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NUMERICAL SOLUTION FOR SOLVING BURGER'S-FISHER EGUATION BY USING ITERATIVE METHODS

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Abstract- In this paper, a Burger's-Fisher equation is solved by using the Adomian's decomposition method (ADM) , modified Adomian's decomposition method (MADM), variational iteration method (VIM), modified variational iteration method (MVIM), modified homotopy perturbation method (MHPM) and homotopy analysis method (HAM). The approximate solution of this equation is calculated in the form of series which its components are computed by applying a recursive relation. The existence and uniqueness of the solution and the convergence of the proposed methods are proved. A numerical example is studied to demonstrate the accuracy of the presented methods.
Key Words: Burger's-Fisher equation, Adomian decomposition method , Modified Adomian decomposition method , Variational iteration method , Modified variational iteration method , Modified homotopy perturbation method , Homotopy analysis method.

1.INTRODUCTION

Burger's-Fisher equation playes an important role in mathematical physics. In recent years some works have been done in order to find the numerical solution of this equation. For example [1-11,35-37]. In this work, we develop the ADM, MADM, VIM, MVIM, MHPM and HAM to solve the Burger's –Fisher equation as follows:

$$\frac{\partial u}{\partial t}+au^{\sigma}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u\left(1-u^{\sigma}\right), 0\le x\le L, 0\le t\le T, \tag{1}$$

with the initial condition given by:

$$u(x,0)=\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-a\sigma}{2(\sigma+1)}x\right)\right)^{\frac{1}{\sigma}}=f(x), \tag{2}$$

and boundary conditions :

$$u(0,t)=\left(\frac{1}{2}+\frac{1}{2}\tanh\left[\frac{-a\sigma}{2(\sigma+1)}\left(\left(\frac{a}{2(\sigma+1)}\left(\left(\frac{a}{\sigma+1}+\frac{\beta(\sigma+1)}{a}\right)t\right)\right]\right]\right)^{\frac{1}{\sigma}}, t\ge 0, \tag{3}$$

$$u(L,t)=\left(\frac{1}{2}+\frac{1}{2}\tanh\left[\frac{-a\sigma}{2(\sigma+1)}\left(1-\left(\frac{a}{\sigma+1}+\frac{\beta(\sigma+1)}{a}\right)t\right)\right]\right)^{\frac{1}{\sigma}}, t\ge 0. \tag{4}$$

Where α, β and σ are constants. When $\alpha=0, \sigma=1$, Eq.(1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibre and wall motion in liquid crystals [12]. Generalized Burger equation will be obtained when $\beta=0$.

This equation when $\beta=0$, has been used to investigate sound waves in a viscous medium by Lighthill [13]. However, it was originally introduced by Burgers [14] to model one-dimensional turbulence and can also be applied to waves in fluid-filled viscous elastic tubes and magnetohydrodynamic waves in a medium with finite electrical conductivity [15].

In order to obtain an approximate solution of Eq.(1), let us integrate one time Eq.(1) with respect to t using the initial conditions we obtain,

$$u(x,t)=f(x)+\beta\int_0^t F(u(x,t))dt-\alpha\int_0^t F_1((u(x,t))dt+\int_0^t D^2(u(x,t)dt, \tag{5}$$

where,

$$F(u(x,t))=u(x,t)(1-u^\sigma(x,t)),\ F_1(u(x,t))=u^\sigma(x,t)\frac{\partial u(x,t)}{\partial x},\ D^2(u(x,t))=\frac{\partial^2 u(x,t)}{\partial x^2}.$$

In Eq.(5), we assume *f(x)* is bounded for all *x* in $J=[0, L]$.

The terms $D^2(u(x,t))$, $F(u(x,t))$ and $F_1(u(x,t))$ are Lipschitz continuous with $|D^2(u)-D^2(u^*)|\le L_3|u-u^*|, |F_1(u)-F_1(u^*)|\le L_2|u-u^*|, |F(u)-F(u^*)|\le L_1|u-u^*|$ and $a_1 := T\,(|\beta|L_1+|a|L_2+L_3),\ \beta_1 := 1-T\,(1-a_1), \beta_2 := 1-TL\alpha_1.$

2 . THE ITERATIVE METHODS

2.1 Description of the MADM and ADM

The Adomian decomposition method is applied to the following general nonlinear equation

$$Lu+Ru+Nu=g, \tag{6}$$

where u is the unknown function, L is the highest order derivative operator which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(6), and using the given conditions we obtain

$$u(x,t)=f(x)-L^{-1}(Ru)-L^{-1}(Nu), \tag{7}$$

where the function *f(x)* represents the terms arising from integrating the source term g . The nonlinear operator $Nu=G_1(u)$ is decomposed as

$$G_1(u)=\sum_{n=0}^{\infty}A_n, \tag{8}$$

where $A_n, n\ge 0$, are the Adomian polynomials determined formally as follows :

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]\lambda=0. \tag{9}$$

Adomian polynomials were introduced in as

$$A_0 = G_1(u_0),\ A_1 = u_1 G_1'(u_0),\ A_2 = u_2 G_1'(u_0) + \frac{1}{2!} u_1^2 G_1''(u_0),$$

$$A_3 = u_3 G_1'(u_0) + u_1 u_2 G_1''(u_0) + \frac{1}{3!} u_1^3 G_1'''(u_0), \ldots \tag{10}$$

2.1.1Adomian decomposition method

The standard decomposition technique represents the solution of *u(x, t)* in (6) as the following series,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{11}$$

where, the components $u_0, u_1, \ldots$ are usually determined recursively by

$$u_0 = f(x),\ u_1 = \beta \int_0^t A_0(x,t)dt - a\int_0^t B_0(x,t)dt + \int_0^t L_0(x,t)dt,$$

$$u_{n+1} = \beta \int_0^t A_n(x,t)dt - a\int_0^t B_n(x,t)dt + \int_0^t L_n(x,t)dt, n \geq 0. \tag{12}$$

Substituting (10) into (12) leads to the determination of the components of u. Having determined the components $u_0, u_1, \ldots$ the solution u in a series form defined by (11) follows immediately.

2.1.2 The modified Adomian decomposition method

The modified decomposition method was introduced by Wazwaz [19]. The modified forms was established based on the assumption that the function *f(x)* can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{13}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part f_1 be assigned to the zeroth component u_0, whereas the remaining part f_2 be combined with the other terms given in (12) to define u_1. Consequently, the modified recursive relation

$$u_0 = f_1(x),\ u_1 = f_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0),\ u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), n \geq 1, \tag{14}$$

was developed.

To obtain the approximation solution of Eq.(1), according to the MADM, we can write the iterative formula (14) as follows:

$$u_0 = f_1(x),\ u_{n+1} = \beta \int_0^t A_n(x,t)dt - a\int_0^t B_n(x,t)dt + \int_0^t L_n(x,t)dt.$$

$$u_1 = f_2(x) + \beta \int_0^t A_0(x,t)dt - a\int_0^t B_0(x,t)dt + \int_0^t L_0(x,t)dt \tag{15}$$

The operators $D^2(u), F(u)$, and $F_1(u)$ are usually represented by the infinite series of the Adomian polynomials as follows :

$$F(u)=\sum_{i=0}^{\infty}A_{i},\ F_1(u)=\sum_{i=0}^{\infty}B_{i},\ D^2(u)=\sum_{i=0}^{\infty}L_{i},$$

where A_i, B_i and $L_i (i \geq 0)$ are the Adomian polynomials .

Also, we can use the following formula for the Adomian polynomials [20]:

$$A_n = F(s_n)-\sum_{i=0}^{n-1}A_{i},\quad B_n = F_1(s_n)-\sum_{i=0}^{n-1}B_{i},\quad L_n = D^2(s_n)-\sum_{i=0}^{n-1}L_{i}. \tag{16}$$

2.2 Description of the VIM and MVIM

To obtain the approximation solution of Eq.(1), according to the VIM[21-24,33-34], we can write iteration formula as follows :

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta L_t^{-1}\left(\lambda\left[u_n(x,t)-f(x)-\beta\int_0^t F(u_n(x,t))dt+\alpha\int_0^t F_1(u_n(x,t))dt-\int_0^t D^2(u_n(x,t))dt\right]\right). \tag{17}$$

Where,

$$L_t^{-1}(.)=\int_0^t(.)d\tau$$

To find the optimal λ, we proceed as

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta L_t^{-1}\left(\lambda\left[u_n(x,t)-f(x)-\beta\int_0^t F(u_n(x,t))dt+\alpha\int_0^t F_1(u_n(x,t))dt-\int_0^t D^2(u_n(x,t))dt\right]\right). \tag{18}$$

From Eq.(18), the stationary conditions can be obtained as follows:

$\lambda'=0$ and $1+\lambda'=0$.

Therefore, the Lagrange multipliers can be identified as λ = -1 and by substituting in (17), the following iteration formula is obtained .

$$u_0(x,t)=f(x),$$
$$u_{n+1}(x,t)=u_n(x,t)-L_t^{-1}\left(u_n(x,t)-f(x)-\beta\int_0^t F(u_n(x,t))dt+\alpha\int_0^t F_1(u_n(x,t))dt-\int_0^t D^2(u_n(x,t)).\right)dt,\ n\geq 0. \tag{19}$$

To obtain the approximation solution of Eq.(1), based on the MVIM [25-27], we can write the following iteration formula :

$$u_0(x,t)=f(x),$$
$$u_{n+1}(x,t)=u_n(x,t)-L_t^{-1}\left(-\beta\int_0^t F(u_n(x,t)-u_{n-1}(x,t))dt\right.$$
$$\left.+\alpha\int_0^t F_1(u_n(x,t)-u_{n-1}(x,t))dt-\int_0^t D^2(u_n(x,t)-u_{n-1}(x,t))dt\right),\ n\geq 0. \tag{20}$$

Relations (19) and (20) will enable us to determine the components $u_n(x, t)$ recursively for $n \geq 0$.

2.3 Description of the HAM

Consider

$$N[u]=0\ ,$$

where N is a nonlinear operator, u(x, t) is unknown function and x is an independent variable. Let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H(x, t) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[s(x,t)] = 0$ when $s(x, t) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows :

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]-qhH(x,t)N[\phi(x,t;q)]=\hat{H}[\phi(x,t;q);u_0(x,t),H(x,t),h,q]. \tag{21}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x, t)$.

Enforcing the homotopy (21) to be zero, i.e ,.

$$\hat{H}\left[\phi(x,t;q);u_0(x,t),H(x,t),h,q\right]=0, \tag{22}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=ghH(x,t)N[\phi(x,t;q)]. \tag{23}$$

When $q = 0$, the zero-order deformation Eq.(23) becomes

$$\phi(x;0)=u_0(x,t), \tag{24}$$

and when $q = 1$, since $h \neq 0$ and $H(x,t) \neq 0$, the zero-order deformation Eq.(23) is equivalent to

$$\phi(x,t;1)=u(x,t). \tag{25}$$

Thus, according to (24) and (25), as the embedding parameter q increases from 0 to 1, $\varphi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x, t)$. Such a kind of continuous variation is called deformation in homotopy .[28,29]

Due to Taylor's theorem, $\varphi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m,\ u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\Big|_{q=0.} \tag{26}$$

Let the initial guess $u_0(x, t)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x, t)$ be properly chosen so that the power series (26) of $\varphi(x, t; q)$ converges at $q = 1$, then, we have under these assumptions the solution series

$$u(x,t)=\phi(x,t;1)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{27}$$

From Eq.(26), we can write Eq.(23) as follows

$$(1-q)L[\phi(x,t,q)-u_0(x,t)]=(1-q)L\left[\sum_{m=1}^{\infty}u_m(x,t)q^m\right]=qhH(x,t)N[\phi(x,t,q)]\Rightarrow \tag{28}$$

$$L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] - qL\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] = qhH(x,t)N[\phi(x,t,q)]$$

By differentiating (28) m times with respect to q, we obtain

$$\{L\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right] - qL\left[\sum_{m=1}^{\infty} u_m(x,t)q^m\right]\}^{(m)} = \{qhH(x,t)N\left[\varphi(x,t,q)\right]\}^{(m)}$$

$$= m!L[u_m(x,t) - u_{m-1}(x,t)] = hH(x,t)m\frac{\partial^{m-1}N[\varphi(x,t,q)]}{\partial q^{m-1}}\Big|_{q=0}.$$

Therefore ,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\Re_m(u_{m-1}(x,t)), \tag{29}$$

where ,

$$\Re_m(u_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\Big|q = 0,\ \chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases} \tag{30}$$

To obtain the approximation solution of Eq.(1), according to HAM, let

$$N\left[u(x,t)\right] = u(x,t) - f(x) - \beta\int_0^t F(u(x,t))dt + \alpha\int_0^t F_1(u(x,t))dt - \int_0^t D^2(u(x,t))dt,$$

so ,

$$\Re_m\left(u_{m-1}(x,t)\right) = u_{m-1}(x,t) - f(x) - \beta\int_0^t F\left(u_{m-1}(x,t)\right)dt + \alpha\int_0^t F_1\left(u_{m-1}(x,t)\right)dt - \int_0^t D^2\left(u_{m-1}(x,t)\right)dt, \tag{31}$$

Substituting (31) into (29)

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hH(x,t)\left[u_{m-1}(x,t) - \beta\int_0^t F(u(x,t))dt + \alpha\int_0^t F_1(u(x,t))dt\right.$$

$$\left. - \int_0^t D^2(u(x,t))dt - (1-\chi_m)f(x)\right]. \tag{32}$$

We take an initial guess $u_0(x,t) = f(x)$, an auxiliary nonlinear operator $Lu = u$, a nonzero auxiliary parameter h = -1, and auxiliary function $H(x,t)$ = 1. This is substituted into (32) to give the recurrence relation

$$u_0(x,t) = f(x),$$
$$u_{n+1}(x,t) = \beta\int_0^t F(u_n(x,t))dt - \alpha\int_0^t F_1(u_n(x,t))dt + \int_0^t D^2(u_n(x,t))dt, n \ge 1. \tag{33}$$

2.4 Description of the MHPM

To explain MHPM, we consider Eq. (1) as

$$L(u(x,t)) = u(x,t) - f(x) - \beta\int_0^t F(u(x,t))dt + \alpha\int_0^t F_1(u(x,t))dt - \int_0^t D^2(u(x,t))dt.$$

Where $F(u(x,t)) = g_1(x)h_1(t), F_1(u(x,t)) = g_2(x)h_2(t)$ and $D^2(u(x,t)) = g_3(x)h_3(t)$. We can define homotopy *H(u(x, t), p, m)* by

$$H(u(x,t),o,m) = f_1(u(x,t)), H(u(x,t),1,m) = L(u(x,t)).$$

Where m is an unknown real number and

$$f_1(u(x,t)) = u(x,t) - f(x).$$

Typically we may choose a convex homotopy by

$$H(u(x,t),p,m) = (1-p)f(u(x,t)) + pL(u(x,t)) + p(1-p)[m(g_1(x)+g_2(x)+g_3(x))] = 0, 0 \le p \le 1. \tag{34}$$

where m is called the accelerating parameters, and for m=0 we define *H(u(x, t, p, 0) = H(u(x, t),p),* which is the standard HPM .

The convex homotopy (34) continuously trace an implicity defined curve from a starting point *H(u(x, t) − f₁(u(x, t)), 0,m)* to a solution function *H(u(x, t), 1,m)*. The embedding parameter p monotonically increase from o to 1as trivial problem $f_1(u(x,t))$ =0. is continuously deformed to original problem *L(u(x, t))* =0. [30-32]

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{35}$$

when $p \to 1$, Eq. (33) becomes the approximate solution of Eq. (1), i.e ,.

$$u = lim_{p\to 1} v = \sum_{n=0}^{\infty} u_n(x,t), \tag{36}$$

where ,

$$\begin{aligned}
&u_0(x,t) = f(x),\\
&u_1(x,t) = \beta\int_0^t F(u_0(x,t))dt - \alpha\int_0^t F_1(u_0(x,t))dt + \int_0^t D^2(u_0(x,t))dt - m(g_1(x)+g_2(x)+g_3(x)),\\
&u_2(x,t) = \beta\int_0^t F(u_1(x,t))dt - \alpha\int_0^t F_1(u_1(x,t))dt + \int_0^t D^2(u_1(x,t))dt + m(g_1(x)+g_2(x)+g_3(x)),\\
&u_m(x,t) = \sum_{k=0}^{m-1} \beta\int_0^t F(u_{m-k-1}(x,t))dt - \alpha\int_0^t F_1(u_{m-k-1}(x,t))dt + \int_0^t D^2(u_{m-k-1}(x,t))dt, m \ge 3.
\end{aligned} \tag{37}$$

3. EXISTENCE AND CONVERGENCY OF ITERATIVE METHODS

Theorem 3.1. Let $0 < \alpha_1 < 1$, then Burger's-Fisher equation (1), has a unique solution .

Proof. Let u and u^* be two different solutions of (5) then

$$\left|u - u^*\right| = \left|\beta\int_0^t F(u(x,t))dt - \alpha\int_0^t F_1(u(x,t))dt + \int_0^t D^2(u_n(x,t))dt\right|$$

$$\le |\beta|\int_0^t\left|F(u(x,t)) - F(u^*(x,t))\right|dt + |\alpha|\int_0^t\left|F_1(u(x,t)) - F_1(u^*(x,t))\right|dt + \int_0^t\left|D^2(u(x,t)) - D^2(u(x,t))\right|dt$$

$$\le T\left(|\beta|L_1 + |\alpha|L_2 + L_3\right)\left|u - u^*\right| = \alpha_1\left|u - u^*\right|$$

From which we get $(1-\alpha_1)\left|u-u^*\right| \le 0$.since $0<\alpha_1<1$.then $\left|u-u^*\right|=0$.Implies $u=u^*$. and completes the proof .

Theorem 3.2. The series solution $u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)$ of problem(1) using MADM convergence when $0<\alpha_1<1, \left|u_1(x,t)\right|<\infty$.

Proof. Denot as ($C[J]$, ||. ||) the Banach space of all continuous functions on J with the norm $\|f(t)\| = \max |f(t)|$, for all t in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \ge m$. We are going to prove that s_n is a Cauchy sequence in this Banach space:

$$\|s_n - s_m\| = \max_{\forall t\in J} |s_n - s_m| = \max_{\forall\, \in J} \left|\sum_{i=m+1}^{n} u_i(x,t)\right|$$

$$= \max_{\forall t\in J}\ \beta\int_0^t \left(\sum_{i=m}^{n-1} A_i\right)dt - \alpha\int_0^t \left(\sum_{i=m}^{n-1} B_i\right)dt + \int_0^t \left(\sum_{i=m}^{n-1} L_i\right)dt\ |.$$

From [20], we have

$$\sum_{i=m}^{n-1} A_i = F(s_{n-1}-s_{m-1}),\ \sum_{i=m}^{n-1} B_i = F_1(s_{n-1}-s_{m-1}),\ \sum_{i=m}^{n-1} L_i = D^2(s_{n-1}-s_{m-1}).$$

So ,

$$\|s_n - s_m\| = \max_{\forall t\in J}\left|\beta\int_0^t\left[F(s_{n-1}-s_{m-1})\right]dt - \alpha\int_0^t\left[F_1(s_{n-1}-s_{m-1})\right]dt + \int_0^t\left[D^2(s_{n-1}-s_{m-1})\right]dt\right| \le$$

$$|\beta|\int_0^t F(s_{n-1}-s_{m-1})|dt + |\alpha|\left|\int_0^t\ \left|F_1(s_{n-1}-s_{m-1})\right| D^2(s_{n-1}-s_{m-1})\,dt\right| \le \alpha_1 \left\|s_n - s_m\right\|.$$

Let $n = m+1$, then

$$\left\|s_n - s_m\right\| \le \alpha_1 \left\|s_m - s_{m-1}\right\| \le \ldots \le \alpha_1^m \left\|s_1 - s_0\ \right\|.$$

From the triangle inquality we have

$$\|s_n - s_m\| \le \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \ldots + \ \|s_n - s_{n-1}\ \| \le \left[\alpha_1^m + \alpha_1^{m+1} + \ldots + \alpha_1^{n-m-1}\right]\|s_1 - s_0\|$$

$$\le \alpha_1^m\left[1+\alpha_1^2+\ldots+\alpha_1^{n-m-1}\right]\|s_1 - s_0\| \le \alpha_1^m\left[\frac{1-\alpha_1^{n-m}}{1-\alpha_1}\right]\|u_1(x,t)\|.$$

Since $0< \alpha_1<1$, we have $(1-\alpha_1^{n-m}) < 1$, then

$$\|s_n - s_m\| \le \frac{\alpha_1^m}{1-\alpha_1}\max_{\forall t\in J}|u_1(x,t)|.$$

But $|u_1(x, t)| < \infty$, so, as $m \to\infty$, then $\|s_n - s_m\| \to 0$.. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete . □

Theorem 3.3. The series solution $u_n(x,t)$ of problem (1) using VIM converges when $0<\alpha_1<1,\ 0<\beta_1<1$.

Proof .

$$u_{n+1}(x,t) = u_n(x,t) - L_t^{-1}\left(\left[u_n(x,t) - f(x) - \beta\int_0^t F(u_n(x,t))dt + \alpha\int_0^t F_1(u_n(x,t)))dt - \int_0^t D^2(u_n(x,t))dt\right]\right) \quad (38)$$

$$u(x,t)=u(x,t)-L_t^{-1}\left(\left[u(x,t)-f(x)-\beta\int_0^t F(u(x,t))dt+\alpha\int_0^t F_1(u(x,t)))dt-\int_0^t D^2(u(x,t))dt\right]\right)\quad(39)$$

By subtracting relation (38) from (39),

$$u_{n+1}(x,t)-u(x,t)=u_n(x,t)-u(x,t)-L_t^{-1}\left(u_n(x,t)-u(x,t)-\beta\int_0^t[F(u_n(x,t))-F(u(x,t))]\right)$$

$$+\alpha\int_0^t\left[F_1(u_n(x,t))dt-F_1(u(x,t))dt-F_1(u(x,t))dt\right]dt-\int_0^t\left[D^2(u_n(x,t))-D^2(u(x,t))\right]dt),$$

if we set, $e_{n+1}(x,t)=u_{n+1}(x,t)-u_n(x,t), e_n(x,t)=u_n(x,t)-u(x,t), \left|e_n(x,t^*)\right|=\max_t|e_n(x,t)|$ then since e_n is a decreasing function with respect to t from the mean value theorem we can write ,

$$e_{n+1}(x,t)=e_n(x,t)+L_t^{-1}\left(-e_n(x,t)+\beta\int_0^t[F(u_n(x,t))-F(u(x,t))]dt\right)$$

$$-\alpha\int_0^t\left[F_1(u_n(x,t))-F_1(u(x,t))\right]dt+\int_0^t\left[D^2(u_n(x,t))-D^2(u(x,t))\right]dt)$$

$$\le e_n(x,t)+L_t^{-1}\left[-e_n(x,t)+L_t^{-1}|e_n(x,t)|(T(|\beta|L_1+|\alpha|L_2+L_3)\right]$$

$$\le e_n(x,t)-Te_n(x,\eta)+T(|\beta|L_1+|\alpha|L_2+L_3)L_t^{-1}L_t^{-1}|e_n(x,t)|\le(1-T(1-\alpha_1)\left|e_n(x,t^*)\right|,$$

$$0\le\eta\le t,\ en+1(x,t)\le\beta\left|en\left(x,t^*\right)\right|.$$

Therefore ,

$$\|en+1\|=\max_{\forall t}\in J|en|\le\beta_1\|en\|.$$

Since $0<\beta_1<0, then\ \|en\|\to 0$. So, the series converges and the proof is complete . □

Theorem 3.4. The series solution $u_n(x,t)$ of problem (1) using MVIM converges when $0<\alpha_1<1$, $0<\beta_2<1$.

Proof. The proof is similar to the previous theorem.

Theorem 3.5. If the series solution (33) of problem (1) using HAM convergent then it converges to the exact solution of the problem (1).

Proof.. We assume :

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t),\ \hat{F}(u(x,t))=\sum_{m=0}^{\infty}F(u_m(x,t)),\ \hat{F}_1(u(x,t))=\sum_{m=0}^{\infty}F_1(u_m(x,t)),$$

$$\hat{D}^2(u(x,t))=\sum_{m=0}^{\infty}D^2(u_m(x,t)).$$

We can write ,

$$\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=u_1+(u_2-u_1)+...+(u_n-u_{n-1})=u_n(x,t).\quad(40)$$

Hence, from (40)

$$\lim_{m\to\infty}u_n(x,t)=0.\quad(41)$$

So, using (41) and the definition of the nonlinear operator L, we have

$$\sum_{m=1}^{\infty} L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = L\left[\sum_{m=1}^{\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]\right] = 0.$$

therefore from (29), we can obtain that ,

$$\sum_{m=1}^{\infty} L(u_m(x,t)-\chi_m u_{m-1}(x,t)) = hH(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0.$$

Since $h \neq 0$ and $H(x, t)= 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = 0. \tag{42}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (42) and simplifying it , we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t)) = \sum_{m=1}^{\infty}\left[u_{m-1}(x,t))dt + \alpha\int_0^t F_1(u_{m-1}(x,t))dt - \int_0^t D^2(u_{m-1}(x,t))dt - (1-\chi_m)f(x)\right] \tag{43}$$

$$= u(x,t) - f(x) - \beta\int_0^t \hat{F}(u(x,t))dt + \alpha\int_0^t \hat{F}_1(u(x,t))dt - \int_0^t \hat{D}^2(u(x,t))dt.$$

From (42) and (43), we have

$$u(x,t) = f(x) + \beta\int_0^t \hat{F}(u(x,t))dt - \alpha\int_0^t \hat{F}_1(u(x,t))dt + \int_0^t \hat{D}^2(u(x,t))dt$$

therefore, $u(x, t)$ must be the exact solution. □

Theorem 3.6. If $|u_m(x, t)| \leq 1$, then the series solution (37) of problem (1) converges to the exact solution by using MHPM .

Proof. We set,

$\phi_n(x,t) = \sum_{i=1}^{n} u_i(x,t), \phi_{n+1}(x,t) = \sum_{i=1}^{n+1} u_i(x,t).$

$|\phi_{n+1}(x,t)-\phi_n(x,t)| = D(\phi_{n+1}(x,t),\phi_n(x,t)) = D(\phi_n + u_n, \phi_n) =$

$D(u_n,0) \leq \sum_{k=0}^{m-1}|\beta|\int_0^t |F(u_{m-k-1}(x,t))|dt + |\alpha|\int_0^t |F_1(u_{m-k-1}(x,t))|dt + \int_0^t |D^2(u_{m-k-1}(x,t))|dt.$

$\leq (m\alpha_1)^n |f(x)|.$

$|\phi_{n+1}(x,t)-\phi_n(x,t)| \leq$

$\sum_{k=0}^{m-1}|\beta|\int_0^t |F(u_{m-k-1}(x,t))|dt + |\alpha|\int_0^t |F_1(u_{m-k-1}(x,t))|dt + \int_0^t |D^2(u_{m-k-1}(x,t))|dt.$

$(m\alpha_1)^n |f(x)|. \to \sum_{n=0}^{\infty} \|\phi_{n+1}(x,t)-\phi_n(x,t)\| \leq m\alpha_1 |f(x)| \sum_{n=0}^{\infty} (m\alpha_1)^n,$

Therefore,

$\lim_{n\to\infty} u_n(x,t) = u(x,t).$ □

4 . NUMERICAl EXAMPLE

In this section, we compute a numerical example which is solved by the ADM, MADM, VIM, MVIM, MHPM and HAM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm :

Step 1. Set n ←0 .**Step 2**. Calculate the recursive relations (12) for ADM, (15) for MADM, (36) for HAM ,(40) for MHPM,. (22) for VIM and (23) for MVIM.
Step 3. If $|u_{n+1}-u_n|<\varepsilon$ then go to step 4 , else n ← n+1 and go to step 2. **Step4**.Print $u(x,t)=\sum_{i=0}^{n}u_i(x,t)$ as the approximate of the exact solution(ADM,MADM,HAM and MHPM) and Print $u_n(x,t)$ as the approximate of the exact solution .(VIM and MVIM) .
Example 4.1. [6] Consider the Burger's-Fisher equation as follows:

$$u_t=u_{xx}+uu_x+u(1-u),$$

subject to the initial condition :

$$u(x,0)=\frac{1}{2}+\frac{1}{2}\tanh(\frac{x}{4}).$$

With the exact solution is $u(x,t)=\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2\sqrt{2}}(x+\frac{5}{2}t)).$

Table 1. Numerical results for Example 1 (t=0.1)

X	Errors					
	ADM(n=10)	MADM(n=8)	VIM(n=6)	MVIM(n=5)	MHPM (n=4)	HAM (n=7)
1	4.53×10^{-11}	2.13×10^{-10}	3.89×10^{-12}	2.73×10^{-12}	1.21×10^{-13}	3.53×10^{-12}
1.5	5.74×10^{-11}	3.27×10^{-11}	2.94×10^{-12}	2.16×10^{-12}	1.71×10^{-13}	2.58×10^{-12}
2.5	6.39×10^{-12}	3.69×10^{-12}	7.16×10^{-13}	4.17×10^{-13}	4.53×10^{-14}	603×10^{-13}
3	4.86×10^{-11}	2.25×10^{-11}	2.32×10^{-12}	1.73×10^{-12}	1.26×10^{-13}	1.22×10^{-12}

5. CONCLUSION

The MHPM has been shown to solve effectively, easily and accurately a large class of

nonlinear problems with the approximations which convergent are rapidly to exact solutions. In this work, the MHPM has been successfully employed to obtain the approximate analytical solution of the Burger's-Fisher equation. For this purpose, we showed that the MHPM is more rapid convergence than the ADM, MADM, VIM, MVIM and HAM .

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