquad+i\mu E_y P_z\,\mathbf{e}_1+i\mu E_x P_y\,\mathbf{e}_3-i\mu E_x P_z\,\mathbf{e}_2+\mu^2E_x^2-i\mu E_y P_x\,\mathbf{e}_3+i\mu E_y P_z\,\mathbf{e}_1+\mu^2E_y^2\\
&\qquad+i\mu E_z P_x\,\mathbf{e}_2-i\mu E_z P_y\,\mathbf{e}_1+\mu^2E_z^2)-\mu.\mathbf{B}\Big)\varphi\end{aligned} \tag{21}$$

$$= \Big(\frac{1}{2m} (P_x^2 + P_y^2 + P_z^2 + i2\mu (P_z E_y - P_y E_z)\mathbf{e}_1 + i2\mu (P_x E_z - P_z E_x)\mathbf{e}_2 + i2\mu (P_y E_x - P_x E_y)\mathbf{e}_3$$
$$+ \mu^2 (E_x^2 + E_y^2 + E_z^2)) - \mu.\mathbf{B} \Big) \varphi$$

where the units are $\hbar = c = 1$. On the right-hand side of eq.(21), the first term is the biquaternionic Hamiltonian describing the interaction between a charged particle and a magnetic moment. At this stage, the sAB and the AC effects to arbitrary directions of the magnetic dipole have been extended with biquaternions. The non-commutative quantum mechanics can be defined with biquaternion by this equation. In the AB and sAB effects the conditions are given in terms of fields whereas in the AC and sAC effects the sources such as charges and currents take the role of fields.

4. CONCLUSIONS

The physics behind Aharonov-Bohm(AB) and Aharonov-Casher(AC) effects has been extensively discussed by using different methods and algebras. Although the quaternionic Dirac Lagrangian and biquaternionic Dirac equation have been expressed in different forms in literature. However, the formulations including sAB and AC effects have not been written by using biquaternionic algebra in literature before. For the first time, the biquaternionic Schrödinger equation (Eq.(21)) governing the sAB and AC effects via biquaternionic Hamiltonian and biquaternionic generalized momentum (Π) has been derived and suggested. The Schrödinger equation has also been transcribed in eight dimensions (8D). The proposed method serves as a compact, useful, comprehensible and elegant formulations and it will also contribute to the existing studies in literature. By this formulation, the usage of different algebras together such as matrices, vectors, complex numbers in combined form is not needed.

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5. REFERENCES

1. W. R. Hamilton, *Elements of Quaternions*, Chelsea Publishing, 1969.
2. A. Gsponer and J. P. Hurni, *Found. Phys. Lett*. **14**, 77-85, 2001.
3. D. Hestenes, *Space-Time Algebra*, Gordon and Breach, 1966.
4. P. Lounesto, *Clifford Algebras and Spinors*, Cambridge Uni. Press, 1997.
5. R. Ablamowicz and G. Sobcyzk (Editors), *Lectures on Clifford Geometric Algebras-Conference Edition*, TTU Press, TN, 2002.
6. M. S. El Naschie, Quantum Gravity, Clifford Algebras, Fuzz Set Theory and The Fundamental Constant of Nature, *Chaos Soliton Fract*., **20**, 437-450, 2004.
7. A. W. Conway, *Phil. Mag*., **24**, 208-216, 1912.
8. J. Lambek, *Math. Intelligencer*, **17**, 7-15, 1995.
9. S. K. Srinivasan and E. C. G. J. Sudarshan, A direct derivation of the Dirac equation via quaternion measures, *Phys. A:Math. Gen.*, **29**, 5181-5186, 1996.

10. M. Tanışlı and G. Özgür, Biquaternionic Representations of Angular Momentum and Dirac Equation, *Acta Phys. Slovaca*, **53**(3), 243-252, 2003.
11. M. Tanışlı, Gauge Transformation and Electromagnetism with Biquaternions, *Europhys. Lett*., **74**(4), 569-573, 2006.
12. S. Demir and M. Tanışlı, Complex Quaternionic Reformulation of Relativistic Elastic Collision Problem, *Phys. Scripta*, **75**, 630-637, 2007.
13. D.Hestenes, Observables, Operators, and Complex Numbers in the Dirac Theory, *J. Math. Phys.*, 16, 556-572, 1975.
14. W. E. Baylis, Classical Eigenspinors and the Dirac equation, Phys. Rev., **A45**, 4293-4302, 1992.
15. S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford Uni. Press, 1995.
16. A. Conte, *Biquaternion Quantum Mechanics*, Pitagora Editrice, 2000.
17. O. P. S. Negi, Bisht S. and P. S. Bisht, *Nuovo Cimento B*, **113**, 1449-1468, 1998.
18. S. De Leo and W. A. Rodrigues, Quaternionic electron theory: Dirac's equation, *Int. J. Theor. Phys*., **37**, 1511-1519, 1998.
19. Y. Aharonov, A. Casher, Topological Quantm effects for neutral particles, *Phys. Rev. Lett*., **53**(4), 319-321, 1984; *Anandan J., Phys. Rev. Lett*., **48**, 1660, 1982; *Phys. Rev. Lett. B*, **138**, 347, 1989.
20. P. Hyllus and E. Sjöqvist, Precession and Interference in the Aharonov-Casher and Scalar Aharonov-Bohm Effects, *Found. Phys*., **33**(7), 1085-1105, 2003 and all references therein.
21. B. Mirza and M. Zarei, *Eur. Phys. J. C*, **32**, 583-586, 2004; S. Dulat and K. Li, *Eur. Phys. J. C*, **54**(2), 333-337, 2008; S. M. Al-Jaber, *Nuovo Cimento B*, **110**, 1003-1005, 1995; J. Maeda and K. Z. Shizuya, *F. Physik C*, **60**, 265-270, 1993; Y. I. Azimov and R. M. Ryndin, The Aharonov-Casher Effect for Particles of Arbitrary Spin, *Preprint* arXiv:hep-ph/9710433v1; K. Li, Wang *J. Eur. Phys. J. C*, **50**, 1007-1011, 2007; H. Xia-Gang and H. Y. Bruce, The Topoligical AC and HMW Effect and the Dual Current in 2+1 Dimensions, *Preprint* arXiv:quant-ph/0009065v1.
22. M. Chaichian et al., *Phys. Lett. B,* **527**, 149-154, 2002.
23. U. Krey, The Aharonov-Bohm Effect, Non-commutative Geometry, Dislocation Theory, and Magnetism, *Preprint* arXiv:0711.085524.
24. S. De Leo and P. Rotelli, *Mod. Phys. Lett. A*, **11**, 357-366, 1996.
25. H. K. Lee and Y. Yoon, Effective Potential for Uniform Magnetic Fields Through Pauli Interaction, *Preprint* arXiv:hep-th/0611134v4.
26. B. R. Pascalutsa, M. V. Holstein and M. Vanderhaeghen, *Phys. Lett. B*, **600**, 239-247, 2004.
27. Y. Aharonov and D. Bohm, *Phys. Rev*., **115**, 485-91, 1959.
28. J. Planelles, J. I. Climente and F. Rajadell, Quantm Rings in Tilted Magnetic Fields, *Physica E*, **33**, 370-375, 2006.
29. D. Dragoman and S. Bogdan, The Aharonov-Bohm Effect in the Momentum Space, *Optics Comm*., **281**, 2685-2689¸2008.
30. L. De Honer, *Ann. Phys*., **323**, 317-331, 2008.
31. C. D. Yang, *Ann. Phys*., **321**, 2876-2926, 2006.
32. M. Lee and M. Y. Choi, Corresponces and Quantum Description of Aharanov-Bohm and Aharanov-Casher Effects, *Preprint* arXiv:quant-ph/0310032v1.