Mathematical and Computational Applications, Vol. 17, No. 1, pp. 56-67, 2012

ON THE STABILITY OF DELAY POPULATION DYNAMICS RELATED WITH ALLEE EFFECTS

O. A. Gumus and H. Kose

Department of Mathematics, Selcuk University, 42070, Konya, Turkey
ozlemak@selcuk.edu.tr, hkose@selcuk.edu.tr

Abstract- In recent years, many scientists have focus on the studies of the Allee effect in population dynamics. This paper presents the stability analysis of equilibrium points of population dynamics with Allee effect which occurs at low population density.

Key Words- Population dynamics, Allee effect, Stability, Equilibrium point

1. INTRODUCTION

When previous studies have been examined on population dynamics including differential and difference equations, it is generally observed that Allee effect can have either a stable or an unstable effect on the system [1,3,5,6,8,9,11-14,16-19]. Nonetheless, discrete-time models are more suitable for numerical solutions and calculations [10,15].

Allee effect was first defined by Allee as negative density dependence when the growth rate of the population decreases in low population density. This effect can consist of social dysfunction at small population size, inbreeding depression, food exploitation, predator avoidance of defence and difficulties finding in mates. Authors have studied the stability of different population models within the framework of these effects and developed similar models. Besides, stability analysis is an important research topic in such studies.

In this present study, our purpose is to investigate and compare the stability of equilibrium point with and without Allee effect by considering a more general state of the model studied in [3]. Let's look at the nonlinear general delay difference equation

$$N_{t+1} = F(\lambda, N_t, N_{t-1}, N_{t-2}) \tag{1}$$

where λ is per capita growth rate which is always positive, N_t represents the population density at time t and T is the time for sexual maturity. Also, F has the following form:

$$F(\lambda, N_t, N_{t-1}, N_{t-2}) = \lambda N_t f(N_{t-1}, N_{t-2}), \quad \lambda > 0$$

where $f(N_{t-1}, N_{t-2})$ is the function describing interactions (competitions) among mature individuals.

We assume that f satisfies the following conditions:

(1) $\partial F / \partial N_{t-1}(N,N) < 0,\ \ \partial F / \partial N_{t-2}(N,N) < 0\ \ for\ N \in [0,\infty).$

(2) f(0,0) is a positive finite number.

This paper is organized as follows: In section 2, first of all, we give a characterization of the stability of the equilibrium points of Eq.(1). In section 3, we work on the stability analysis of the equilibrium points in Eq.(1) with the Allee effect. In section 4, we present numerical simulations that support the analytical result. Finally the last section of the paper includes conclusions.

2. STABILITY ANALYSIS OF Eq.(1)

Before we give the main results of this paper, we shall remind the following Schur-Cohn criterion (see references [2,4,15]).

Theorem 1. (Schur-Cohn Criteria**)** The roots of the characteristic polynomial,
$g(\sigma)=\sigma^k+a_1\sigma^{k-1}+a_2\sigma^{k-2}+...+a_k$
lie inside the unit circle if and only if the following hold:
(i) $g(1)>0$
(ii) $(-1)^k g(-1)>0,$
(iii) the $(k-1)\times(k-1)$ matrices

$$B_{k-1}^{\pm}=\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ a_1 & 1 & \cdots & & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{k-3} & & & 1 & 0 \\ a_{k-2} & a_{k-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & a_k \\ 0 & 0 & \cdots & a_k & a_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_k & & a_4 & a_3 \\ a_k & a_{k-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

are positive innerwise.
The characteristic polynomial which is getting from linearization of Eq.(1) around N^* will be
$g(\sigma)=\sigma^3-p\sigma^2-q\sigma-r.$

Assume that Eq.(1) has an equilibrium points as N^*. Then we get the following theorem.

Theorem 2. N^* is locally stable if and only if the inequalities

$$N^*\left[\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}-\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right]<2 \tag{2}$$

$$N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\left[N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}-1\right]-N^*\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}<1 \tag{3}$$

$$N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\left[N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+1\right]+N^*\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}<1 \tag{4}$$

hold.

Proof. From the equilibrium point definition of Eq.(1), we have

$$1=\lambda f(N^*,N^*). \tag{5}$$

Let's take $p=F_{N_t}(\lambda,N^*,N^*)$, $q=F_{N_{t-1}}(\lambda,N^*,N^*)$ and $r=F_{N_{t-2}}(\lambda,N^*,N^*)$. Given Eq.(5), $p=1.$ From Eq.(1), the values of q and r are

$$q = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}, \quad r = N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)}.$$

We get that N^* is locally stable if and only if

$$|p+r| < 1-q \text{ and } |pr+q| < 1-r^2 \tag{6}$$

by Theorem 1. If we write the values of p, q and r in the first inequality of Eq.(6), we obtain

$$\left| 1 + N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - 1 < 1 + N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

It is easy to see that in this case

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 2$$

and

$$N^* \left[\frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} - \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right] < 0.$$

Since $f_{N_{t-2}}$, $f_{N_{t-1}}$ are negative values for $N \in [0, \infty)$, the last inequality is always provided. Therefore, (2) is confirmed.

Now, if the values of p, q and r are written in the second inequality in (6), we get

$$\Leftrightarrow \left| N^* \left[\frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right] \right| < 1 - N^{*2} \frac{f^2_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)}$$

$$\Leftrightarrow N^{*2} \frac{f^2_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)} - 1 < N^* \left[\frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right] < 1 - N^{*2} \frac{f^2_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)}$$

If the last expressions is written in the form of two inequalities, we can write

$$\Leftrightarrow N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^{**}, N^*)} \left[N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} - 1 \right] - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

$$\Leftrightarrow N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \left[N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + 1 \right] + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

as confirmed.

3. ALLEE EFFECTS ON THE DISCRETE DELAY MODEL (1)

In this section, we study the local stability analysis of the equilibrium points of Eq.(1) with the addition of Allee effect at time $t-2,$ t and $(t,t-2).$

3.1. Allee effect at time t-2

We consider the following non-linear delay difference equation by the addition of Allee effect to discrete delay model Eq.(1)

$$N_{t+1}=F^{(\alpha^{-})}(\lambda^{*},N_{t},N_{t-1},N_{t-2})=\lambda^{*}N_{t}\alpha(N_{t-2})f(N_{t-1},N_{t-2}),\quad \lambda^{*}>0 \tag{7}$$

where the function f satisfies the properties (1) and (2). The conclusion of the biological facts requires the following assumption on α.

(3) if $N=0,$ then $\alpha(N)=0;$ that is, there is no reproduction without partners.

(4) $\partial\alpha/\partial N>0$ for $N\in(0,\infty);$ that is, Allee effect decreases as density increases.

(5) $\lim_{N\to\infty}\alpha(N)=1;$ that is, Allee effect vanishes at high densities.

Eq.(7) has the same positive equilibrium points with (1), since λ^{*} is normalized growth rate such that $\lambda^{*}=\lambda/\alpha.$ Then we get the following theorem.

Theorem 3. N^{*} is locally stable if and only if the inequalities

$$N^{*}\left[\frac{f_{N_{t-1}}(N^{*},N^{*})}{f(N^{*},N^{*})}-\frac{\alpha'(N^{*})}{\alpha(N^{*})}-\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right]<2, \tag{8}$$

$$N^{*}\left[\frac{f_{N_{t-1}}(N^{*},N^{*})}{f(N^{*},N^{*})}+\frac{\alpha'(N^{*})}{\alpha(N^{*})}+\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right]<0$$

$$N^{*2}\left[\frac{\alpha'^{2}(N^{*})}{\alpha^{2}(N^{*})}+\frac{f^{2}_{N_{t-2}}(N^{*},N^{*})}{f^{2}(N^{*},N^{*})}+2\frac{\alpha'(N^{*})}{\alpha(N^{*})}\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right] \tag{9}$$

$$-N^{*}\left[\frac{\alpha'(N^{*})}{\alpha(N^{*})}+\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}+\frac{f_{N_{t-1}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right]<1$$

$$N^{*2}\left[\frac{\alpha'^{2}(N^{*})}{\alpha^{2}(N^{*})}+\frac{f^{2}_{N_{t-2}}(N^{*},N^{*})}{f^{2}(N^{*},N^{*})}+2\frac{\alpha'(N^{*})}{\alpha(N^{*})}\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right] \tag{10}$$

$$+N^{*}\left[\frac{\alpha'(N^{*})}{\alpha(N^{*})}+\frac{f_{N_{t-2}}(N^{*},N^{*})}{f(N^{*},N^{*})}+\frac{f_{N_{t-1}}(N^{*},N^{*})}{f(N^{*},N^{*})}\right]<1$$

hold.

Proof. From the equilibrium point definition for Eq.(7), it is clear that $p=F_{N_t}(\lambda^{*},N^{*},N^{*})=1.$ Likewise, if the values of q and r are calculated for Eq.(7), we get

$$q = F_{N_{t-1}}(\lambda^*, N^*, N^*) = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

$$r = F_{N_{t-2}}(\lambda^*, N^*, N^*) = N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right].$$

The first inequality in (6) leads the following inequality

$$\left| 1 + N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] \right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - 1 < 1 + N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

or equivalently

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 2$$

and

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 0.$$

For the second inequality in (6), we arrive at

$$N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] \left(N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] - 1 \right) - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

$$N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] \left(N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] + 1 \right) + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

From the last two inequality, we can write

$$N^{*2} \left[\frac{\alpha'^2(N^*)}{\alpha^2(N^*)} + \frac{f^2_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)} + 2 \frac{\alpha'(N^*)}{\alpha(N^*)} \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right]$$

$$- N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right] < 1,$$

$$N^{*2} \left[\frac{\alpha'^2(N^*)}{\alpha^2(N^*)} + \frac{f^2_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)} + 2 \frac{\alpha'(N^*)}{\alpha(N^*)} \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right]$$

$$+ N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right] < 1$$

as required.

3.2. Allee effect at time t-1

Let us consider the following non-linear delay difference equation by the addition of Allee effect to discrete delay model (1)

$$N_{t+1} = F^{(o)}(\lambda^*, N_t, N_{t-1}, N_{t-2}) = \lambda^* N_t \alpha(N_{t-1}) f(N_{t-1}, N_{t-2}). \tag{11}$$

N^* equilibrium point of Eq.(11) is positive equilibrium point of Eq.(7). Then we have the following theorem.

Theorem 4. N^* is locally stable if and only if the inequalities

$$N^*\left(\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} - \frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right) < 2, \tag{12}$$

$$N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)} + \frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right] < 0$$

$$N^{*2}\,\frac{f^2_{N_{t-2}}(N^*,N^*)}{f^2(N^*,N^*)} - N^*\left(\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right) < 1 \tag{13}$$

$$N^{*2}\,\frac{f^2_{N_{t-2}}(N^*,N^*)}{f^2(N^*,N^*)} + N^*\left(\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right) < 1 \tag{14}$$

hold.

Proof. According to Eq.(11), the values of p, q and r are as follows

$$p = F_{N_t}(\lambda^*, N^*, N^*) = 1$$

$$q = F_{N_{t-1}}(\lambda^*, N^*, N^*) = N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right]$$

$$r = F_{N_{t-2}}(\lambda^*, N^*, N^*) = N^*\,\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}.$$

Firstly, from the first inequality in (6), we obtain

$$\left|1 + N^*\,\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right| < 1 - N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right]$$

$$N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right] - 1 < 1 + N^*\,\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)} < 1 - N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right].$$

or equivalently,

$$N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)} - \frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right] < 2$$

and

$$N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}+\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right]<0.$$

If we consider the other inequality in (6), we get

$$\left|N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+N^*\left[\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right]\right|<1-N^{*2}\frac{f^2_{N_{-2}}(N^*,N^*)}{f^2(N^*,N^*)}$$

$$N^{*2}\frac{f^2_{N_{-2}}(N^*,N^*)}{f^2(N^*,N^*)}-1<N^*\left(\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right)<1-N^{*2}\frac{f^2_{N_{-2}}(N^*,N^*)}{f^2(N^*,N^*)}$$

Last inequality leads the following inequality

$$N^{*2}\frac{f^2_{N_{-2}}(N^*,N^*)}{f^2(N^*,N^*)}-N^*\left(\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right)<1$$

$$N^{*2}\frac{f^2_{N_{-2}}(N^*,N^*)}{f^2(N^*,N^*)}+N^*\left(\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}\right)<1.$$

3.3. Allee effect at time t

We now incorporate an Allee effect into the discrete delay model as follows:

$$N_{t+1}=F^{(\alpha^+)}(\lambda^*,N_t,N_{t-1},N_{t-2})=\lambda^*N_t\alpha(N_t)f(N_{t-1},N_{t-2}) \tag{15}$$

N^* equilibrium point of Eq.(15) is positive equilibrium point of Eq.(7). Then we can state the following theorem.

Theorem 5. N^* is locally stable if and only if the inequality

$$N^*\left[\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}-\frac{\alpha'(N^*)}{\alpha(N^*)}-\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right]<2, \tag{16}$$

$$N^*\left[\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}+\frac{\alpha'(N^*)}{\alpha(N^*)}+\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}\right]<0$$

$$N^{*2}\frac{f^2_{N_{t-2}}(N^*,N^*)}{f^2(N^*,N^*)}-N^{*2}\frac{\alpha'(N^*)}{\alpha(N^*)}\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)} \tag{17}$$

$$-N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}-N^*\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}<1$$

$$N^{*2}\frac{f^2{}_{N_{t-2}}(N^*,N^*)}{f^2(N^*,N^*)}+N^{*2}\frac{\alpha'(N^*)}{\alpha(N^*)}\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)} \tag{18}$$

$$+N^*\frac{f_{N_{t-2}}(N^*,N^*)}{f(N^*,N^*)}+N^*\frac{f_{N_{t-1}}(N^*,N^*)}{f(N^*,N^*)}<1$$

hold.

Proof. If the values of p, q and r are written in Eq.(15), we have

$$p = F_{N_t}(\lambda^*, N^*, N^*) = 1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)}$$

$$q = F_{N_{t-1}}(\lambda^*, N^*, N^*) = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

$$r = F_{N_{t-2}}(\lambda^*, N^*, N^*) = N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)}$$

Let's consider stability conditions in (6). Thus, we get

$$\left| 1 + N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] \right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - 1 < 1 + N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}.$$

The last inequality can be written as follows,

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{\alpha'(N^*)}{\alpha(N^*)} - \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 2$$

and

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 0.$$

Now, when the process is regulated for the second inequality in (6), we obtain

$$\left| \left(1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} \right) \left(N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right) + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right| < 1 - N^{*2} \frac{f^2{}_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)}$$

$$N^{*2} \frac{f^2{}_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)} - N^{*2} \frac{\alpha'(N^*)}{\alpha(N^*)} \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} - N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

$$N^{*2} \frac{f^2{}_{N_{t-2}}(N^*, N^*)}{f^2(N^*, N^*)} + N^{*2} \frac{\alpha'(N^*)}{\alpha(N^*)} \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 1$$

Consequently, (16), (17) and (18) are confirmed.

Corallary 6. Allee effects at time $t-2$, $t-1$ and t decreases the stability of Eq.(1)

Proof. Let's take

$$x = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 0, \; y = N^* \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} < 0 \text{ and } z = N^* \frac{\alpha'(N^*)}{\alpha(N^*)} > 0.$$

Eq.(1) is stable if and only if

$$\begin{aligned} & x-y<2 \\ & y^2-y-x<1 \\ & y^2+y+x<1 \end{aligned} \tag{19}$$

If the values of x, y and z are written in the stability conditions of Eq.(7), Eq.(11) and Eq.(15), we obtain

$$\begin{aligned} & x-z-y<2, \;\; x+y+z<0 \\ & z^2+y^2+2yz-z-y-x<1 \\ & z^2+y^2+2yz+z+y+x<1, \end{aligned} \tag{20}$$

$$\begin{aligned} & x+z-y<2, \quad x+y+z<0 \\ & y^2-y-z-x<1 \\ & y^2+y+z+x<1 \end{aligned} \tag{21}$$

and

$$\begin{aligned} & x-z-y<2, \quad x+y+z<0 \\ & y^2-yz-y-x<1 \\ & y^2+yz+y+x<1 \end{aligned} \tag{22}$$

respectively. It is clear that for each value of x and y, which provides inequality (19), at least one of the conditions (20), (21) and (22) is not satisfied (for each z>0). In other words, stable equilibrium point of Eq.(2) is not stable for equations (7), (11), (15).

4. NUMERICAL SIMILATIONS

In this section, we numerically present our the analytical result obtained in the former sections by using MATLAB programming. We graph the 2D trajectories of the population dynamics model (1) with and without Allee effect at time $t-2$, $t-1$ and t in Fig. 1, Fig. 2 and Fig. 3, respectively. In this figures we take the function $f(N_{t-1}, N_{t-2}) = (1-N_{t-1}-N_{t-2})$ (see, for instance [7]) and the Allee function $\alpha(N_i) = N_i/(\alpha+N_i)$, $i=t-2$, $t-1$ and t, where α is a positive constant. It is obvious from the graph that the comparisons of the population density diagrams also verify the stabilizing impact of the Allee effects. In these computations, the initial conditions are taken as $N_{-2}=0.2$, $N_{-1}=0.3$, $N_0=0.4$ and $\lambda=1.9$ that yield the corresponding equilibrium point as $N^* \cong 0.4737$. In addition, the parameter value is taken as $\alpha=0.03$. Normalized growth rate is $\lambda^*=2.0204$ such that $\lambda=\lambda^*\alpha(N^*)$.

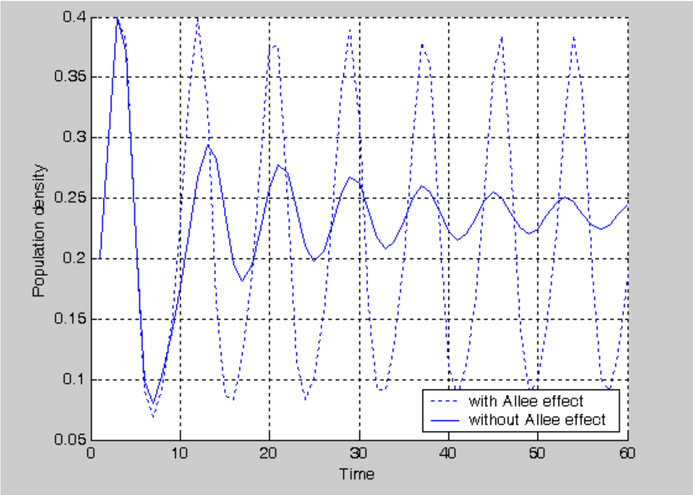

Fig. 1. Density-time graphs of the models
$N_{t+1} = \lambda N_t(1 - N_{t-1} - N_{t-2})$ and $N_{t+1} = \lambda^* N_t \alpha(N_{t-2})(1 - N_{t-1} - N_{t-2})$ with
$\lambda = 1.9,\ \alpha(N_{t-2}) = N_{t-2} / (\alpha + N_{t-2}),\ \alpha = 0.03,\ \lambda = \lambda^* \alpha(N^*)$.

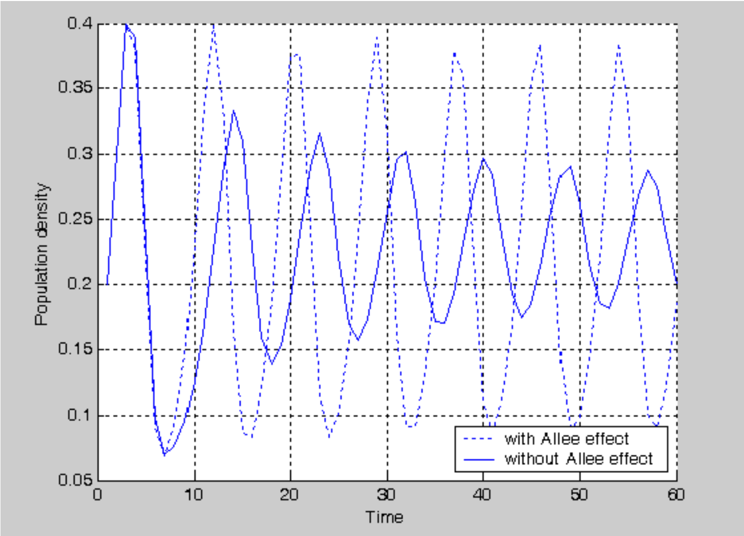

Fig. 2. Density-time graphs of the models
$N_{t+1} = \lambda N_t(1 - N_{t-1} - N_{t-2})$ and $N_{t+1} = \lambda^* N_t \alpha(N_{t-1})(1 - N_{t-1} - N_{t-2})$ with
$\lambda = 1.9,\ \alpha(N_{t-1}) = N_{t-1} / (\alpha + N_{t-1}),\ \alpha = 0.03,\ \lambda = \lambda^* \alpha(N^*)$.

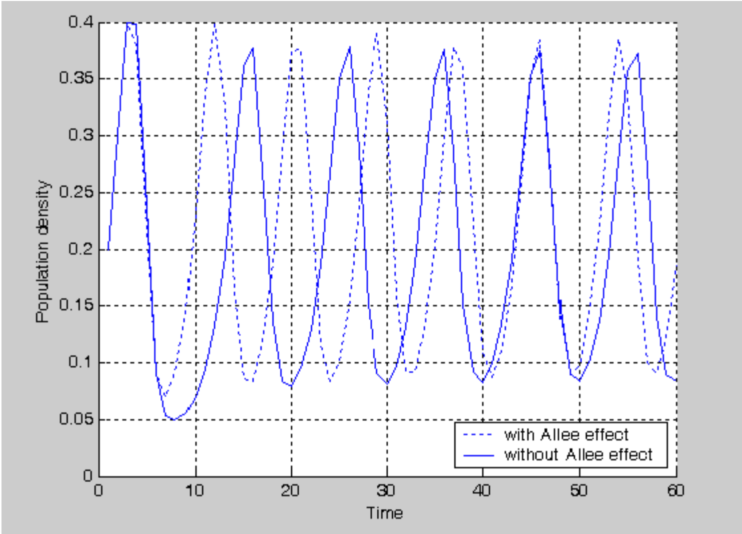

Fig. 3. Density-time graphs of the models
$N_{t+1} = \lambda N_t(1 - N_{t-1} - N_{t-2})$ and $N_{t+1} = \lambda^* N_t \alpha(N_t)(1 - N_{t-1} - N_{t-2}$ with
$\lambda = 1.9,\ \alpha(N_t) = N_t/(\alpha + N_t),\ \alpha = 0.03,\ \lambda = \lambda^* \alpha(N^*)$

These numerical simulation are consistent with the analytical result obtain in the former sections and supports the mathematical analysis.

5. CONCLUSION

Former studies indicate that Allee effect has different effects on different populations. Mathematical formulations of the population will provide information to us about the factors effecting population and the development of that group of living beings in the future. This situation is important in that it contributes to the establishment of equilibrium in life cycle, which is situated in Biology.

In this paper, we studied on a third degree delay difference model under a competitive effect. Firstly, we obtained the stability conditions of the equilibrium point of this model. Then, we investigated the stability of the equilibrium point of the model together with Allee effect. We compared the stability of these models with and without Allee effect. In conclusion, we observed that Allee effect reduced stability in the model class that we studied.

6. REFERENCES

1. W. C. Allee, Animal Aggretions: A Study in General Sociology. University of Chicago Press, Chicago, 1931.
2. L. J. S. Allen, An Introduction to Mathematical Biology, Pearson, New Jersey, 2007.
3. C. Çelik, H. Merdan, O. Duman, Ö. Akın, Allee effects on population dynamics with delay, Chaos Solitons & Fractals **37**, 65-74, 2008.
4. S. Elaydi, An Introduction to Difference Equations. Springer, New York, 2006.
5. S. Elaydi, R. J. Sacker, Population models with Allee effect: A new model, Journal of

Biological Dynamics **0,** 1-11, 2009.
6. M. S. Fowler, GD. Ruxton, Populations dynamics consequences of Allee effects, J. Theor Biol. **215**, 39-46, 2002.
7. K. Gopalsamy, Global stability in the delay-logistic equation with discrete-delays, Hauston J. Math. **16**, 347-356, 1990.
8. D. Hadjiavgousti, S. Ichtiaroglou, Existence of stable localized structures in population dynamics through the Allee effect, Chaos Solutions & Fractals **21,** 119-131, 2004.
9. S. R. J. Jang, Allee effects in a discrete-time host parasitoid model with stage structure in the host, Discrete Contin Dyn Syst Ser B **8,** 145-159, 2007.
10. Y. Kuang, Delay Differential Equations with Applications in Population Dynamics. Academic Press, New York, 1993.
11. M. A. McCarthy, The Allee effect, finding mates and theoretical models, Ecological Modeling **103,** 99-102, 1997.
12. H. Merdan, O. Duman, Ö. Akın, C. Çelik, Allee effects on population dynamics in continuous (overlapping) case, Chaos Solutions & Fractals **39**, 1994-2001, 2009.
13. H. Merdan, O. Duman, On the stability analysis of a general discrete-time population model involving predation and Allee effects, Chaos Solitons & Fractals **40**, 1169-1175, 2009.
14. H. Merdan, O. A. Gumus, Stability analysis of a general discrete-time population model involving delay and Allee effects, Mathematical and Computational Applications, in press, 2011
15. J. D. Murray, Mathematical Biology, Springer-Verlag, New York, 1993.
16. O. Duman, H. Merdan, Stability analysis of continuous population model involving predation and Allee effects, Chaos Solitons & Fractals **41**, 1218-1222, 2009.
17. I. Scheuring, Allee effect increases the dynamical stability of populations, J. Theor. Biol. **199**, 407-414, 1999
18. PA. Stephens, WJ. Sutherland, Consequences of the Allee effect for behavior, ecology and conservation, Trends Ecol. Evol. **14**, 401-405, 1999.
19. SR. Zhou, YF. Liu, G. Wang, The stability of predator-prey systems subject to the Allee effects, Theor. Popul. Biol. **67**, 23-31, 2005.