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EXACT SOLVABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FINITE ACTIVITY LEVY PROCESSES

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Abstract- We consider linearizing transformations of the one-dimensional nonlinear stochastic differential equations driven by Wiener and compound Poisson processes, namely finite activity Levy processes. We present linearizability criteria and derive the required transformations. We use a stochastic integrating factor method to solve the linearized equations and provide closed-form solutions. We apply our method to a number ofstochastic differential equations including Cox-Ingersoll-Ross short-term interest rate model, log-mean reverting asset pricing model and geometric Ornstein-Uhlenbeck equation all with additional jump terms. We use their analytical solutions to illustrate the accuracy of the numerical approximations obtained from Euler and Maghsoodi discretization schemes. The means of the solutions are estimated through Monte Carlo method.

Key Words- Stochastic differential equations, Levy processes, Stochastic integrating factors, Linearization

1. INTRODUCTION

The theory of stochastic differential equations has recently enjoyed significant reputation as a result of its impact on physics, finance and engineering [3,11,22,23,4,27]. Analytical solutions of stochastic differential equations not only allow us to study the underlying stochastic processes, but also provide the means to test the numerical schemes [10,19]. Therefore, analytical methods for the integration of nonlinear stochastic differential equations are of paramount importance.

Stochastic differential equations with jump terms, driven by Levy processes in general, are more realistic in cases where sudden events play prominent role [4,12,14,17,21,24,27]. Levy processes are basically stochastic processes with stationary and independent increments. They are analogues of the random walks in continuous time. Moreover, they form a subclass of semi-martingales and Markov processes which include very important special cases such as Brownian motion, Poisson process and subordinators. Although much of the basic theory was established earlier, a great deal of new theoretical development as well as novel applications in diverse areas has emerged in recent years [1,25]. Let $W = \{W_t, t \geq 0\}$ be a Wiener process and $N^j = \{N_t^j, t \geq 0\}$ be Poisson processes with arrival rates $\lambda_j, j = 1, 2, \ldots, m$ on a complete probability space (Ω, F, P). Let $V^j = \{V_{i\ t}^j, i = 1, 2, \ldots\}$ be independent and identically distributed real-valued random variables to form the compound Poisson process

$$C_t^j = \sum_{i=1}^{N_t^j} V_j^i \tag{1}$$

for each $j \in \{1,2,\ldots,m\}$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, P)$ with every $\mathcal{F}_t$ containing P-null subsets of $\mathcal{F}$. We consider a real-valued adapted stochastic process X, starting at time $t=0$ satisfying the nonlinear stochastic differential equation with jump terms of the form

$$dX_t = f(X_{t-},t)dt + g(X_{t-},t)dW_t + \sum_{j=1}^{m} r_j(X_{t-},t)dC_t^j \tag{2}$$

with $X_0 = 0$ where dW_t is the infinitesimal increment of the Wiener process, independently dC_t^j is the infinitesimal increment of the compound Poisson processes for $j = 1,2,\ldots,m$ and f, g and r_j are Borel functions on $\mathbb{R}\times\mathbb{R}_+$ [12,23]. Since a finite activity Levy process can be decomposed into a Wiener and a compound Poisson process as a special case of Levy-Itô decomposition [1]; we say that (2) is driven by a Levy process. Note that (2) is more general than a stochastic differential equation which involves the increment of a Levy process only as a single term.

In this paper, we derive the conditions for linearizability of (2) and solve the arising linear stochastic differential equation using integrating factor method. Linearization problem has been considered in [29,30] for equations including a single Poisson jump term, which serve as preliminaries of the present work. We extend and generalize these results by considering Equation (2) and demonstrate identification of a stochastic integrating factor for solving the linearized equation. Integrating factors have been used earlier for linear stochastic differential equations driven by a Wiener process [23]. On the other hand, the linearized version of (2) includes jump terms inherited from the compound Poisson processes (1) and is in the form

$$dY_t = dH_t + Y_{t-}dZ_t \tag{3}$$

where H and Z are semimartingales satisfying the condition that the jump $\Delta Z_t := Z_t - Z_{t-} \neq -1$ for all $t \in [0,\infty]$ and $Z_0 = 0$. The conditions imposed on the coefficients of (2) for the existence of X imply that H and Z are semimartingales. More references for the solution of related linear equations based on semimartingales and their generalization with adapted cadlag processes can be found in [7]. Linearization of purely continuous diffusion processes has been considered in [10, Chp.4]. Independently from the present work, linearizability conditions for a jump equation similar to (2) are studied for the time homogeneous case, that is, when $f(x,t) = f(x), g(x,t) = g(x)$ and $r(x,t) = r(x)$ by Gapeev [9]. However, the sufficient conditions for exact solvability of (2) have not been obtained by Gapeev in his study. Therefore, we present our rigorous derivations, which are obtained independently from [9], for the more general time inhomogeneous case and with the further generalization to several compound processes rather than a single Poisson random measure.

As applications, we show that Cox-Ingersoll-Ross short-term interest rate model [3], log-mean reverting asset pricing model [5,26,31], geometric Ornstein-Uhlenbeck equation [6,8,23,28] with additional jump terms and one other example [12,20] are linearizable under specified conditions on the functions f, g and r. Exact solutions of these linearizable equations are obtained. We then compare our analytical solutions with the numerical approximations found by Euler and Maghsoodi schemes to demonstrate the agreement.

The paper is organized as follows. In Section 2, we present our results about the linearization of nonlinear stochastic differential equations and give the linearizability criteria. The analytical solution of the resulting linear stochastic differential equation is found in Section 3. Several examples are given in Section 4 to demonstrate the linearizability conditions of several well-known equations with additional jump terms.

2. LINEARIZATION

We are interested in stochastic differential equations of the form (2), which can be transformed into a linear equation and solved as a result. Although one can impose certain conditions on the functions f, g and r_j, $j = 1, \ldots, m$ that are sufficient for the existence and uniqueness of the solution X, one can instead concentrate on the linearized equation and the conditions from there on.

We seek a sufficiently smooth Borel function $h : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of X_t to yield a change of variable

$$Y_t = h(X_t, t)$$

which will transform the nonlinear stochastic differential equation given in (2) into a linear equation of the form

$$dY_t = (a_1(t)Y_{t-} + a_2(t))dt + (b_1(t)Y_{t-} + b_2(t))dW_t + \sum_{j=1}^{m} (c_1^j(t, V_{N_t^j}^j)Y_{t-} + c_2^j(t, V_{N_t^j}^j))dN_t^j \tag{4}$$

since

$$dC_t^j = V_{N_t^j}^j dN_t^j$$

in (2). Note that the magnitude of the jump at time t can be found as $X_t - X_{t-}$ from the cadlag property, where $X_{t-} = \lim_{s \uparrow t} X_s$. If a unique solution of (4) exists and h is invertible, then Equation (2) also has a unique solution X. Therefore, we can focus on sufficient conditions for the existence of solutions of (4). We assume sufficiently smooth continuous functions a_1, a_2 and b_1, b_2 below and state the implied differentiability conditions on the original functions f and g. Linearizability conditions are found under smoothness assumptions on $r_j, j = 1, \ldots, n$ as well. Technical conditions on all functions are stated during the derivations, for the linear coefficients of (4) to be well-defined.

Suppose that $h \in C^{1,2}(\mathbb{R} \times \mathbb{R}_+)$ is a Borel function having an inverse k such that $h(k(t, x), t) = x$ [10]. Then, Itô's formula [1,23] for $h(X_t, t)$ leads to

$$dY_t = [\partial_t h(X_{t-}, t) + f(X_{t-}, t)\partial_x h(X_{t-}, t) + \frac{1}{2} g^2(X_{t-}, t)\partial_{xx} h(X_{t-}, t)]dt$$
$$+ g(X_{t-}, t)\partial_x h(X_{t-}, t)dW_t + \sum_{j=1}^{m} [h(X_{t-} + r_j(X_{t-}, t)V_{N_t^j}^j, t) - h(X_{t-}, t)]dN_t^j.$$

Using Equations (2) and (4), we obtain,

$$\partial_t h(X_{t-},t) + f(X_{t-},t)\partial_x h(X_{t-},t) + \frac{1}{2}g^2(X_{t-},t)\partial_{xx}h(X_{t-},t) = a_1(t)h(X_{t-},t) + a_2(t), \quad (5)$$

$$g(X_{t-},t)\partial_x h(X_{t-},t) = b_1(t)h(X_{t-},t) + b_2(t) \quad (6)$$

and

$$\sum_{j=1}^{m}[h(X_{t-} + r_j(X_{t-},t)V^j_{N^j_t},t) - h(X_{t-},t)]dN^j_t = \sum_{j=1}^{m}(c^j_1(t,V^j_{N^j_t})Y_{t-} + c^j_2(t,V^j_{N^j_t}))dN^j_t \quad (7)$$

Ordinary differential equation (6) has two distinct solutions for i) $b_1(t) = 0$ and ii) $b_1(t) \neq 0$. We now consider each case separately.

Case 1 $b_1(t) = 0$ and $b_2(t) \neq 0$

In this case, (10) is satisfied if

$$g(x,t)\partial_x h(x,t) = b_2(t),$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Then the transformation h can be found as

$$h(x,t) = \int^x \frac{b_2(t)}{g(\tilde{x},t)}d\tilde{x} \quad (8)$$

where we have chosen the arbitrary function of integration to be zero and assumed $g(\tilde{x},t) \neq 0$. Substituting (8) into (5) and differentiating with respect to x gives

$$\frac{b'_2(t)}{g(x,t)} + b_2(t)(L(x,t) - \frac{a_1(t)}{g(x,t)}) = 0 \quad (9)$$

where

$$L(x,t) = \partial_t(\frac{1}{g(x,t)}) + \partial_x(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)). \quad (10)$$

Multiplying both sides of (9) by $g(x,t)$ and differentiating with respect to x leads to

$$\partial_x[g(x,t)L(x,t)] = 0 \quad (11)$$

as we have assumed $b_2(t) \neq 0$. Then, we can choose $b_2 \in C^1$, set

$$a_1(t) = g(x,t)L(x,t) + \frac{b'_2(t)}{b_2(t)}$$

and find $a_2(t)$ for $f \in C^{2,0}(RxR_+)$ and $g \in C^{3,1}(RxR_+)$.

Now we consider (7), which is satisfied if

$$h(X_{t-} + r_j(X_{t-},t)V^j_{N^j_t},t) = (c^j_1(t,V^j_{N^j_t}) + 1)h(X_t,t) + c^j_2(t,V^j_{N^j_t})$$

$j = 1,\ldots,m$, for all $x \in \mathbb{R}, z \in \mathbb{R}, t \in \mathbb{R}_+$. Equivalently, we must have

$$\int^{x+r_j(x,t)z} \frac{b_2(t)}{g(\tilde{x},t)}d\tilde{x} = (c^j_1(t,z)+1)\int^x \frac{b_2(t)}{g(\tilde{x},t)}d\tilde{x} + c^j_2(t,z) \quad (12)$$

and differentiating it with respect to x, we obtain

$$\frac{b_2(t)}{g(x+r_j(x,t)z,t)}(\partial_x r_j(x,t)z + 1) = \frac{b_2(t)}{g(x,t)}(c^j_1(t,z)+1)$$

provided that $g(x + r_j(x,t)z,t) \neq 0$ for all $x, z \in \mathbb{R}$ We rewrite the above equation as

$$A_j(x,t) := (\partial_x r_j(x,t)z+1)\frac{g(x,t)}{g(x+r_j(x,t)z,t)} \tag{13}$$

$$= c_1^j(x,t)+1, \tag{14}$$

Now, $c_1(t)$ can be found from (14) and $c_2(t)$ can be found from (12). Differentiating (14) with respect to x yields

$$\partial_x A_j(x,t) = 0 \tag{15}$$

$j = 1,\ldots,m$ for $r_j \in C^{2,0}(\mathbb{R}\times\mathbb{R}_+)$. Therefore, (11) and (15) are the linearization conditions.

Case 2 $b_1(t) \neq 0$ and $b_2(t) = 0$

The solution of (10) in this case is in the form $h(x,t) = K(t)e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}}$. We simply choose $K(t)=1$ and seek $b_1(t)$. Thus, we get

$$h(x,t) = e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}}. \tag{16}$$

Substitution of (16) into (9) yields

$$a_2(t) = e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}}\left[\int^x \partial_t\left(\frac{b_1(t)}{g(\tilde{x},t)}\right)d\tilde{x})\right] + b_1(t)\left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right) + \frac{b_1^2(t)}{2} - a_1(t)]. \tag{17}$$

Differentiation of both sides of (22) with respect to x and simplification lead to

$$b_1(t)[g(x,t)L(x,t) + \int^x \partial_t\left(\frac{b_1(t)}{g(\tilde{x},t)}\right)d\tilde{x}) - a_1(t)] + b_1^2(t)\left[\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right]$$

$$+\frac{b_1^3(t)}{2} + b'_1(t) = 0 \tag{18}$$

where L is as in (10). We aim to find $b_1(t)$ first. Differentiating with respect to x and cancelling $b_1(t)$ as it is nonzero, we get

$$[\partial_x[g(x,t)L(x,t)] + \frac{b'_1(t)}{g(x,t)}] + b_1(t)[\partial_x\left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right) + \partial_t\left(\frac{1}{g(x,t)}\right)] = 0.$$

Therefore, we have

$$b'_1(t) + b_1(t)g(x,t)L(x,t) = -g(x,t)\partial_x[g(x,t)L(x,t)]. \tag{19}$$

Differentiating (19) with respect to x, we get

$$b_1(t) = -\frac{\partial_x(g(x,t)\partial_x[g(x,t)L(x,t)])}{\partial_x[g(x,t)L(x,t)]} \tag{20}$$

where we assume that $\partial_x[g(x,t)L(x,t)] \neq 0$. Differentiation of (20) with respect to x yields

$$\partial_x M(x,t) = 0 \tag{21}$$

where we have introduced

$$M(x,t) := \frac{\partial_x(g(x,t)\partial_x[g(x,t)L(x,t)])}{\partial_x[g(x,t)L(x,t)]}$$

for $f \in C^{2,0}(RxR_+)$, $g \in C^{3,1}(RxR_+)$ as a term in the linearization criterion (21).

Differentiation of (16) with respect to x yields

$$\partial_x h(x,t) = \frac{b_1(t)}{g(x,t)} e^{b_1(t)\int^x \frac{1}{g(\tilde{x},t)} d\tilde{x}} . \tag{22}$$

Substitution of (22) into (7) and then differentiating with respect to x and cancelling $b_1(t)$, we get

$$A_j(x,t) e^{\left(\int^{x+r_j(z,t)z} \frac{b_1(t)}{g(\tilde{x},t)} d\tilde{x} - \int^x \frac{b_1(t)}{g(\tilde{x},t)} d\tilde{x}\right)} = (c_1^j(t,z)+1) \tag{23}$$

$j = 1,\ldots,m$ in terms of A_j of (13). Differentiation of (23) with respect to x yields

$$\partial_x A_j(x,t) + A_j(x,t) b_1(t) \left(\frac{A_j(x,t)-1}{g(x,t)}\right) = 0 \tag{24}$$

as the linearization condition involving $r_j \in C^{2,0}(RxR_+)$.

Therefore, Equations (21) and (24) are the linearization conditions in Case 2. Now, one can obtain $a_1(t)$ from (18), $a_2(t)$ from (17), $b_1(t)$ from (20) with (10), $c_1(t)$ from (23) and $c_2(t)$ by substitution of (22) into (7) finally.

We now state our findings as a theorem.

Theorem 1 *A nonlinear stochastic differential equation (2) is linearizable via the transformation*

$$h(x,t) = \int^x \frac{d\tilde{x}}{g(\tilde{x},t)}$$

if (11) and (15) are satisfied or via

$$h(x,t) = e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)} d\tilde{x}}$$

if (21) and (24) are satisfied where $b_1(t)$ *is given in (18).*

Note that the two sets of conditions in Theorem 1 are mutually exclusive. If (11) is satisfied, then (21) is not possible as it originates from (19) where $b_1(t)$ is nonzero and (11) cannot hold. Clearly, this argument holds both ways. The results of Theorem 1 are similar to [9, Eqns.(3.10)-(3.12)], which are in particular based on time-homogeneous functions f and g and a single Poisson random measure.

3. SOLUTION OF THE LINEAR EQUATION

In this section, we use the integrating factor method for solving linear ODEs to the linear SDEs driven by finite activity Levy processes and find the solution of (7). The solution of a linear jump-diffusion equation has been considered in [9] as cited by [18], which is also based on an integrating factor. More generally, the solution to linear system of SDEs based on semi-martingales [15], and more recently including cadlag processes [7], is well-known. We demonstrate the integrating factor method. Our solution appears in a more compact form mainly because the specific form of the semimartingales are used during the derivation.

A process μ_t serves as a stochastic integrating factor for solving (7) if the product $\mu_t Y_t$ is not a function of Y. To find such a μ, we start by applying the product rule for semimartingales [1] as

$$\mathrm{d}(\mu_t \mathrm{Y}_t) = \mu_{t^-}\mathrm{dY}_{t^-} + \mathrm{Y}_{t^-}\mathrm{d}\mu_t + \mathrm{d}[\mu, \mathrm{Y}]_t.$$

Itô's formula for μ reads,

$$\mathrm{d}\mu_t = (\partial_t \mu_{t-} + \frac{1}{2}\partial_{xx}\mu_{t^-})\mathrm{d}_t + \partial_x \mu_{t-}\mathrm{dW}_t + \sum_{j=1}^{m}(\mu_t \text{ -}\mu_{t-})\mathrm{dN}_t^j \quad . \tag{25}$$

Therefore, from (7) and (25) we can write the terms which will form the differential product $d\left(\mu_t Y_t\right)$ as

$$\mu_\downarrow(t-)dY_\downarrow t = \mu_\downarrow(t-)[\![a]\!]_\downarrow 1(t)Y_\downarrow(t-) + a_\downarrow 2(t))dt + \mu_\downarrow(t-)(b_\downarrow 1(t)Y_\downarrow(t-) + c_\downarrow 2^\uparrow j(t, V_\downarrow(N_\downarrow t^\uparrow j)^\uparrow j))dN_\downarrow t^\uparrow j$$

as well as

$$Y_{t^-} d\mu_t = Y_{t^-}\left(\partial_t \mu_{t^-} + \frac{1}{2}\partial_{xx}\mu_{t^-}\right)dt + Y_{t^-}\partial_x \mu_{t^-} dW_t + Y_{t^-}\sum_{j=1}^{m}\left(\mu_t - \mu_{t^-}\right)dN_t^j$$

and the quadratic variation

$$d\left[\mu, Y\right]_t = \partial_x \mu_{t^-}\left[b_1(t)Y_{t^-} + b_2(t)\right]dt + \sum_{j=1}^{...}\left[\mu_t - \mu_{t^-}\right]\left[c_1^j\left(t, V_{N_t^j}^j\right)Y_{t^-} - c_2^j\left(t, V_{N_t^j}^j\right)\right]dN_t^j.$$

These terms should not involve the variable Y_{t^-} for μ_t to comply with the definition of an integrating factor. Arranging all the terms yields

$$\begin{aligned} d\mu_t Y_t = {} & Y_{t^-}\left(\partial_t \mu_{t^-} + \frac{1}{2}\partial_{xx}\mu_{t^-} + b_1(t)\partial_x \mu_{t^-} + a_1(t)\mu_{t^-}\right)dt + Y_{t^-}\left(\partial_x \mu_{t^-} + b_1(t)\mu_{t^-}\right)dW_t \\ & + Y_{t^-}\left(\sum_{j=1}^{m}\left(1 + c_1^j\left(t, V_{N_t^j}^j\right)\right)\left(\mu_t - \mu_{t^-}\right)dN_t^j + \mu_{t^-}\sum_{j=1}^{m} c_1^j\left(t, V_{N_t^j}^j\right)dN_t^j\right) \\ & + \left[\mu_{t^-}a_2(t) + \partial_x \mu_{t^-}b_2(t)\right]dt + \mu_{t^-}b_2(t)dW_t + \sum_{j=1}^{m}\mu_t c_2^j(t, V_{N_t^j}^j)dN_t^j \end{aligned} \tag{26}$$

Hence, we must have

$$\partial_t \mu_{t^-} + \frac{1}{2}\partial_{xx}\mu_{t^-} + b_1(t)\partial_x \mu_{t^-} + a_1(t)\mu_{t^-} = 0 \tag{27}$$

$$\partial_x \mu_{t^-} + b_1(t)\mu_{t^-} = 0 \tag{28}$$

and

$$\sum_{j=1}^{m}\left[\left(1 + c_1^j\left(t, V_{N_t^j}^j\right)\right)\left(\mu_t - \mu_{t^-}\right)\right]dN_t^j = 0 \tag{29}$$

Let us seek a solution to the system of PDEs (27) and (28) together with (29). Equation (29) implies that

$$\mu_t = \frac{1}{1+c_1^j\left(t,V_{N_t^j}^j\right)}\mu_{t^-} \tag{30}$$

if there is a jump of the compound Poisson process C^j at time *t*, for some $j \in \{1,\cdots,m\}$, provided that $1+c_1^j(t,z) \neq 0$. For this condition to hold, we need that

$$\partial_x r_j(x,t) z \neq -1 \tag{31}$$

by (14) if $b_1(t)=0$ as we have already assumed $b(x,t)\neq 0$, and by (13) and (29) if $b_2(t)=0$. Clearly, (31) cannot hold for all $x,z\in R$ and $t\in R_+$. That is why we require that the jump amount V_i^j of the compound Poisson processes satisfy

$$\partial_x r_j(x,t)V_i^j \neq -1 \quad a.s.$$

for each $x\in R$ and $t\in R_+$. A sufficient condition is to have an absolutely continuous measure $F_j(dv)$.

Since μ_t has a continuous part, (30) implies that it is in the form

$$\mu_t = M_1(W)_t M_2\left(C^1,\cdots,C^m\right)_t \tag{32}$$

where the functional M_1 and M_2 denote the continuous and discontinuous parts of μ_t, respectively. Note that since the Poisson processes N^j, $j=1,\ldots,m$ are independent, at most one jump occurs from only one of $N^1,\ldots,N^m$ with probability 1 at any time. Therefore, the discontinuous part M_2 can be written by (30) as

$$M_2\left(C^1,\ldots,C^m\right)_t = \prod_{j=1}^{m}\prod_{S_i^j\le t}\frac{1}{1+c_1^j\left(S_i^j,V_i^j\right)} \tag{33}$$

where $\left(S_i^j,V_i^j\right), i=1,\ldots,N_t^j$ denote the pump denote the jump times and jump amounts of the compound Poisson process C^j up to time *t*, for $j=1,\ldots,m$.

Now, we will fill find M_1 using (27) and (28). Substitution of (32) into (28) leads to

$$\partial_x M_1(W)_t + b_1(t)M_1(W)_t = 0 \tag{34}$$

By Itô's formula and in view of (34), we have

$$dM_1(W)_t = \partial_t M_1(W)_t\,dt - b_1(t)M_1(W)_t\,dW_t + \frac{1}{2}b_1^2(t)\,dM_1(W)_t\,dt. \tag{35}$$

To solve (35), we try M_1 which satisfies

$$\partial_t M_1(W)_t = M_1(W)_t\,q'(t) - \frac{1}{2}b_1^2(t)M_1(W)_t \tag{36}$$

for some Stieltjes function *q* and we have taken q' for the sake of brevity in the sequel. Hence, (35) reduces to

$$dM_1(W)_t = -b_1(t)M_1(W)_t\,dW_t + M_1(W)_t\,q'(t)\,dt \tag{37}$$

and solution to (37) is

$$M_1(W)_t = e^{-\int_0^t b_1(s)dW_s - \frac{1}{2}\int_0^t b_1^2(s)ds + q(t)} \tag{38}$$

where the integral $\int_0^t b_1(s)\,dW_s$ is a Riemann-Stieltjes integral since $b_1(t)$ is a deterministic function [22]. Note that solution (38) indeed satisfies (36). Thus, the stochastic integrating factor takes the form

$$\mu_t = e^{-\int_0^t b_1(s)dW_s - \frac{1}{2}\int_0^t b_1^2(s)ds + q(t)} \prod_{j=1}^{m} \prod_{S_i^j \le t} \frac{1}{1 + c_1^j\left(S_i^j, V_i^j\right)} \tag{39}$$

We now use μ_t of (39) in (27) to obtain $-\frac{1}{2}b_1^2(t) + q'(t) + \frac{1}{2}b_1^2(t) - b_1^2(t) + a_1(t) = 0.$

Then, we get $q(t) = \int_0^t \left(b_1^2(s) - a_1(s)\right)ds.$ Therefore, (39) becomes

$$\mu_t = \exp\left(-\int_0^t b_1(s)\,dW_s + \int_0^t \left[\frac{1}{2}b_1^2(s) - a_1(s)\right]ds\right) \prod_{j=1}^{m} \prod_{S_i^j \le t} \frac{1}{1 + c_1^j\left(S_i^j, V_i^j\right)}. \tag{40}$$

For determining Y_t, consider equations (26) and (28) which now imply

$$Y_t = \mu_t^{-1} \left(\begin{array}{l} \int\limits_0^t \mu_{s^-}[a_2(s) - b_1(s)b_2(s)]ds + \int\limits_0^t \mu_{s^-} b_2(s)dW_s \\ + \sum_{j=1}^{m} \int_0^t \mu_s c_2^j\left(s, V_{N_s^j}^j\right) dN_s^j \end{array} \right). \tag{41}$$

Substituting (40) into Equation (41) yields the explicit solution of the linear SDE (7)

$$\text{as } Y_t = \exp\left(M(t)\right) \prod_{j=1}^{m} \prod_{S_i^j \le} \left(1 + c_1^j\left(S_i^j, V_i^j\right)\right) . \left\{ \begin{array}{l} \int_0^t e^{-M(s)} M_2(s^-)\left[a_2(s) - b_1(s)b_2(s)\right]ds \\ + \int_0^t e^{-M(s)} M_2(s^-) b_2(s) dW_s \\ \prod_{j=1}^{m} \prod_{S_i^j \le t} \frac{e^{-M(S_i^j)} c_2^j\left(S_i^j, V_i^j\right)}{1 + c_1^j\left(S_i^j, V_i^j\right)} \end{array} \right\}$$

where

$$M_t = \int_0^t b_1(s)dW_s + \int_0^t \left[a_1(s) - \frac{1}{2}b_1^2(s)\right]ds$$

and M_2 is as in (33).

4. ANALYTICAL SOLUTIONS OF SPECIFIC EXAMPLES

We now consider some linearizable SDEs driven by Wiener and compound Poisson processes and compare analytical solutions with the Euler and Maghsoodi numerical approximations [13,19,20]. All examples of this section satisfy criteria (11) and (15).

4.1. Example 1

The second example is taken from [12], with extra jump terms given by

$$dX_t = (\alpha(t)X_{t-}^{\frac{3}{4}} + \frac{3}{8}\beta^2 X_{t-}^{\frac{1}{2}})dt + \beta X_{t-}^{\frac{3}{4}} dW_t + \sum_{j=1}^{m} \gamma_j X_{t-} dC_t^j \tag{42}$$

with $X_0 = x_0$, where $f(x,t) = \alpha(t)x^{\frac{3}{4}} + \frac{3}{8}\beta^2 x^{\frac{1}{2}}, g(x) = \beta x^{\frac{3}{4}}, r_j(x) = \gamma_j x$ and $\alpha(t), \beta, \gamma_j, j = 1, \ldots, m$ are positive real valued parameters.

The transformation

$$Y_t = \frac{4}{\beta}(X_t^{\frac{1}{4}} - x_0^{\frac{1}{4}}) \tag{43}$$

linearizes Equation (42) into

$$dY_t = \frac{\alpha(t)}{\beta}dt + dW_t + \sum_{j=1}^{m}[((1+\gamma_j V_{N_t^j}^j)^{\frac{1}{4}} - 1)Y_{t-} + \frac{4}{\beta}x_0^{\frac{1}{4}}((1 \quad + \ \gamma_j V_{N_t^j}^j)^{\frac{1}{4}} - 1)dN_t^j)$$

which corresponds to (7) with $a_1(t) = 0$, $a_2(t) = \frac{\alpha(t)}{\beta}$, $b_1(t) = 0$, $b_2(t) = 1$ $c_1(t,z) = (1+v_j z)^{\frac{1}{4}} - 1$ and $c_2(t,z) = \frac{4}{\beta}x_0^{\frac{1}{4}}((1+\gamma_j z)^{\frac{1}{4}} - 1)$. The solution is

$$Y_t = \mu_t^{-1}(\int_0^t \mu_{s-}\frac{\alpha(s)}{\beta}ds + \int_0^t \mu_{s-}dW_s + \frac{4}{\beta}x_0^{\frac{1}{4}}\sum_{j=1}^{m}\int_0^t \mu_s((1 \quad + \ \gamma_j V_{N_s^j}^j)^{\frac{1}{4}} - 1)dN_s^j)$$

where the integrating factor is

$$\mu_t = \prod_{j=1}^{m}\prod_{i=1}^{N_t^j}(1+\gamma_j V_i^j)^{-\frac{1}{4}}.$$

Transformation (43) leads to the solution

$$X_t = \left(x_0^{\frac{1}{4}} + \frac{\beta}{4}\mu_t^{-1}(\int_0^t \mu_{s-}\frac{\alpha(s)}{\beta}ds + \int_0^t \mu_{s-}dW_s \ + \frac{4}{\beta}x_0^{\frac{1}{4}}\sum_{j=1}^{m}\int_0^t \mu_s((1+\ \gamma_j V_{N_s^j}^j)^{\frac{1}{4}} - 1)dN_s^j) \right)^4.$$

4.2. Cox Ingersoll Ross Model

The third equation is taken from [3] but with an additional jump term as

$$dX_t = \alpha(\beta - X_{t-})dt + \sigma X_{t-}^{\frac{1}{2}} dW_t + \sum \gamma_j X_{t-} dC_t^j \tag{44}$$

with $X_0 = x_0$, where $\alpha, \beta, \sigma, \gamma_j, j = 1, \ldots, m$ are positive real valued. This equation is know as the Cox-Ingersoll-Ross interest rate model, before the jump terms are added. Here, X_t represents the mean-reverting short-term interest rate. In this model, β is the long-term average value of interest rate with jumps, α is the intensity (strength) of mean reversion, σ is the interest rate volatility where X_t is the instantaneous interest rate at period *t*, maturing at period *T*.

In (44), we have $f(x)=\alpha\left(\beta-x\right), g(x)=\sigma x^{\frac{1}{2}}, r_j(x)=\gamma_j x.$ These functions satisfy the linearization criteria (11) and (15) when $\sigma=2\sqrt{\alpha\beta}$. Indeed,

$$\partial_x\left[g(x,t)L\right]=\partial_x\left[\sigma x^{\frac{1}{2}}\left(\partial_x\left[\left(\frac{\alpha\beta}{\sigma}-\frac{1}{4}\sigma\right)x^{-\frac{1}{2}}-\frac{\alpha}{\sigma}x^{\frac{1}{2}}\right]\right)\right]=0$$

is satisfied when $\sigma=2\sqrt{\alpha\beta}.$ Using the transformation, we get

$$Y_t=\int_{x_0}^{x}\frac{d\tilde{x}}{\sigma\tilde{x}^{\frac{1}{2}}}=\frac{1}{\sqrt{\alpha\beta}}\left(X_t^{\frac{1}{2}}-x_0^{\frac{1}{2}}\right). \tag{45}$$

We see that equation (44) transforms into

$$dY_t=-\frac{\alpha}{2}Y_{t^-}dt+dW_t+\sum_{j=1}^{m}\left[\begin{array}{l}\left(\left(1+\gamma_j V_{N_t^j}^j\right)^{\frac{1}{2}}-1\right)Y_{t^-}\\+\frac{1}{\sqrt{\alpha\beta}}x_0^{\frac{1}{2}}\left(\left(1+\gamma_j V_{N_t^j}^j\right)^{\frac{1}{2}}-1\right)\end{array}\right]dN_t^j$$

where we have $a_1(t)=-\frac{\alpha}{2}, a_2(t)=0, b_1(t)=0, b_2(t)=1, c_1(t,z)=\left(1+\gamma_j z\right)^{\frac{1}{2}}-1$ and $c_2(t,z)=\frac{1}{\sqrt{\alpha\beta}}x_0^{\frac{1}{2}}\left(\left(1+\gamma_j z\right)^{\frac{1}{2}}-1\right)$. Integration yields

$$Y_t=\mu_t^{-1}\left(\int_0^t\mu_{s^-}dW_s+\frac{1}{\sqrt{\alpha\beta}}x_0^{\frac{1}{2}}\sum_{j=1}^{m}\int_0^t\mu_s((1+\ \gamma_j V_{N_s^j}^j)^{\frac{1}{2}}-1)dN_s^j)\right)\frac{1}{\sqrt{\alpha\beta}}\left(X_t^{\frac{1}{2}}-x_0^{\frac{1}{2}}\right)$$

where $\mu_t=e^{-\frac{\alpha}{2}t}\prod_{j=1}^{m}\prod_{i=1}^{N_t^j}\left(1+\gamma_j V_i^j\right)^{-\frac{1}{2}}$. Transformation (45) leads to the solution

$$X_t=\left(x_0^{\frac{1}{2}}+\mu_t^{-1}\left(\sqrt{\alpha\beta}\int_0^t\mu_{s^-}dW_s+x_0^{\frac{1}{2}}\sum_{j=1}^{m}\int_0^t\mu_{s^-}dN_s^j\right)\right)^2.$$

4.3. Log Mean-Reverting Model

The following example is a log-mean-reverting Black-Karasinski interest rate model [2,5,26,31] with a jump term. We have

$$dX_t=\eta X_{t-}(\theta(t)-\ln X_{t-})dt+\rho X_{t-}dW_t+\sum_{j=1}^{m}\zeta_j X_{t-}dC_t^j \tag{46}$$

with $X_0=x_0$, where $\eta,\theta(t),\rho,\zeta_j, j=1,\ldots,m$ are positive real valued. This log mean-reverting equation with jumps is commonly used in modeling assets subject to supply and demand such as commodities. Due to advantage of ease of simulation, modeling and parameters estimation, this model is widely preferred. Therefore, X_t now corresponds to the spot price of the commodity. In this model, θ is the long-run mean of the logarithm of the price with jumps, η is the mean reversion speed (intensity) of the price and ρ is the price volatility.

We have $f(x,t)=\eta x(\theta(t)-\ln x), g(x)=\rho x, r_j(x)=\zeta_j x$ from (46). The transformation given by

$$Y_t=\frac{1}{\rho}\ln\frac{X_t}{x_0}, \tag{47}$$

linearizes Equation (46) into

$$dY_t=(-\eta Y_{t-}+(\frac{\rho}{2}+\frac{\mu\theta(t)}{\rho}))dt+dW_t+\sum_{j=1}^{m}\frac{1}{\rho}\ln(1+\zeta_j V^j_{N^j_t})dN^j_t$$

with $a_1(t)=-\eta$, $a_2(t)=\frac{\rho}{2}+\frac{\mu\theta(t)}{\rho}$, $b_1(t)=0$, $b_2(t)=1$ $c_1(t,z)=\frac{1}{\rho}\ln(1+\zeta_j z)$ and $c_2(t,z)=0$. The solution is found as

$$Y_t=\mu_t^{-1}\left(\int_0^t\mu_{s^-}(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho})ds+\int_0^t\mu_{s^-}dW_s+\sum_{j=1}^{m}\int_0^t\mu_s\frac{1}{\rho}\ln(1+\zeta_j V^j_{N^j_s})dN^j_s\right)$$

where $\mu_t=e^{\eta t}$.

Finally, the use of (47) leads to the solution

$$X_t=x_0\exp\left[\rho\mu_t^{-1}\left(\int_0^t\mu_{s^-}(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho})ds+\int_0^t\mu_{s^-}dW_s+\sum_{j=1}^{m}\int_0^t\mu_s\frac{1}{\rho}\ln(1+\zeta_j V^j_{N^j_s})dN^j_s\right)\right].$$

4.4. Geometric Ornstein-Uhlenbeck Equation

The last example is from [6,8,23,28], a geometric Ornstein-Uhlenbeck equation with an additional jump term. We have

$$dX_t=\xi(t)X_{t-}(\eta(t)-X_{t-})dt+\delta X_{t-}dW_t+\sum_{j=1}^{m}\lambda_j X_{t-}dC^j_t \tag{48}$$

with $X_0=x_0$, where $\xi(t),\eta(t),\delta,\lambda_j, j=1,\ldots,m$ are positive real valued. Equation (48) is also known as geometric Ornstein-Uhlenbeck or Dixit & Pindyck model, now including additional jump terms. This model is based on a mean-reverting commodity price or interest rate X_t. In this equation, the mean reversion component is governed by the difference between the current price and the mean η as well as by the mean reversion rate ξ where δ is the volatility of the spot price. Note that, spot price X_t is always positive.

Here, the functions in (2) correspond to

$$f(x)=\xi(t)x(\eta(t)-x), g(x)=\delta x, r_j(x)=\lambda_j x$$

The transformation given by

$$Y_t=(\frac{X_t}{x_0})^{-1} \tag{49}$$

linearizes Equation (48) into

$$dY_t = [(\delta^2 - \xi(t)\eta(t))Y_{t-} + \xi(t)x_0]dt + (-\delta)Y_{t-}dW_t + \sum_{j=1}^{m}\left(\frac{-\lambda_j V^j_{N^j_t}}{1+\lambda_j V^j_{N^j_t}}\right)Y_{t-}dN^j_t$$

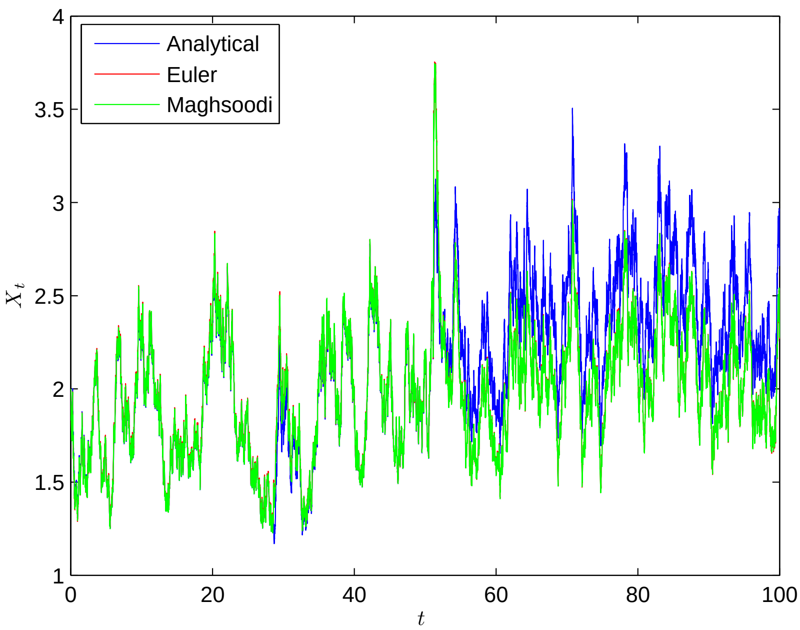
with $a_1(t) = \delta^2 - \xi(t)\eta(t)$, $a_2(t) = \xi(t)x_0$, $b_1(t) = -\delta$, $b_2(t) = 0$ $c_1(t,z) = \dfrac{-\lambda_j z}{1+\lambda_j z}$ and $c_2(t,z) = 0$. Integration gives,

$$Y_t = \mu_t^{-1} x_0 \int_0^t \mu_{s^-}\xi(s)ds$$

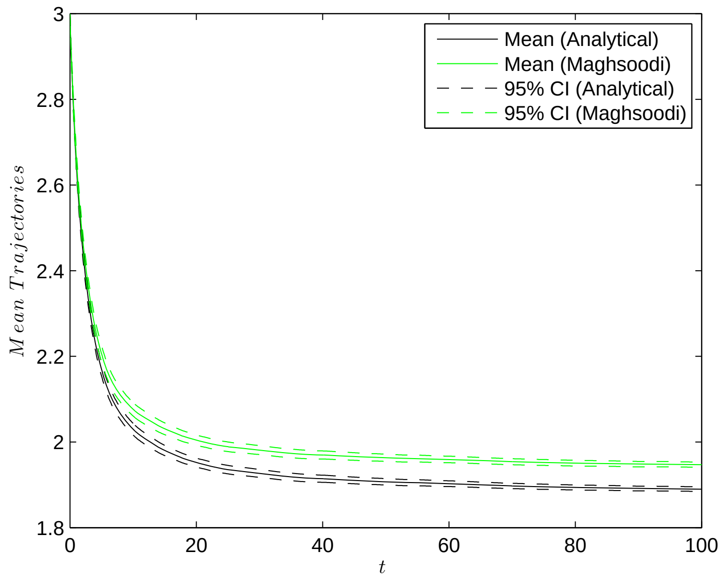
where $\mu_t = \exp(-\delta W_t - \frac{1}{2}\delta^2 t + \int_0^t \xi(s)\eta(s)ds)\prod_{j=1}^{m}\prod_{i=1}^{N_t^j}(1+\lambda_j V_i^j)$. As before, (49) leads to the solution

$$X_t = \mu_t x_0^{-1}\left[\int_0^t \mu_{s^-}\xi(s)ds\right]^{-1}$$

We have simulated $N = 10000$ trajectories for all numerical approximations with $x_0 = 3,, T = 100, \xi(t) = 0.3, \eta = 1.9, \delta = 0.2, \lambda_j = 1$, jump intensity $\lambda = 0.03$ and $\Delta t = 10^{-3}$. It can be seen that the analytical sample trajectories in Fig. 1a) nearly coincides with the mean estimated from the numerical approximations. However, the means are statistically significantly different as shown in Fig. 1b).



a)



b)

Fig. 1 a) Simulation of the exact solution and the numerical approximations. b) Mean trajectories estimated from 10000 independent replications for the geometric O-U equation.

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