



ON CONSTITUTIVE EQUATIONS FOR ANISOTROPIC NONLINEARLY PIEZOELECTRIC MATERIALS

Melek Usal*, Lokman Yünlü

*Department of Manufacturing Engineering, Technology Faculty,
Süleyman Demirel University, 32260, Isparta, Turkey

Abstract: In this paper, the elastic-piezoelectric continuum has been investigated theoretically and its non-linear constitutive equations have been defined. The theory is formulated in the context of continuum electrodynamics. The solid medium is assumed to be non-linear, homogeneous, compressible and isothermal, has elastic and piezoelectric anisotropy. Basic principles of modern continuum mechanics and balance equations of electrostatic have provided guidance and have been determining in the process of this study. From the formulation belonging to the constitutive equations, it has been observed that the symmetric stress and polarization have been derived from a scalar-valued thermodynamic potential defined in calculations. As a result of thermodynamic constraints, it has been determined that the free energy function is dependent on a symmetric tensor and a vector. The free energy function has been represented by a power series expansion and the type and number of terms taken into consideration in this series expansion has determined the non-linearity of the medium. Finally, the quasi-linear constitutive equations are substituted in the balance equations to obtain the field equations.

Key Words: Piezoelectric, Continuum electrodynamics, Constitutive equations, Anisotropy.

1. INTRODUCTION

Piezoelectric materials have widespread applications in modern technical areas such as mechatronics, electromechanical devices, microelectromechanical systems or smart structures, serving as sensors, actuators or transducers [1, 2, 3]. In order to assess the strength and durability of those materials, exhaustive theoretical investigations have been performed recently [3, 4, 5]. Piezoelectric materials produce electric charges when mechanically deformed and an electric potential causes a mechanical deformation. This property makes them suitable for sensor and transducer applications [6]. The aim of the paper is to determine the electro-stressed state of an elastic-piezoelectric material under mechanical and electric loads.

The constitutive equations are one of the main aspects continuum mechanics. Many attempts have been made in order to write the constitutive equations in terms of invariants [7, 8, 9, 10]. Material in this study has been assumed to be a general anisotropic medium. In the framework of this approximation, stress potential Σ function has been expanded into power series in terms of components of the arguments it depends on, yielding the elastic behavior and polarization reaction of the medium. Kind and number of terms considered in the series expansion have determined the non-linearity grade of the medium. First, it has been assumed that both mechanic and electrical interactions are non-linear. Afterwards, assuming that deformation gradients,

(or displacement gradients) are very small and electrical interactions are non-linear; the constitutive equations of polarization field and symmetric stress have been linearized to a degree. As a result, the quasi-linear constitutive equations are substituted in the balance equations to obtain the field equations.

Our early studies have provided a basis for the conduction of this study. In all of our previous studies [11,12,13,14], since the material is viscoelastic, the symmetric stress was obtained as the sum of elastic and dissipative stresses. In our study [15], the constitutive equations determining the linear electro-thermomechanical behavior of a thermoelastic-piezoelectric medium have been obtained. In this study, however, the constitutive equations have been obtained that determine the non-linear electro-thermomechanical behavior of elastic-piezoelectric medium. In this study, the material is elastic-piezoelectric but the material isn't composite. Besides the medium is homogeneous, compressible and isothermal. Since the temperature has been assumed to be constant, no temperature change has been considered leading to the omission of the heat flux vector formulation.

2. ELECTROSTATIC AND THERMO-MECHANIC BALANCE EQUATIONS

Local Electrostatic Thermomechanical balance equations can be summarized as follows [16, 17].

Coulomb-Gauss Law:

$$\nabla \cdot \mathbf{D} = \mathbf{0} \quad \text{in } V(t), \quad [[\mathbf{D}]] \cdot \mathbf{n} = w_f \quad \text{on } \sigma(t) \quad (1)$$

Faraday Law:

$$\nabla \times \mathbf{E} = \mathbf{0} \Rightarrow \mathbf{E} = -\nabla \phi \quad \text{in } V(t), \quad \mathbf{n} \times [[\mathbf{E}]] = \mathbf{0} \quad \text{on } \sigma(t), \quad \mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (2)$$

Where \mathbf{D} is electric displacement vector; \mathbf{n} outward unit normal vector of discontinuity surface σ , \mathbf{E} electric field vector, ϕ electric scalar potential, ε_0 electric permittivity of vacuum, \mathbf{P} polarization field vector, w_f free surface charge distribution, $[[\mathbf{D}]] \equiv \mathbf{D}^+ - \mathbf{D}^-$ jump of \mathbf{D} across the discontinuity surface [16].

Consevation of Mass:

$$\rho(\mathbf{x}, t) = \frac{\rho_0(\mathbf{X})}{J(\mathbf{x}, t)}, \quad \dot{\rho} + \rho v_{k,k} = 0 \quad \text{in } V(t), \quad [[U\rho]] = 0 \quad \text{on } \sigma(t) \quad (3)$$

Balance of Linear Momentum:

$$\rho \dot{v}_p = \rho f_p + \bar{t}_{r p, r} - P_{r, r} E_p \quad \text{in } V(t), \quad [[n_l t_{lk} + \rho v_k U]] = 0 \quad \text{on } \sigma(t) \quad (4)$$

Balance of Moment of Momentum:

$$\varepsilon_{krp}(t_{rp} + P_r E_p) = 0 \Rightarrow \varepsilon_{krp} \bar{t}_{rp} = 0 \Rightarrow \bar{t}_{rp} = \bar{t}_{pr} \quad \text{in } V(t), \quad \varepsilon_{klp} x_l [[n_r t_{rp} + \rho U v_p]] = 0 \quad \text{on } \sigma(t) \quad (5)$$

Conservation of Energy:

$$\rho \dot{\varepsilon} = t_{kl} v_{l, k} - q_{k, k} + \rho h + \rho \mathbf{E} \cdot \dot{\mathbf{P}} \quad \text{in } V(t), \quad \rho U [[\varepsilon + 1/2 |\mathbf{v}|^2]] + n_k [[t_{kl} v_l - q_k]] = 0 \quad \text{on } \sigma(t) \quad (6)$$

Clausius-Duhem Inequality:

$$\rho \dot{\eta} - \rho \frac{h}{\theta} + \frac{1}{\theta} \nabla \cdot \mathbf{q} - \frac{1}{\theta^2} \mathbf{q} \cdot \nabla \theta \equiv \rho \gamma \geq 0 \text{ in } V(t), \quad \rho U [[\eta]] - [[(\mathbf{n} \cdot \mathbf{q})/\theta]] \leq 0 \text{ on } \sigma(t) \quad (7)$$

In these and following equations physical meanings of various symbols are: U relative displacement velocity of the discontinuous surface according to the medium, \mathbf{v} velocity vector of continuous media, \mathbf{u} velocity vector of discontinuity surface, ρ_0 mass density before deformation, ρ mass density after deformation, J jacobian, $\dot{\mathbf{v}}$ acceleration vector, t_{lk} asymmetric stress tensor, \bar{t}_{pr} symmetric stress tensor, t_{ij}^E polarization stress tensor, d_{kl} rates of deformation tensor, w_{kl} spin tensor, $v_{l,k}$ velocity gradient tensor, f_k mechanical body force per unit mass, F_k^E electrostatic body force per unit volume, C_k^E intrinsic body couple per unit volume, ε internal energy per unit mass, q_k heat flux vector, Π polarization vector per unit of mass, h energy source per unit mass, h^E electrostatic energy source, w_k angular velocity, η entropy per unit mass, $\theta(\mathbf{X}, t)$ the absolute temperature distribution, $\rho \gamma$ entropy production per unit mass and ε_{ijk} permutation tensor. These symbols have been expressed as follows [18]:

$$\begin{aligned} U &\equiv u_n - v_n, & F_i^E &= P_j E_{i,j}, & \rho h^E &= E_k P_k^* + t_{kl}^E d_{kl}, & C_i^E &= \varepsilon_{ijk} P_j E_k, & \mathbf{w} &= \frac{1}{2} \nabla \times \mathbf{v} \\ \mathbf{P} &\equiv \dot{\mathbf{P}} - P_k \mathbf{v}_{,k} + \mathbf{P} v_{k,k}, & t_{ij}^E &= P_i E_j, & v_{l,k} &= d_{lk} + w_{lk}, & d_{kl} &\equiv \frac{1}{2} (v_{k,l} + v_{l,k}) = d_{lk}, \\ w_{kl} &\equiv \frac{1}{2} (v_{k,l} - v_{l,k}) = -w_{lk}, & \rho h^E + \mathbf{C}^E \cdot \mathbf{w} &= \rho \mathbf{E} \cdot \dot{\mathbf{\Pi}}, & \mathbf{\Pi} &\equiv \frac{\mathbf{P}}{\rho}. \end{aligned} \quad (8)$$

3. THERMODYNAMIC CONSTRAINTS AND MODELING CONSTITUTIVE EQUATIONS

The local energy equation (6) is then suitably combined with the entropy inequality (7) and, using a Legendre transformation such as $\psi \equiv \varepsilon - \theta \eta - \mathbf{E} \cdot \mathbf{\Pi}$ for the free energy, the entropy inequality is obtained as follows in the material form [19]:

$$-(\dot{\Sigma} + \rho_0 \eta \dot{\theta}) + \frac{1}{2} \bar{T}_{KL} \dot{C}_{KL} - \frac{1}{\theta} \theta_{,K} Q_K - \Pi_K \dot{E}_K \geq 0 \quad (9)$$

Relationships between material and spatial forms of values appearing in this inequality have been presented as follows [18,19]:

$$\begin{aligned} \Sigma &\equiv \rho_0 \psi, & \dot{C}_{KL} &= 2 d_{kl} x_{k,K} x_{l,L} \Rightarrow d_{kl} = \frac{1}{2} \dot{C}_{KL} X_{K,k} X_{L,l}, & \theta_{,K} &= x_{k,K} \theta_{,k} \Rightarrow \theta_{,k} = X_{K,k} \theta_{,K}, \\ \bar{T}_{KL} &\equiv J X_{K,k} X_{L,l} \bar{t}_{kl} \Rightarrow \bar{t}_{kl} = J^{-1} x_{k,K} x_{l,L} \bar{T}_{KL}, & \Pi_K &\equiv \frac{\rho_0}{\rho} X_{K,k} P_k \Rightarrow P_k = J^{-1} x_{k,K} \Pi_K, \\ Q_K &\equiv J X_{K,k} q_k \Rightarrow q_k = J^{-1} x_{k,K} Q_K, & E_K &\equiv x_{k,K} E_k \Rightarrow E_k = X_{K,k} E_K \end{aligned} \quad (10)$$

Where Σ is thermodynamical stress potential, ψ generalized free energy density, $X_{K,k}$ deformation gradient of inverse motion.

Formulation of the constitutive equations for the elastic-piezoelectric continuum under consideration can be summarized as follows [20].

$$\begin{aligned} \Sigma &= \Sigma(C_{KL}, E_K, \theta), \quad \Pi_K = -\frac{\partial \Sigma}{\partial E_K}, \quad \bar{T}_{KL} = 2 \frac{\partial \Sigma}{\partial C_{KL}}, \quad \eta = -\frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta}, \quad Q_K = 0, \\ T_{KL} &= \bar{T}_{KL} - \Pi_K E_M C_{ML}^{-1}, \quad t_{kl} = \bar{t}_{kl} - P_k E_l, \quad \varepsilon = \frac{1}{\rho_0} \left(\Sigma - \frac{\partial \Sigma}{\partial \theta} \theta - \frac{\partial \Sigma}{\partial E_K} E_K \right) \end{aligned} \quad (11)$$

From the equations provided in the expressions (11)₂ and (11)₃, it is understood that the polarization and the stress are derived from the stress potential Σ . Thus, the explicit form of Σ , which appear as constitutive function with definite arguments, should be determined. The material in this study has been assumed to be a general anisotropic medium. In the framework of this approximation the stress potential Σ function has been expanded into power series in terms of components of the arguments it depends on, yielding the elastic behavior and polarization reaction of the medium. Kind and number of terms considered in the series expansion has determined the non-linearity grade of the medium. First, it has been assumed in this study that both mechanic interactions and electrical interactions are non-linear. This situation will be taken into consideration in operations pertaining to the sections below.

4. DETERMINATION OF SYMMETRIC STRESS AND POLARIZATION CONSTITUTIVE EQUATIONS

Due to the existence of the relationship $C_{KL} = \delta_{KL} + 2E_{KL}$ between the Green deformation tensor and the strain tensor, arguments of the stress potential given in the expression (11)₁ can be recorded in the following form:

$$\Sigma = \Sigma(E_{KL}, E_K, \theta) \quad (12)$$

Assuming that this function (12) is analytic in terms of values E_{KL}, E_K , and expanding the Taylor series around $E_{KL} = 0$ and $E_K = 0$ (all to be represented with the symbol $|_0$), the following expression can be found for the stress potential.

$$\begin{aligned} \Sigma(E_{KL}, E_Q) &= \Sigma_0 + \Sigma_{KL} E_{KL} + \beta_Q E_Q + \frac{1}{2} \Sigma_{KLMN} E_{KL} E_{MN} + \frac{1}{2} \beta_{QN} E_Q E_N + \lambda_{KLQ} E_{KL} E_Q + \\ &\frac{1}{3} \Sigma_{KLMNSQ} E_{KL} E_{MN} E_{SQ} + \frac{1}{3} \beta_{QNS} E_Q E_N E_S + \frac{1}{2} \lambda_{KLMNQ} E_{KL} E_{MN} E_Q + \lambda_{KLQN} E_{KL} E_Q E_N \end{aligned} \quad (13)$$

It is clear that the coefficients in these equations depend only on the temperature.

$$\begin{aligned} \Sigma_0 &\equiv \Sigma(\underline{0}, \underline{0}), \quad \Sigma_{KL} \equiv \left. \frac{\partial \Sigma}{\partial E_{KL}} \right|_0, \quad \beta_0 \equiv \left. \frac{\partial \Sigma}{\partial E_Q} \right|_0, \quad \Sigma_{KLMN} \equiv \left. \frac{\partial^2 \Sigma}{\partial E_{KL} \partial E_{MN}} \right|_0, \quad \beta_{QN} \equiv \left. \frac{\partial^2 \Sigma}{\partial E_Q \partial E_N} \right|_0, \\ \lambda_{KLQ} &\equiv \left. \frac{1}{2} \frac{\partial^2 \Sigma}{\partial E_{KL} \partial E_Q} \right|_0, \quad \Sigma_{KLMNSQ} \equiv \left. \frac{1}{2} \frac{\partial^3 \Sigma}{\partial E_{KL} \partial E_{MN} \partial E_{SQ}} \right|_0, \quad \beta_{QNS} \equiv \left. \frac{1}{6} \frac{\partial^3 \Sigma}{\partial E_Q \partial E_N \partial E_S} \right|_0 \end{aligned}$$

$$\lambda_{KLMNQ} \equiv \frac{1}{3} \frac{\partial^3 \Sigma}{\partial E_{KL} \partial E_{MN} \partial E_Q} \Big|_0, \quad \lambda_{KLQN} \equiv \frac{1}{6} \frac{\partial^3 \Sigma}{\partial E_{KL} \partial E_Q \partial E_N} \Big|_0 \quad (14)$$

Due to the symmetry of the tensor E_{KL} and independence of derivatives in the definitions in expressions (14) from the order, these coefficients bear the symmetry characteristics given below:

$$\begin{aligned} \Sigma_{KL} &= \Sigma_{LK}, \quad \Sigma_{KLMN} = \Sigma_{LKMN} = \Sigma_{KLN M} = \Sigma_{MNKL}, \quad \beta_{QN} = \beta_{NQ}, \quad \lambda_{KLQ} = \lambda_{LKQ}, \\ \Sigma_{KLMNSQ} &= \Sigma_{LKMNSQ} = \Sigma_{KLNMSQ} = \Sigma_{MNKLSQ} = \Sigma_{KLSQMN} = \Sigma_{SQMNKL}, \\ \beta_{QNS} &= \beta_{NQS} = \beta_{QSN} = \beta_{SNQ}, \\ \lambda_{KLMNQ} &= \lambda_{LKMNQ} = \lambda_{KLNMQ} = \lambda_{MNKLQ}, \quad \lambda_{KLQN} = \lambda_{LKQN} = \lambda_{KLNQ} \end{aligned} \quad (15)$$

Since $2 \frac{\partial \Sigma}{\partial C_{KL}} = \frac{\partial \Sigma}{\partial E_{KL}}$, the constitutive equation of symmetric stress can be written in the following form.

$$\bar{T}_{PR} = \frac{\partial \Sigma}{\partial E_{PR}} \quad (16)$$

If derivatives in (16) and (11)₂ are taken from (13) and used in substitution, the following is obtained.

$$\bar{T}_{PR} = \Sigma_{PRMN} E_{MN} + \lambda_{PRQ} E_Q + \Sigma_{PRMNSQ} E_{MN} E_{SQ} + \lambda_{PRMNQ} E_{MN} E_Q + \lambda_{PRQN} E_Q E_N \quad (17)$$

$$\Pi_R = - \left[\beta_{RQ} E_Q + \lambda_{KLR} E_{KL} + \beta_{RQN} E_Q E_N + \frac{1}{2} \lambda_{KLMNR} E_{KL} E_{MN} + 2 \lambda_{KLQR} E_{KL} E_Q \right] \quad (18)$$

Thus, in a elastic-piezoelectric anisotropic medium, in a situation where both mechanical interactions and electrical interactions are assumed non-linear, the constitutive equations for symmetric stress and polarization on material coordinates in terms of their components can be expressed through expressions (17) and (18). First term on the right part of the the constitutive equation for the symmetric stress (17) is the classical term of the Hooke law and contributes to the stress through the strain tensor. The second term shows the stress arising out of piezoelectric effect. The third term shows the non-linear effect of the strain tensor. The fourth term shows mutual interaction between the strain tensor and the electric field. The last term shows the stress formed by the non-linear effect of the electric field. Considering expression (18), which yields the constitutive equation of the polarization field, linear effects of the electric field and the strain tensor, non-linear effects of the electric field and the strain tensor, mutual interaction between the strain tensor and the electric field contribute to the formation of polarization field in the said medium.

Substituting symmetric stress given by the expression (17) and polarization field given by the expression (18) into the equation (11)₆, the asymmetric stress (total stress) is found as follows:

$$\begin{aligned}
T_{PR} = & \Sigma_{PRMN} E_{MN} + \lambda_{PRQ} E_Q + \lambda_{PRMNQ} E_{MN} E_Q + \lambda_{PRQ_N} E_Q E_N + \Sigma_{PRMNSQ} E_{MN} E_{SQ} + \\
& \beta_{RQ} E_Q E_M C_{MR}^{-1} + \lambda_{KLR} E_{KL} E_M C_{MR}^{-1} + \beta_{RQN} E_Q E_N E_M C_{MR}^{-1} + \frac{1}{2} \lambda_{KLMNR} E_{KL} E_{MN} E_M C_{MR}^{-1} + \\
& 2\lambda_{KLQR} E_{KL} E_Q E_M C_{MR}^{-1}
\end{aligned} \quad (19)$$

This equation is a non-linear expression of the stress obtained for the material in consideration under the said assumptions. Because fifth and sixth grade material tensors appear in this expression, it will be difficult to solve problems using the constitutive equation. Therefore, quasi-linear constitutive equations are to be obtained as follows.

5. QUASI – LINEAR THEORY

Assuming that deformation gradients are very small and electrical interactions are non-linear; the constitutive equations of polarization field and the stress have been linearized to a degree. To obtain the linear theory it is known that the strain tensor conform to constraint: $\|\underline{\underline{E}}\| \ll 1$ [21]. Furthermore, $E_Q E_N$ term in the constitutive equation of polarization field given by the expression (18) can be omitted because of that this term shows up third grade electrical field vector in the asymmetric stress equation. $E_{MN} E_Q$ term in the constitutive equation of the symmetric stress given by the expression (17) can be omitted because of that the coefficient of this term is fifth grade material tensor. In this case, the constitutive equations of the symmetric stress and the polarization field will be reduced to the following forms:

$$\bar{T}_{PR} = \Sigma_{PRMN} E_{MN} + \lambda_{PRQ} E_Q + \lambda_{PRQN} E_Q E_N \quad (20)$$

$$\Pi_R = -[\beta_{RQ} E_Q + \lambda_{KLR} E_{KL} + 2\lambda_{KLQR} E_{KL} E_Q] \quad (21)$$

Substituting equations (20) and (21) into the expression (11)₆, the quasi-linear material stress tensor occurring asymmetrically in the considered anisotropic material has been obtained as follows, based on the assumptions previously mentioned.

$$\begin{aligned}
T_{PR} = & \Sigma_{PRMN} E_{MN} + \lambda_{PRQ} E_Q + \lambda_{PRQN} E_Q E_N + \beta_{RQ} E_Q E_M C_{MR}^{-1} + \\
& \lambda_{KLR} E_{KL} E_M C_{MR}^{-1} + 2\lambda_{KLQR} E_{KL} E_Q E_M C_{MR}^{-1}
\end{aligned} \quad (22)$$

The following relations can be written down for the linear theory in continuum mechanics [21]:

$$\lambda_{kK} \lambda_{lK} = \delta_{kl}, \quad \lambda_{kK} \lambda_{kL} = \delta_{KL}, \quad x_{k,K} = \lambda_{kK} + \lambda_{kL} U_{L,K}, \quad X_{K,k} = \lambda_{kK} - \lambda_{lK} u_{l,k},$$

$$U_{K,L} = \lambda_{kK} \lambda_{lL} u_{k,l}, \quad x_{k,K} x_{l,L} = \lambda_{kK} \lambda_{lL}, \quad X_{K,k} X_{L,l} = \lambda_{kK} \lambda_{lL},$$

$$x_{p,P} x_{r,R} E_K \equiv \lambda_{pP} \lambda_{rR} \lambda_{kK} E_k, \quad E_{KL} \equiv \tilde{E}_{KL} \equiv \lambda_{kK} \lambda_{lL} \tilde{e}_{kl} = \frac{1}{2} \lambda_{kK} \lambda_{lL} (u_{k,l} + u_{l,k}),$$

$$e_{kl} \equiv \tilde{e}_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k}), \quad J^{-1} \equiv 1 - u_{k,k}, \quad \rho \equiv \rho_0 (1 - u_{k,k}) \quad (23)$$

In this case, substituting equations (20) and (21) into equations (10)₄ and (10)₅, using expressions (23), the quasi-linear constitutive equations of the symmetric stress and the polarization field on spatial coordinates as follows:

$$\bar{t}_{pr} = (1 - u_{k,k}) [\Sigma_{prmn} \tilde{e}_{mn} + \lambda_{prq} E_q + \lambda_{prqn} E_q E_n] \quad (24)$$

$$P_r = -(1 - u_{k,k}) (\beta_{rq} E_q + \lambda_{klr} \tilde{e}_{kl} + 2\lambda_{klqr} \tilde{e}_k E_q) \quad (25)$$

The spatial material tensors Σ_{prmn} , λ_{prq} , λ_{prqn} , β_{rq} , λ_{klr} and λ_{klqr} in these equations bear the same symmetry characteristics as the material tensors of materials Σ_{PRMN} , λ_{PRQ} , λ_{PRQN} , β_{RQ} , λ_{KLR} and λ_{KLQR} , and are defined as follows:

$$\begin{aligned} \Sigma_{prmn} &\equiv \lambda_{pP} \lambda_{rR} \lambda_{mM} \lambda_{nN} \Sigma_{PRMN}, & \lambda_{prq} &\equiv \lambda_{pP} \lambda_{rR} \lambda_{qQ} \lambda_{PRQ}, \\ \lambda_{prqn} &\equiv \lambda_{pP} \lambda_{rR} \lambda_{qQ} \lambda_{nN} \lambda_{PRQN}, & \beta_{rq} &\equiv \lambda_{rR} \lambda_{qQ} \beta_{RQ}, & \lambda_{klr} &\equiv \lambda_{kK} \lambda_{lL} \lambda_{rR} \lambda_{RKL} \\ \lambda_{klqr} &\equiv \lambda_{kK} \lambda_{lL} \lambda_{qQ} \lambda_{rR} \lambda_{RK LQ} \end{aligned} \quad (26)$$

Due to symmetries $\Sigma_{prmn} = \Sigma_{prnm}$, $\lambda_{klr} = \lambda_{lkr}$ and $\lambda_{klqr} = \lambda_{lkqr}$ of the coefficients Σ_{prmn} , λ_{klr} and λ_{klqr} , the equations (24) and (25) have been obtained as follows in terms of linear constituents of displacement gradient.

$$\bar{t}_{pr} = \Sigma_{prmn} u_{m,n} + \lambda_{prq} E_q + \lambda_{prqn} E_q E_n - \lambda_{prq} u_{k,k} E_q - \lambda_{prqn} u_{k,k} E_q E_n \quad (27)$$

$$P_r = -\beta_{rq} E_q - \lambda_{klr} u_{k,l} - 2\lambda_{klqr} u_{k,l} E_q + \beta_{rq} u_{m,m} E_q \quad (28)$$

Substituting the equations (27) and (28) into the expression (11)₇, the quasi-linear spatial constitutive equation of the asymmetric stress is obtained as follows.

$$\begin{aligned} t_{pr} &= \Sigma_{prmn} u_{m,n} + \lambda_{prq} E_q + \lambda_{prqn} E_q E_n - \lambda_{prq} u_{k,k} E_q - \lambda_{prqn} u_{k,k} E_q E_n + \\ &\beta_{pq} E_q E_r + \lambda_{klp} u_{k,l} E_r + 2\lambda_{klqp} u_{k,l} E_q E_r - \beta_{pq} u_{m,m} E_q E_r \end{aligned} \quad (29)$$

In operations conducted up to now, the symmetric stress, the asymmetric stress and the polarization field on spatial coordinates have been expressed in terms of displacement gradient and electric field vector in the equations (27), (28) and (29). These spatial constitutive equations obtained will be substituted into the balance equations, yielding the field equations. If the equation (28) is substituted into the equation (2)₃ and the expression (2)₁ is used, the total electrical displacement vector can be found as follows:

$$D_r = -\epsilon_{rq} \phi_{,q} - \lambda_{klr} u_{k,l} + \mu_{klqr} u_{k,l} \phi_{,q} - \beta_{rq} u_{m,m} \phi_{,q} \quad (30)$$

The coefficients in these equations have been defined as: $\epsilon_{rq} \equiv \epsilon_0 \delta_{rq} - \beta_{rq}$, $\mu_{klqr} \equiv 2\lambda_{klqr}$. Considering that the medium is homogeneous and isothermal, if divergence of the expression (30) is taken and substituted into the equation (1)₁ the following expression is obtained.

$$0 = D_{r,r} = -\epsilon_{rq} \phi_{,q} - \lambda_{klr} u_{k,lr} + \mu_{klqr} (u_{k,lr} \phi_{,q} + u_{k,l} \phi_{,qr}) - \beta_{rq} (u_{m,mr} \phi_{,q} + u_{m,m} \phi_{,qr}) \quad (31)$$

If the expression (2)₁ is substituted into the constitutive equations of the symmetric stress and the polarization field given in equations (27) and (28), and, considering the fact that the medium is homogeneous and isothermal, its divergence is taken and it is afterwards substituted into the equation (4)₁, the following field equation is obtained under the said assumptions.

$$\begin{aligned} \rho_0 \frac{\partial^2 u_p}{\partial t^2} = & \rho_0 f_p - u_{k,k} f_p + \Sigma_{prmn} u_{m,nr} - \lambda_{prq} \phi_{,qr} - \lambda_{prqn} (\phi_{,qr} \phi_{,n} + \phi_{,q} \phi_{,nr}) + \\ & \lambda_{prq} (u_{k,kr} \phi_{,q} + u_{k,k} \phi_{,qr}) - \lambda_{prqn} [u_{k,kr} \phi_{,q} \phi_{,n} + u_{k,k} (\phi_{,qr} \phi_{,n} + \phi_{,q} \phi_{,nr})] + \\ & \beta_{rq} \phi_{,qr} \phi_{,p} - \lambda_{klr} u_{k,lr} \phi_{,p} + \mu_{klqr} (u_{k,lr} \phi_{,q} \phi_{,p} + u_{k,l} \phi_{,qr} \phi_{,p}) - \beta_{rq} (u_{m,mr} \phi_{,q} \phi_{,p} + u_{m,m} \phi_{,qr} \phi_{,p}) \end{aligned} \quad (32)$$

The expressions (31) and (32) yield the field equations containing the unknowns u_k and ϕ . Solution of these field equations under initial and boundary conditions forms the mathematical structure of a boundary value problem to consider. Thus, the system comprised of field equations (31) and (32) along with the boundary equations contained in jump conditions (1)₂, (4)₂ and (2)₂ constitutes governing equations for boundary value problems related with anisotropic, quasi-linear, elastic and piezoelectric media. The said boundary conditions can be openly expressed as follows:

$$D_n = D_n^+ - \omega_f, \quad n_l t_{lk} = t_k, \quad E_k = E_k^+ \quad (33)$$

6. CONCLUDING REMARKS

In this paper a nonlinear constitutive model for elastic-piezoelectric materials has been presented. We have proposed and analyzed a nonlinear elasticity model of deformation induced by electrostatic forces. The coupling of elastic deformation to the electrostatic field is great importance in modeling the electro-mechanical behavior of elastic-piezoelectric materials subject to external loading. In this study, a mathematical model has been formulated in the context of continuum electrodynamics. The model is embedded in a thermodynamic consistent framework, which is based on the definition of a free energy function. Fundamental principles and axioms of modern Continuum Mechanics, Clausius-Duhem inequality, general thermodynamic balance equations and balance equations of electrostatic have provided guidance and have been determining in modeling the nonlinear electro-mechanical behavior of material. The nonlinearity according to the electric field of the torque applied on the electric dipoles by the electric field applied on polarizable media causes the asymmetry of the mechanical stress tensor. To calculate the asymmetric stress, the symmetric stress and the polarization field should be known. From the constitutive equations, we have seen that the symmetric stress and the polarization field are derived from the stress potential Σ . In this case, open form of Σ , which is known as constitutive function with clear arguments, should be found.

As an approach it has been assumed that the stress potential function is analytic, leading to their expansion in Taylor series in terms of the arguments it depends on. Kind and number of terms in series expansion determined the order of non-linearity of the

medium. Besides, both mechanical interactions and electrical interactions were assumed first to be non-linear. In this case non-linear constitutive equations of the symmetric stress and the polarization field have been expressed by (17) and (18), respectively. By using these equations, non-linear constitutive equation of the asymmetric stress has been obtained in the material form with expression (19). Because fifth and sixth grade material tensors appear in this expression, it will be difficult to solve problems using the constitutive equations. Therefore, the quasi-linear constitutive equations were to be obtained. Assuming that deformation gradients are very small and electrical interactions are non-linear; the constitutive equations of the polarization field and the stress have been linearized to a degree. The quasi-linear constitutive equations which on material coordinates have been obtained by expressions (20)-(22). The quasi-linear constitutive equations have been given in expressions (27)-(29) on spatial coordinates. To obtain the field equations, constitutive equation of the polarization given by the equation (28) has been substituted into $(2)_3$ and, in the balance equation provided, the constitutive equation of the symmetric stress given by the equation (27) has been substituted into the Cauchy motion equation $(4)_1$, yielding field equations (31) and (32). Solution of the field equations along with initial and boundary conditions in conformity with the structure of the problem to be used in practice will constitute the structure of a boundary value problem to consider.

Besides, considering the motion equation (32), we can see the internal electromechanical forces affecting the medium. Type of the terms on the right is in the dimension of force per unit of volume. The first term on the right represents mechanical body force. The second term represents force caused by interaction of the deformation field with mechanical body force. The third term represents force created by the elastic deformation. The fourth term represents force created by the electric field gradient. The fifth and eighth terms represent second-grade electrostatic forces. The sixth and ninth terms are force terms caused by interaction of the deformation field with linear electric field. The seventh, tenth and eleventh terms represent force caused by interaction of the deformation field with non-linear electric field. In a future work we will study the development of numerical methods for this model where electrostatic forces a dominant role.

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