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EXP-FUNCTION METHOD FOR SOLVING NONLINEAR EVOLUTION EQUATIONS

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Abstract- In this paper, we applied Exp-function method to some nonlinear evolution equations. The solution procedure of this method, by the help of symbolic computation of Matlab, Mathematica or so on, is of utter simplicity. The obtained results show that Exp-function method is very powerful and convenient mathematical tool for nonlinear evolution equations in science and engineering.

Key Words- Exp-function Method, Symbolic Computation, 2-D Bratu Type Equation, Generalized Fisher Equation

1.INTRODUCTION

The investigation of the wave solutions for nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. In recent years, many effective methods have been proposed for solving the nonlinear differential equations, such that tanh-sech method [1], extended tanh method [2], sine-cosine method [3], homogenous balance method [4], Jacobi elliptic function method [5], F-expansion method [6], homotopy perturbation method [7], variational iteration method [8], Hirotas bilinear methods [9], bifurcation method [10] and so on. In 2006, a new method, called Exp-function method, was first introduced by He and Wu [11], and was successfully studied in a lot of problems [12-20] and so on.

In this study, we apply Exp-function method to the two-dimensional Bratu-type equation given in [21] as

$$u_{xx} + u_{yy} + \lambda \exp(su) = 0, \tag{1}$$

and the generalized Fisher equation with higher order nonlinearity given in [22,23] as

$$u_t = u_{xx} + u\left(1 - u^n\right). \tag{2}$$

Two-dimensional Bratu model stimulates a thermal reaction process in a rigid material, where the process depends on a balance between chemically generated heat addition and heat transfer by conduction [24]. The nonlinear reaction-diffusion equation was first introduced by Fisher as a model for the propagation of a mutant gene. It has wide application in the fields of logistic population growth, flame propagation, neurophysiology, autocatalytic chemical reactions, and nuclear reactor theory. It is well

known that wave phenomena of plasma media and fluid dynamics are modelled by kink-shaped and tanh-solution or bell-shaped sech-solutions [25].

2. EXP-FUNCTION METHOD AND APPLICATION TO THE TWO-DIMENSIONAL BRATU TYPE EQUATION

Using a complex variation η defined as $\eta = kx + wy$ and the transformation $u = (1/s)\ln v$, we can convert Eq. (1) into ordinary different equation, which reads

$$\left(k^2 + w^2\right)vv'' - \left(k^2 + w^2\right)\left(v'\right)^2 + \lambda s v^3 = 0, \tag{3}$$

where the prime denotes the derivative with respect to η.

We assume that the solution of Eq. (3) can be expressed in the form

$$v(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{4}$$

where c, d, p and q are positive integers, a_n and b_m are unknown constants.

For simplicity, we set $p = c = 1$ and $q = d = 1$, then Eq. (4) reduced to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{5}$$

Substituting Eq. (5) into Eq. (3), we have

$$\frac{1}{A}\left[C_4 \exp(4\eta) + \cdots C_0 + \cdots + C_{-4}\exp(-4\eta)\right] = 0. \tag{6}$$

Equating the coefficients of $\exp(n\eta)$ in Eq. (6) to be zero yields a set of algebraic equations

$$C_4 = 0, \quad C_3 = 0, \quad C_2 = 0, \quad C_1 = 0, \quad C_0 = 0,$$

$$C_{-1} = 0, \quad C_{-2} = 0, \quad C_{-3} = 0, \quad C_{-4} = 0.$$

Solving the system of algebraic equation given above, with the aid of symbolic computation system of Mathematica, we obtain

$$a_{-1} = 0, \quad a_0 = a_0, \quad a_1 = 0, \quad b_{-1} = \frac{b_0^2}{4},$$

$$b_0 = b_0, \qquad w = \mp\sqrt{\left(\lambda n a_0 - k^2 b_0\right)/b_0}. \tag{7}$$

Substituting Eq. (7) into Eq. (5), we obtain the following exact solution

$$v(x,y) = \frac{a_0}{\exp\left(kx+wy\right) + b_0 + \left(b_0^2/4\right)\exp\left(-\left(kx+wy\right)\right)}. \tag{8}$$

So,

$$u(x,y) = \ln\left[\frac{a_0}{\exp\left(kx+wy\right) + b_0 + \left(b_0^2/4\right)\exp\left(-\left(kx+wy\right)\right)}\right]^{\frac{1}{s}}. \tag{9}$$

Using the properties

$$\exp\left(kx+wy\right) + \exp\left(-\left(kx+wy\right)\right) = 2\cosh\left(kx+wy\right), \tag{10}$$

when $b_0 = \mp 2$, it is easy to see that Eq. (9) can reduce to travelling wave solution as follows:

$$u(x,y) = \ln\left[\frac{a_0}{2\cosh\left(kx \mp \sqrt{\left(\mp\lambda s a_0 - 2k^2\right)/2}\,y\right) \mp 2}\right]^{\frac{1}{s}}. \tag{11}$$

3. THE GENERALIZED FISHER EQUATION

We now consider the generalized Fisher equation (2). Introducing the complex variation η defined as $\eta = kx + wt$, we have

$$k^2 u'' - wu' - u^{n+1} + u = 0, \tag{12}$$

where k, w are real parameters and the prime denotes the derivative with respect to η.

Making the transformation

$$u = v^{\frac{1}{n}}, \tag{13}$$

Eq. (12) becomes

$$k^2 n v v'' + k^2\left(1-n\right)\left(v'\right)^2 - wnvv' - n^2 v^3 + n^2 v^2 = 0. \tag{14}$$

We assume that the solution of Eq. (14) can be expressed in the form

$$v(\eta)=\frac{a_1\exp(\eta)+a_0+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \tag{15}$$

Substituting Eq. (15) into Eq. (14), and by the help of Mathematica, equating to zero the coefficients of all powers of $\exp(n\eta)$, $(n=-4,-3,\cdots,3,4)$ yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, b_0, k, w. Solving this system of algebraic equations by using Mathematica, we obtain the following results:

Case 1:

$$a_{-1}=0,\qquad a_0=0,\qquad a_1=1,\qquad b_{-1}=\frac{b_0^2}{4},\qquad b_0=b_0,$$

$$k=\mp\frac{n}{\sqrt{2(n+2)}},\qquad w=-\frac{n(n+4)}{2(n+2)}, \tag{16}$$

where b_0 is a free parameter. Substituting these results into Eq. (15), we obtain the following exact solutions

$$u(x,t)=\left[\frac{e^{kx+wt}}{e^{kx+wt}+b_0+\left(b_0^2/4\right)e^{-kx-wt}}\right]^{\frac{1}{n}}. \tag{17}$$

Using the properties

$$\exp(kx+wt)+\exp\left(-(kx+wt)\right)=2\cosh(kx+wt), \tag{18}$$

$$\exp(kx+wt)-\exp\left(-(kx+wt)\right)=2\sinh(kx+wt), \tag{19}$$

when $b_0=\mp 2$, k and w, Eq. (17) becomes

$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\mp Bt)\right)+\sinh\left(\mp A(x\mp Bt)\right)}{\mp 2+2\cosh\left(\mp A(x\mp Bt)\right)}\right]^{\frac{1}{n}}, \tag{20}$$

where $A=\dfrac{n}{\sqrt{2(n+2)}}$ and $B=\dfrac{n+4}{\sqrt{2(n+2)}}$.

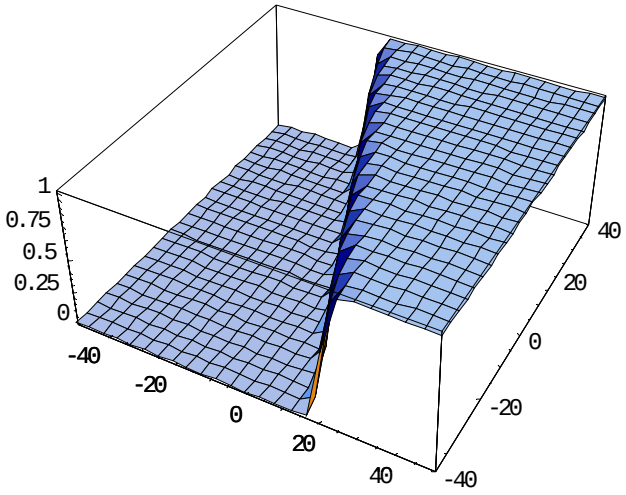

Figure 1 – Solution u is shown at $b_0 = 2$ and $n=1$.

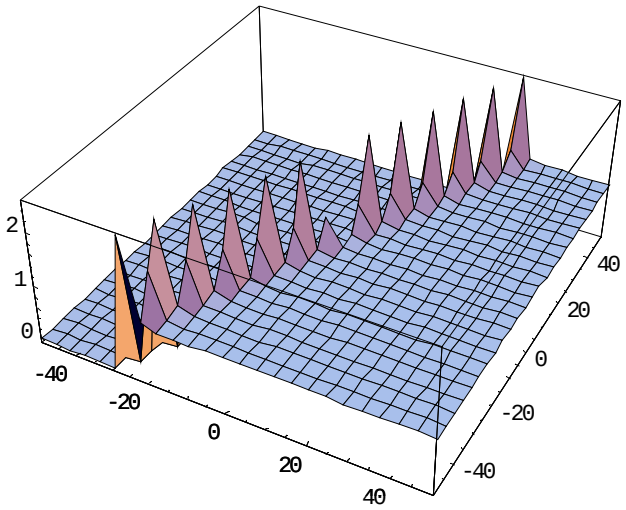

Figure 2 – Solution u is shown at $b_0 = -2$ and $n=1$.

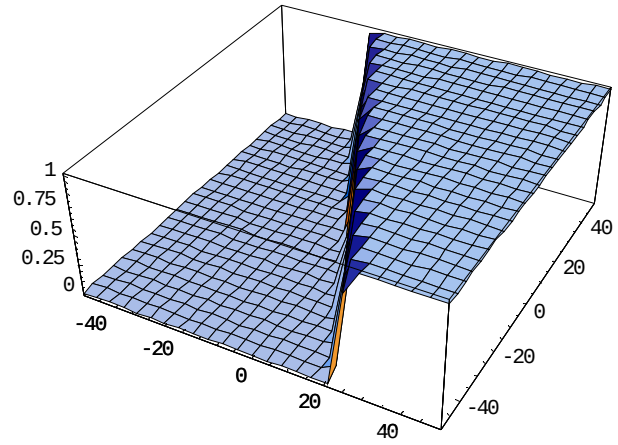

Figure 3 – Solution u is shown at $b_0 = 2$ and $n=2$.

Case 2:

$$a_{-1}=b_{-1},\qquad a_0=0,\qquad a_1=0,\qquad b_{-1}=\frac{b_0^2}{4},\qquad b_0=b_0,$$

$$k=\mp\frac{n}{\sqrt{2(n+2)}},\qquad w=-\frac{n(n+4)}{2(n+2)}, \tag{21}$$

where b_0 is a free parameter. Substituting these results into Eq. (15), we obtain the following exact solutions

$$u(x,t)=\left[\frac{a_{-1}e^{-kx-wt}}{e^{kx+wt}+b_0+a_{-1}e^{-kx-wt}}\right]^{\frac{1}{n}}, \tag{22}$$

or

$$u(x,t)=\left[\frac{\left(b_0^2/4\right)e^{-kx-wt}}{e^{kx+wt}+b_0+\left(b_0^2/4\right)e^{-kx-wt}}\right]^{\frac{1}{n}}. \tag{23}$$

Using the properties (18)-(19), when $b_0=\mp 2$, k, w, Eq. (23) becomes

$$u(x,t)=\left[\frac{\cosh\left(\mp A(x\pm Bt)\right)-\sinh\left(\mp A(x\pm Bt)\right)}{\mp 2+2\cosh\left(\mp A(x\pm Bt)\right)}\right]^{\frac{1}{n}}, \tag{24}$$

where $A=n/\sqrt{2(n+2)}$ and $B=(n+4)/\sqrt{2(n+2)}$.

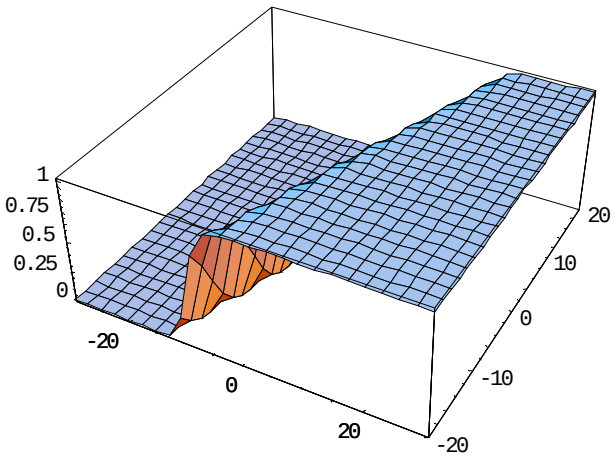

Figure 4 – Solution u is shown at $b_0=2$ and $n=1$.

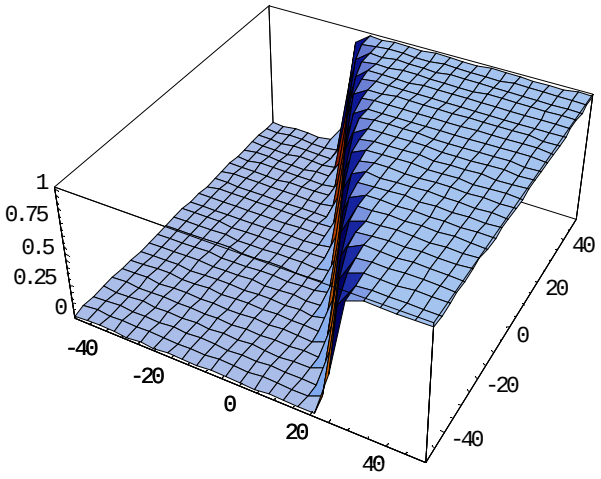

Figure 5 – Solution u is shown at $b_0 = 2$ and $n=1$.

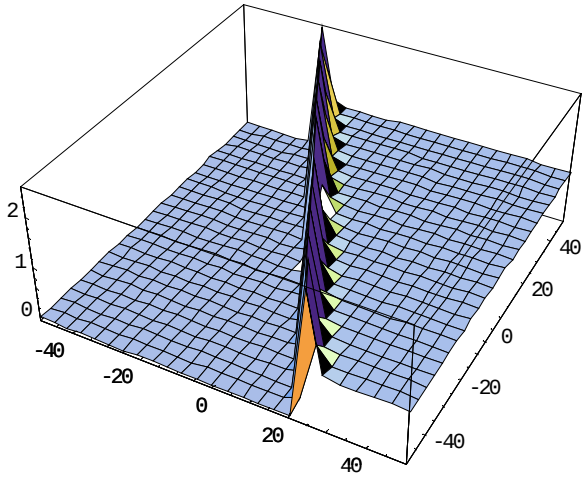

Figure 6 – Solution u is shown at $b_0 = -2$ and $n=1$.

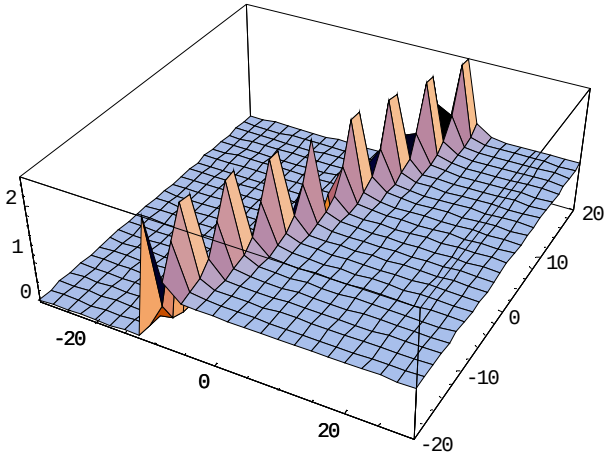

Figure 7 – Solution u is shown at $b_0 = -2$ and $n=1$.

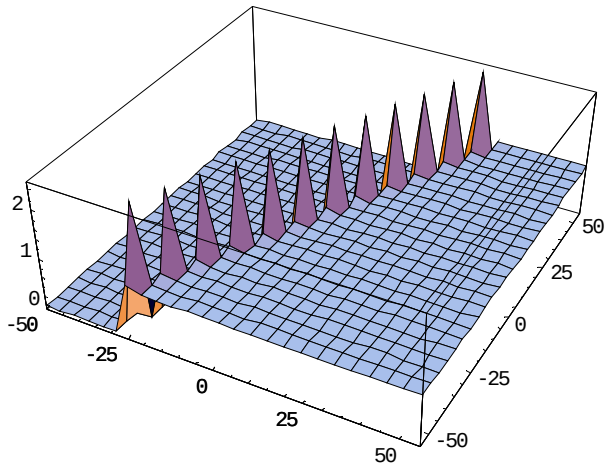

Figure 8 – Solution u is shown at $b_0 = -2$ and $n=2$.

4. CONCLUDING REMARKS

In this study, we applied Exp-function method for obtaining the exact solutions of the two-dimensional Bratu-type equation and the generalized Fisher equation. The results show that this method is a powerful and effective mathematical tool for solving nonlinear evolution equations in science and engineering.

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