Mathematical and Computational Applications, Vol. 16, No. 3, pp. 738-747, 2011.

ANALYTICAL REGULARIZATION METHOD FOR RIGOROUS DIFFRACTION ANALYSIS OF KNIFE EDGES FOR TM WAVES

Hüseyin Yiğit, Fatih Dikmen
Department of Electronics Engineering
Gebze Institute of Technology, 41400
Gebze, Kocaeli, Turkey
hyigit@gyte.edu.tr, dikmen@gyte.edu.tr

Abstract- A direct full-wave solution via the Analytical Regularization Method (ARM) is proposed for the diffraction problem concerning knife edges under TM wave excitation. This problem is reduced to solving an infinite linear algebraic equation system of the second kind which, in principle, can be solved with arbitrary predetermined accuracy via a truncation procedure.
Key Words- Analytical Regularization Method, Electromagnetic Wave Scattering, Radio Wave Propagation

1.INTRODUCTION

Multi-strip models are helpful for analyzing radio wave propagation over hills (in rural areas) or over buildings (in urban areas), where they are widely known as “knife edges” [1]. An analytical solution for single-strip diffraction is numerically impractical to implement [2, 3], especially when the model requires multi-strip systems. For this reason and because of the enormous range of the problem’s domain, either physically corrected Fresnel or Kirchhoff diffraction models [4, 5] or high frequency approximations [1, 6] and marching methods applied to the parabolic approximate of the wave equation [7] are used to accurately investigate the related physics.

In order to investigate the transverse magnetic (TM, or E-Polarized) wave diffraction by a knife edge model constructed by an electrically perfect-conductor multi-strip system, we propose a direct, mathematically rigorous and numerically efficient alternative approach that revives a version of the Analytical Regularization Method (ARM) [12]. Direct-numerical methods such as the Moment Method, the Finite Difference Method, and others, usually require additional supporting verification since these methods entail solving a system of linear algebraic equations of the first kind (SLAE1) [8]. In using ARM, we present a full-wave alternative formulation for such problems which leads to solving a set of linear algebraic equations of the second kind (SLAE2) [12]. The direct implementation of said formulation without using fast calculation techniques for the matrix entries (e.g. [9]) (incorporated with an appropriate iterative linear solver whose performance is in favor of the SLAE2 we aim to construct using ARM for knife edge diffraction (see [10])), is easily applicable up to the VHF band. Given that the SLAE2 solutions provided by ARM can be considered a perfect tool for the verification of approximate, direct-numerical and experimental methods, as well as for the analysis of their errors, ARM provides a basic and mathematically rigorous template suited for the knife edge diffraction problem under investigation even at higher frequencies. Consequently, and in principle, the diffraction boundary value problem can be solved numerically and with arbitrary accuracy, limited solely by memory volume and efficient computer use.

We start with a rather classical solution scheme for the considered problem which arrives at an SLAE1 in space l_2 and use ARM for its regularization, i.e., for the mathematically rigorous construction of an SLAE2 that is equivalent to the original boundary value problem (BVP) in space l_2. (The space l_2 corresponds best to the space of numbers and operations in a finite precision computational environment, i.e., the computer (see [11]).) Thus, the ARM approach we propose builds on the Orthogonal Polynomials Method (OPM) invented in [14], one and a half decades prior to [17], and on the ARM ideas presented in [12] through [16]. Implementation of a special case of the ARM (even though not named there with this terminology-see the discussion after (9)) for a more generally defined geometry of S can also be found in [18].

The remainder of this piece unfolds as follows: First, we formulate the problem under consideration and delineate the corresponding ARM algorithm leading to an SLAE2. Next, we provide the numerical results for near field and far field distribution of the knife edges under incidence of a plane wave (post rigorous simulation by ARM) and the numerical results for the current density distribution (which are compared to those obtained by point-matching (PM) solution of the integral equation under consideration). Subsequently, we compare the ARM condition numbers to the condition numbers of the PM solution and to those of two more regularizing options that also lead to an SLAE2 of the integral equation under consideration, and thus elucidate the significance of such algebraic systems (AS).

2. FORMULATION

2.1. TM Wave Diffraction Boundary Value Problem

Consider the diffraction of an incident $e^{-i\omega t}$ time-harmonic (ω:angular frequency) electromagnetic TM wave with E-field $E^i(q)$ by perfectly conductive and infinitely thin screens defined with the parameterized surface union $S=\bigcup S_j$, where $S_j\bigcap S_m=\varnothing$ for $m\neq j$ by means of an injective parameterization $S_j=\{\eta_j(u)=(x_j(u),y_j(u))\in R^2{:}u\in[-1,1]\}$, $j=1,2,\ldots,N$. In general, the coordinates of η_j belong to Hölder class $C^{1,\alpha}(S_j)$, ($0<\alpha\leq 1$ henceforth) [19], but for the knife-edge model, the parameterization is linear, so $\eta_j\in C^{\infty}$. The scattering E-field satisfies the homogenous Helmholtz equation in free space except S: $(\Delta+k^2)E^s(q)=0$, $q\in R^2\backslash S$. $E^s(q)=\int_S J(p)G(q,p)dS$ $q\in R^2\backslash S$ is the integral representation of such a field for the Dirichlet condition, $E^{total}(q)=E^i(q)+E^s(q)=0$, $q\in S$, with the fundamental solution $G(q,p)=[-i/4]\ H_0^{(1)}(k|q-p|)$ (or the Green's function) of the Helmholtz equation in free space in R^2 and $H_0^{(1)}$ is the Hankel function of the first kind and order zero, where k is the free space wave number and $|q-p|$ is the distance between points p and q. $|q|^{1/2}(\partial/\partial|q|-ik)U(q)\to 0$, with $|q|\to\infty$, i.e. the Sommerfeld radiation condition is satisfied where U can either be E^s or G for either of its arguments. $J(p)$ is the current density on S, which actually is the jump between the normal derivative values of $E^s(q)$ on both sides of S. On each S_j the Meixner edge condition $J_j(p)=[d_{1j}(p)d_{2j}(p)]^{-1/2}w_j(p)$ is satisfied, where $d_{1j}(p)$ and $d_{2j}(p)$ are the distances to the two edges and $w_j(p)\in C^{0,\alpha}(S_j)$, i.e. belongs to corresponding Hölder class [19]. As mentioned above, the Dirichlet boundary condition requires that the total E-field on S be zero in order to arrive at the integral equation $\int_S J(p)G(q,p)dS=-E^i(q)\ q\in S$ which, for the problem being considered, is of the first kind. For the elements of the surface union S, $G(q,p)$ has logarithmic singularity when $j=m$ and $p_j=q_j$, otherwise, is infinitely smooth.

In the next sub-section we cover the canonical integral equation of the TM wave diffraction boundary value problem for ARM when S consists of a single strip.

2.2. OPM and ARM for the Canonical Integral Equation with Logarithmic Singularity

Consider the following canonical integral equation,

$$\int_{-1}^{1}\left\{-\frac{1}{\pi}\ln|u-v|+\mathrm{K}(u,v)\right\}z(v)\,dv=b(u),\quad u\in[-1,1],\tag{1}$$

with unknown function $z(v)$. Suppose all other functions in (1) are known and smooth enough for our purposes. In particular, $\mathrm{K}(u,v)$ is continuous with its first derivatives and its mixed derivatives of the second order are square-integrable. We are looking for solutions of the kind $z(v)=(1-v^2)^{-1/2}w(v)$ in $v\in[-1,1]$ with $w(v)\in C^{0,\alpha}([-1,1])$, where $C^{0,\alpha}$ is the Hölder class [19]. In fact, the properties of the functions $\mathrm{K}(u,v)$, $z(v)$ and $b(u)$ are quite natural to the diffraction problem (see their definitions in next section). Moreover, $z(v)\in L^q([-1,1])$, for any $q<2$ owing to its singularity while v is approaching to ±1, and for $p\geq1$ satisfying $q^{-1}+p^{-1}=1$, $\{-\ln|u-v|/\pi+\mathrm{K}(u,v)\}\in L^p([-1,1])$. For fixed u, changing the integration variable as $v=\cos\theta$, $\theta\in[0,\pi]$ and using the identity $2\cos\theta=(e^{i\theta}+e^{-i\theta})$, (1) now can be written as a Fourier convolution integral over the interval $[-\pi,\pi]$ and we can see that Generalized Parseval Equality (GPE) is valid for the considered functions and their Fourier coefficients, as has been proved in 10.5.3 and 12.10 of [24].

Below we use the orthonormal Chebyshev's polynomials of the first kind with the following orthonormality relation (norm of orthogonal Chebyshev's polynomials of the first kind is defined as $\|T_n(x)\|=(\pi/p)^{1/2}$, where $p=1$ for $n=0$ and $p=2$ otherwise):

$$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}\hat{T}_n(x)\hat{T}_s(x)\,dx=\delta_{s,n},\quad s,n=0,1,2,3...\tag{2}$$

with the Kronecker delta δ_{sn}. The subsequent expansion is given in [13]:

$$-\pi^{-1}\ln|u-v|=\sum_{n=0}^{\infty}\gamma_n^{-2}\hat{T}_n(u)\hat{T}_n(v),\ u,v\in[-1,1];\tag{3}$$

with $\gamma_0=(\ln 2)^{-1/2}$, $\gamma_n=|n|^{1/2}$ for $n\neq0$. The properties of the functions $z(v)$, $b(u)$ and $\mathrm{K}(u,v)$ permit their respective expansions to the Fourier-Chebyshev series:

$$\begin{gathered}\begin{bmatrix}z\\ b\end{bmatrix}(v)=\begin{Bmatrix}1/\sqrt{1-v^2}\\ 1\end{Bmatrix}\sum_{n=0}^{\infty}\begin{bmatrix}z_n\\ b_n\end{bmatrix}\hat{T}_n(v),\ v\in[-1,1]\\ K(u,v)=\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}k_{sn}\hat{T}_s(u)\hat{T}_n(v),\ u,v\in[-1,1]\end{gathered}\tag{4}$$

In (4) z_n are unknown coefficients and b_n and k_{sn} are Fourier-Chebyshev coefficients of the functions $b(u)$ and $\mathrm{K}(u,v)$ respectively. Moreover, coefficients k_{sn} satisfy the following inequality [13, 15]:

$$\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\left(1+n^2\right)\left(1+s^2\right)\left|k_{sn}\right|^2 < \infty \tag{5}$$

This means that k_{sn} at the very least decays faster than $O((ns)^{-3/2})$ and provides a basis for understanding how to build a system like (9) as follows below.

Regarding the discussion about GPE about (1) above, we can view (1) also as a Fourier-Chebyshev convolution integral and corresponding GPE is valid also between the functions in (1) and their Fourier-Chebyshev coefficients in (4). Therefore substituting the right hand sides of series (4) into equation (1), we can change the order of integration and summation, and the orthogonal property (2) gives us the equalities of the Fourier-Chebyshev coefficients of the left and right hand sides of equation (1) as,

$$\gamma_n^{-2} z_n + \sum_{s=0}^{\infty} k_{ns} z_s = b_n,\ n = 0,1,2,... \tag{6}$$

This is equivalent to (1) because of the completeness of orthonormal Chebyshev polynomials of the first kind. Infinite SLAE1 (6) is the final system of the entire domain Galerkin procedure [17]. As a simple scaling of unknowns in (6) by the factors in front of them evidently leads to a SLAE2, this may give, at first glance, the false impression of a well-conditioned AS for (6). However, before going any further, we have to understand the quality of the SLAE1 in (6). Such a system exhibits a behavior consisting of a linearly growing condition number which depends on the truncation number (see [23]). This is in accordance with the fact that, with regard to (5), the equations in the system in (6) is of order $\gamma_n^{-2}=|n|^{-1}$ with $n\to\infty$. Therefore the order of the norm of the inverse of the infinite matrix operator in (6) is $O(n)$ with $n\to\infty$, as well as its condition number. Thus (6) is an ill conditioned system. The ARM scheme provides an analytical two sided preconditioning to equation systems such as (6) making it an SLAE2. In order to apply the corresponding ARM to (6), whose coefficients have the property in (5), we define the matrix-operators L, R, K, and H (*diag*: diagonal matrix) as follows:

$$L = R = diag\{\gamma_n\}_{n=0}^{\infty},\ K = \{k_{sn}\}_{s,n=0}^{\infty},\ H = LKR = \{h_{sn}\}_{s,n=0}^{\infty} \tag{7}$$

with $h_{sn}=\gamma_n\gamma_s k_{sn}$, vector-columns z, b, y, g, with $g_n=\gamma_n b_n$ and

$$z = \{z_n\}_{n=0}^{\infty},\ b = \{b_n\}_{n=0}^{\infty},\ y = R^{-1}z = \{y_n\}_{n=0}^{\infty},\ g = Lb = \{g_n\}_{n=0}^{\infty} \tag{8}$$

Using the new unknowns $y_n=z_n/\gamma_n$, and multiplying each nth equation of (6) by γ_n, we obtain an infinite AS which has the following functional representation (I: identity operator):

$$(I+H)y = g,\quad y,g \in l_2 \tag{9}$$

We can prove that (9) is an SLAE2, i.e., H is a compact operator in l_2. This means that we have constructed the (L,R) regularizing pair which solves the initial problem: to obtain an SLAE2 in l_2. It is worth mentioning here that the regularization procedure defined via (7) and (8) is not the only one that exists for obtaining such an SLAE2 as in (9). Consider the diagonal matrix operator $T= diag\{(\gamma_n)^2\}$. The predecessor of ARM, Semi-Inversion (SI) [15, 16, 18], reads as a line-wise regularization where $L=T$ and $R=I$. What is more, the very simple scaling (SS) mentioned above by the choice of $L=I$ and $R=T$ can be considered a column-wise regularization option. Nevertheless, neither of these regularizations leading to a functional equation as in (9) can produce a matrix system with the property,

$$\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(1+n)(1+s)|h_{ns}|^2 < \infty \tag{10}$$

which follows, in particular, from formulae (5) and (7). This is because the ARM performed by the (L,R) regularizing pair defined in (7) is native to (6) (whose coefficients have the property in (5)), leading to both line-wise and column-wise regularizations and, therefore, to a higher quality of well conditioning. Consequently, H in (9) is evidently a compact operator (as all other operators emerging after the said regularization options), but much more than merely that: its coefficients are decaying much faster than necessary even compared to those of a Hilbert-Schmidt operator. When we compare the condition numbers of the algebraic systems obtained by the aforementioned regularization options, we find proof that the one utilized here is the best conditioned one amongst them (see below in the numerical results).

One can prove (see [15]) that equations (1) and (9) are equivalent in the sense of a one to one correspondence between solutions for both equations (in the relevant functional spaces). By applying a truncation procedure to the SLAE2 as in (9), we are able to obtain a solution to both equations with, in principle, the requisite arbitrary accuracy as the condition numbers of such systems are bounded by increasing truncation numbers, contrary to the system in (6).

In the next sub-section we describe the path from the diffraction problem to the canonical problem just outlined, and construct the AS for a multi-strip system.

2.2. Steps Between The Diffraction Problem And The Canonical Integral Equation

After η parameterization of the integral equation (see their definitions in section 2.1.), we arrive at $\int_{-1}^{1} Z(v)\Gamma(u,v)dv=B(u)$, $u\in[-1,1]$, where we define the unknown function $Z(v)=J(\eta(v))|\eta'(v)|$ (derivative vector of parameterization of order i: $\eta^{(i)}(v)=(x^{(i)}(v), y^{(i)}(v))\in R^2$, $|\eta'(v)|$ is the arc-length at v on S), the right hand side of the integral equation $B(u)=-E^i(\eta(u))$, and $\Gamma(u,v)=G(\eta(u),\eta(v))=[-i/4]H_0^{(1)}(k|\eta(u)-\eta(v)|)$. We can seek for $Z(v)$ in the form of $z(v)$ of (1) since the edge conditions are satisfied with that representation. Writing $\delta=u-v$, one can show as in [15] that the local singular expansion of $\Gamma(u,v)$ by means of its Taylor series expansion in the vicinity $|\delta|=0$ for $N\geq 2$ is $\Gamma_\delta(u,v)=(2\pi)^{-1}\ln|\delta|[1+\Sigma_{n=2}^{N} A_n(u)\delta^n]+F_N(u,v)$, where $F_N\in C^N([-1,1]\times[-1,1])$ (for proof, start with definition of $\Gamma(u,v)$ above, and resort to power series expansion of $H_0^{(1)}(z)$, in e.g.[20]). Functions $A_n\in C^\infty(-1,1)$, do not depend on N, and $A_2(u)=-(k|\eta'(u)|/2)^2$ and $A_3(u)=-k^2(\eta'(u)\bullet\eta''(u))/4$ in particular, where $\bullet$ is scalar product in R^2. Thus in (1), $b(u)=-2B(u)$, $K(u,v)=-2\Gamma(u,v)+\ln|u-v|/\pi$ when $u\neq v$, and $K(u,v)=-\ln(k|\eta'(u)|)/\pi$, if $u=v$.

As stated before, $G(q,p)$ is an infinitely smooth function when $q\neq p$. The same is true for the interaction terms when we consider a multi-strip surface union. Thus, the final SLAE2 has the following configuration:

$$y_j + \sum_{m=1}^{N} H_{jm}\, y_m = g_j, \quad \forall y, g \in l_2,\ j=1,\dots,N \tag{11}$$

H_{jm} are matrices calculated with the definitions in previous paragraph and with the algorithm given in the previous sub-section for self terms (i.e., $j=m$), where no difference occurs in the algorithm when calculating the interaction terms (i.e., $j \neq m$) because they are infinitely smooth kernels. The unknown vectors are y_m and the known right hand sides are g_j and they are obtained for each strip as explained in the previous paragraph and previous sub-section. Let us now turn to the numerical results.

3. NUMERICAL RESULTS

This section includes the results of the numerical experiments done in order to (i) verify the physical relevance of the ARM approach to knife edge diffraction, and (ii) expand on the well-conditioned feature of the resulting SLAE2 obtained by said ARM approach.

We compared the results obtained using PM as implemented in [21] and found that the size of the AS for ARM is at least two times less than what PM requires for accurate enough results. The reason for this is set forth below in the discussion concerning Fig. 2 - right. Specifically, we present the current density results of a 10λ and a 2λ strip illuminated by a unit amplitude plane wave at normal incidence (Fig. 1-left). We also compared our results to those of a multiple knife edge study presented in [7] concerning the total near field results of a 160λ strip under plane wave incident at a five degrees deviation from strip surface's normal direction. We obtained excellent agreement between the two, as can be seen in Fig. 1 - right.

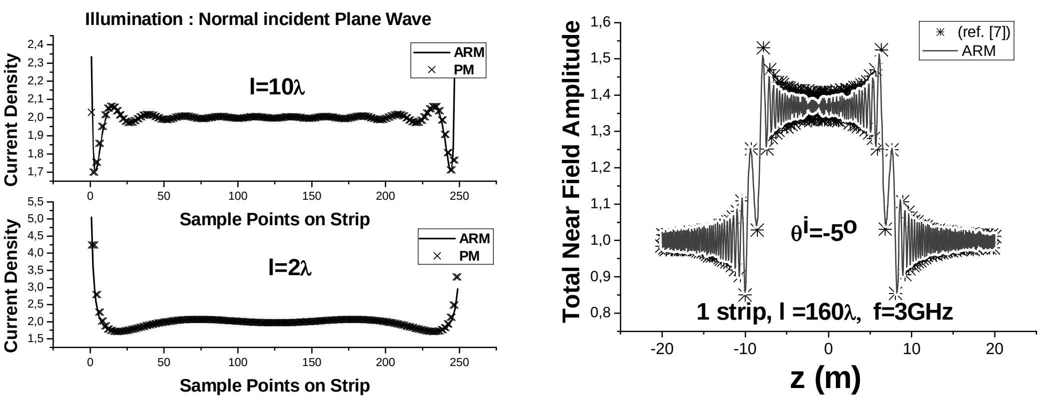

Figure 1: ARM verification using Point Matching (PM) for current densities (left). ARM verification using results of the method in [7], for total near field at f=3 GHz, and 10 m in front of a 16m (160λ) strip (right)

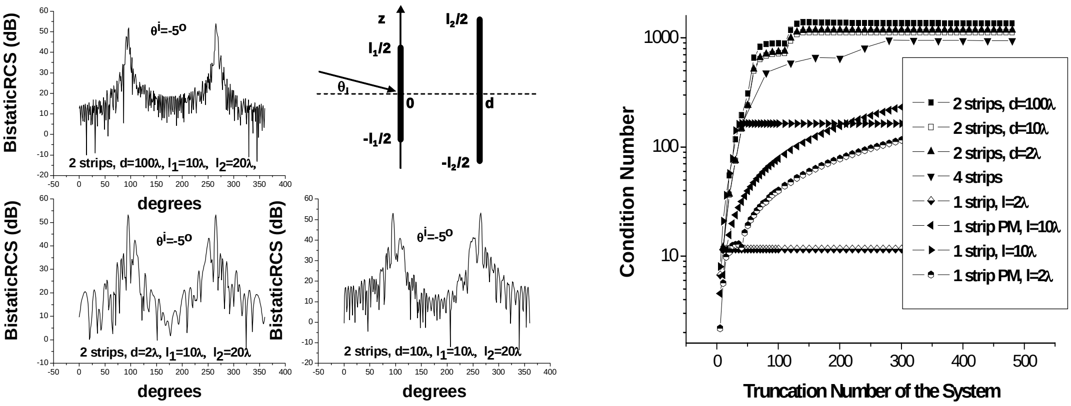

Figure 2: Calculation via ARM of BRCS scan starting from the tangential direction of a basic two strip knife edge model sited at various distances from each (left). Logarithmic plot of condition numbers of the algebraic systems illustrated in the previous Figures 1-right, 2- left, and 3 (PM: point matching, ARM: others) (right).

Fig. 2 – left, illustrates the bistatic radar cross section (BRCS) results of a two strip model of l1=10λ, l2=20λ under the same illumination as in Fig. 2. Strips are sited at three different distances from each other (d=2λ, 10λ, 100λ) and the physical relevance of the BRCS results (i.e., the greater the d, the more ripples) is ascertained at these three separate distances. Fig. 2 – right, depicts logarithmically the condition numbers ($\nu=\|A\|_\infty\|A^{-1}\|_\infty$) in l_∞ norm [11], "A" being the matrix of each AS used. Despite direct methods like PM, ν of which is a linear function of system size [23], ν of ARM-SLAE2 as in (9) tends toward its limit value (Fig. 2 -right), showing its qualitative property. Increasing the length of the strips as well as the presence of resonances in the multi-strip system leads to a greater limit value for ν of ARM-SLAE2. According to [11], the greater the ν, the less accurate are the computer obtained solutions. But even when we consider the 160λ long strip in Fig. 1 - right (ν~1605) or the cases in Fig. 2 - right, a stabilized $\nu \sim 10^3$-10^4 detracts at most four significant figures (SF) in the calculations made in a 16 SF environment as in a 53 bit mantissa. This means that the round-off error for solving the AS is negligible compared to the truncation error for the infinite AS. The latter follows from experiences showing that the size "kl+(10÷20)" of the AS is sufficient to achieve an accuracy with relative error that is less than 0.1% for the current density, and is much less for the far field pattern.

It is worth stating here that the requirement for the truncation number obtained above is characteristically similar to that of the analytical series solution for the scattering by the canonical objects, e.g., the circle [21]. Hence, in conjunction with the mathematical rigor of ARM, the above-mentioned property can also be regarded as a feature that results from the capability of the ARM approach to extend the set of canonical problems that can be solved analytically.

Fig. 3 illustrates a knife edge model on the ground illuminated at the HF band. The illumination utilized in the rigorous simulation is a unit amplitude plane wave impinging on strips with an angle of fifteen degrees to the strip surface's normal axis. The amplitude of the total near field (Fig. 3 above) is depicted, as is the path loss related to this near field amplitude distribution (Fig. 3 below). The ground is assumed to be perfectly conductive and the image principle (see [7]) allows a system size reduction to half because Chebyshev polynomials are symmetrical, as in $T_n(x)= (-1)^n T_n(-x)$, leading to an essential reduction in computational time. A finite-loss ground can also be modeled via our method by simply replacing the Green function of free space with the Green function of lossy dielectric half space (e.g., [22]).

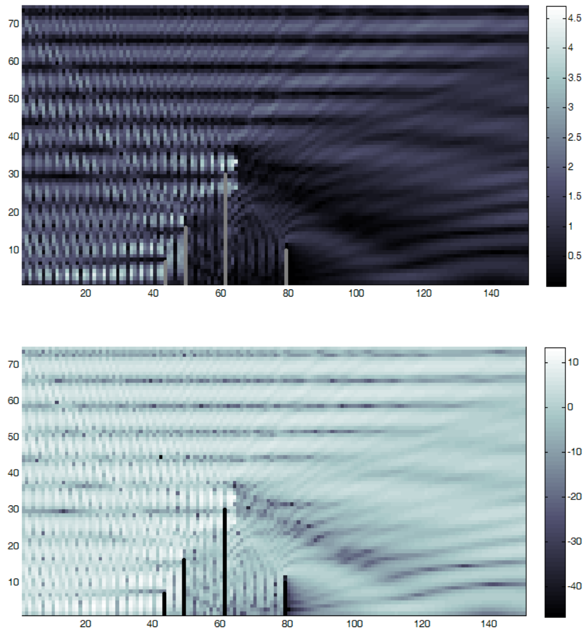

Figure 3: An HF band simulation: Near field amplitudes (|NF|) (above) and path loss (20log|NF|) (3 samples/λ) at a 50λ×25λ frame (below), including four strips on a ground horizontally sited from origin at 14λ, 16λ, 21λ, 26λ with heights l=2λ, 5λ, 10λ, 3λ respectively (shown by grey and black lines respectively). The incident field is a unit amplitude plane TM wave propagating along vector **k**= (cos(-15^o), sin(-15^o)) from origin.

Table
Stabilized condition numbers of the matrices of SLAE2 obtained by the regularization types mentioned in the discussion following equation(9) illustrated in the previous Figures 1 - right, 2- left, and 3.

Regularization Type	Figure 1 - right	Figure 2- left			Figure 3
		d=2λ	d=10λ	d=100λ	
ARM	1604.694	1978.984	1634.348	2362.043	943.672
SS	1725.046	2259.999	1685.013	2373.853	1964.725
SI	1922.423	2725.061	2072.353	2688.005	1314.535

Finally, in Table 1, we compared the stabilized SLAE2 matrix condition numbers from the cases considered in Figures 1 - right, 2 - left, and 3, which were obtained via the matrices resulting from the application of the regularization types mentioned in the discussion following equation (9). One clearly sees that the ARM application proposed produces the best conditioned SLAE2 amongst the various regularization options considered. This is because the corresponding regularizing pair (7) is constructed inherently from the properties in (5) of the SLAE1 obtained by the OPM in (6), and thus, has the quality in (10).

4. CONCLUDING REMARKS

In this paper we focus on ARM as a tool for the analysis of knife edge diffraction problems. ARM is the procedure for constructing, with mathematical equivalence and physical relevance to the original BVP, the proper (L,R) regularizing operator pair, in order to reduce the BVP to an SLAE2 in space l_2 (which is the most proximate candidate to represent the space of numbers and operations in a computer). The theoretical and numerical investigations of the convergence properties as well as the behavior of the condition numbers of the obtained ARM-SLAE2 provide an insight into the excellent qualitative properties of ARM. It is a full-wave, mathematically rigorous approach that is superior to traditional full-wave methods such as PM and, at the same time, it is capable of extending the set of canonical problems that can be solved analytically. The comparisons made with other methods' obtained results show that ARM is relevant to the physical nature of the problems modeled by knife edges. Additional assumptions in modeling the areas through which the radio waves propagate (i.e., whether the ground is perfectly conductive, lossy or of an impedance type) can modify the Green's function used. Futhermore, additional requirements for problems at higher frequencies can be incorporated via the use of fast calculation techniques based on the discrete Chebyshev transform and the appropriate iterative linear solvers. Finally, we intuitively state that the mathematically rigorous and numerically efficient ARM approach embodies great potential for further research on 2D radio wave propagation models.

ACKNOWLEDGEMENTS

The authors would like to thank Yury A. Tuchkin, who has been an essential contributor to ARM theory, for his invaluable discussions and contributions vis-à-vis this study. The authors would also like to thank E. Zoe Schutzman for her priceless editorial commentary.

5. REFERENCES

1. P. D. Holm, Calculation of Higher Order Diffracted Fields for Multiple-Edge Transition Zone Diffraction, *IEEE Trans Antennas Propag*, 52-5 (2004), 1350-1355.
2. L. A. DeAcetis, F. S. Einstein, R. A. Juliano, Jr., and İ. Lazar, Single strip diffraction: comparison of Kirchhoff theory and geometrical theory with the exact solution in the limit of small glancing angle and width; perpendicular polarization, *Applied Optics*, 15-11, (1976), 2866-2870.
3. R. Holland and V. P. Cable, Mathieu Functions and Their Applications to Scattering by a Coated Strip, *IEEE Trans Electromag Comp*,34-1 (1992), 9-16.
4. J. F. Legendre and G. Chassay, Correction Factors for Fresnel Uniform Theory Of Diffraction Results, *Microwave And Opt Tech Let* 7-15, (1994), 702-704.

5. L. Juan-Li´acer and L. Ramos, An Explicit Solution For The Diffraction Of Cylindrical Waves By Multiple Knife Edges Based On The Vogler Attenuation Function, *Microwave And Opt Tech Lett*, 27-4 (2000), 264-265.
6. M. B. Tabakçıoğlu, A. Kara, Comparison of Improved Slope Uniform Theory of Diffraction with Some Geometrical Optic and Physical Optic Methods for Multiple Building Diffractions, *Electromagnetics*, 29 (2009), 303–320.
7. H. Oraizi and S. Hosseinzadeh, Radio-Wave-Propagation Modeling in the Presence of Multiple Knife Edges by the Bidirectional Parabolic-Equation Method, *IEEE Trans Vehic Tech*, 56-3 (2007), 1033-1040.
8. G.C. Hsiao, R.E. Kleinman, Mathematical Foundations for Error Estimation In Numerical Solutions Of Integral Equations In Electromagnetics, *IEEE Trans Antennas and Propag*, 45-3 (1997), 316–328.
9. P.Corr, D.Stewart, P.Hanna, J.Ming, F.J.Smith, Discrete Chebyshev transform. A natural modification of the DCT**,** *Proc. 15th Int. Conf. on Pattern Recognition*, 3 (2000), 1142-1145.
10. P. Ylä-Oijala, M. Taskinen, S. Järvenpää, Analysis Of Surface Integral Equations In Electromagnetic Scattering And Radiation Problems, *Engineering Analysis with Boundary Elements* 32 (2008), 196–209
11. J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.
12. A. Ye. Poyedinchuk, Y. A. Tuchkin and V. P. Shestopalov, New numerical-analytical methods in diffraction theory, *Math.&Comp. Modeling*, 32-9 (2000), 1029-1046.
13. F. Dikmen, E. Karaçuha, Yu. A. Tuchkin, Scalar Wave Diffraction By A Perfectly Soft Infinitely Thin Circular Ring, *Turkish J. of Electrical Eng. & Comp. Sci.*, 9-2 (2001) 199-220.
14. G. Ya. Popov, On Orthogonal Polynomials Method in contact problem of the elasticity theory, *Prikladnaya Matematika i Mekhanica*, 33-3 (1969), (in Russian).
15. V. P. Shestopalov, Yu. A. Tuchkin, A. Ye. Poyednichuk, and Yu. K. Sirenko, *Novel Methods for Solving Direct and Inverse Problems of Diffraction Theory, vol 1: Analytical Regularization of Electromagnetic Boundary Value Problems*, Kharkov: Osnova, 1997 (in Russian).
16. V. P. Shestopalov, and Yu. V. Shestopalov, *Spectral Theory and Excitation of Open Structures*, Peter Peregrinus, London, 1996.
17. C. M. Butler, General Solutions of the Narrow Strip (and Slot) Integral Equations, *IEEE Trans. Antennas Propag*, 33-10 (1985), 1085-1090.
18. R. Kress, Inverse Scattering from an Open Arc, *Math. Meth. In the App. Sci.*Vol:18, 267-293, 1995.
19. D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory,* Krieger, 1992
20. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*. Dover, 1972.
21. K.Umashankar, A.Taflove, *Computational Electromagnetics*, Artech House, 1993.
22. Y. Wang and I. D. Longstaff, Fast computation of Green's functions for a lossy dielectric half-space. *Microwave and Opt. Tech Lett*, 25-4 (2000), 287-289.
23. K.F.Warnick, W.C.Chew, On the spectrum of the electric field integral equation and the convergence of the moment method, *Int.J.Numer.Meth. Engng* 51 (2001), 31-56.
24. R.E.Edwards, *Fourier Series-A Modern Introduction, Vol.I-II*, Springer-Verlag, 1967