Mathematical and Computational Applications, Vol. 15, No. 5, pp. 953-956, 2010.

EXACT THREE-WAVE SOLUTIONS FOR THE (3+1)-DIMENSIONAL BOUSSINESQ EQUATION

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Abstract: In this paper, the three-wave method is used for seeking periodic kink-wave and cross-kink soliton solutions. The (3 + 1)-dimensional Boussinesq equation is chosen as an example to illustrate the effectiveness and convenience the proposed method.

Keywords: Boussinesq equation, Hirota, Three-wave Method, Periodic, Soliton

1. INTRODUCTION

In this paper, we will study the (3+1)-dimensional Boussinesq equation

$$u_{tt} = u_{xx}+u_{yy}+u_{zz}+u_{xxxx}+3(u^2)_{xx} . \qquad (1)$$

which admits many remarkable properties such as N-soliton solutions, N-breather solutions, periodic solutions, symmetries. In this paper the three–wave method[1] will be used to elucidate the inner interaction.

2. THREE-WAVE METHOD

The three-wave method was proposed by Dai et al. in [1] to find coupled wave solutions, the method caught an immediate attention and it was widely used to search for generalized solitary solutions and periodic solutions[2-5].

By using the transformation

$$u=2\ln(f)_{xx}, \qquad (2)$$

where f(x, y, z, t) is an unknown real function.
Eq. (1) is transformed into the bilinear form

$$(D_t^2 - D_x^2 - D_y^2 - D_z^2 - D_x^4)\, f \cdot f = 0, \qquad (3)$$

According to the three-wave method[1], we suppose that the real function f(x,y,z,t) has the following ansatz:

$$f(x,y,z,t)= e^{-\eta_1} +L \cos(\eta_2)+H \cosh(\eta_3)+Ke^{\eta_1}, \qquad (4)$$

where $\eta_i = a_i x+b_i y+c_i z+d_i t$ (i=1,2,3) and a_i,b_i, c_i,d_i are constants to be determined later. Substituting Eq. (4) into Eq. (3) and equating the coefficients of all powers of

$\cosh(\eta_3)\cos(\eta_2)$, $e^{(-\eta_1)}\cos(\eta_2)$, $e^{\eta_1}\cos(\eta_2)$, $e^{(-\eta_1)}\cosh(\eta_3)$, $e^{\eta_1}\cosh(\eta_3)$, $\sin(\eta_2)\sinh(\eta_3)$, $e^{(-\eta_1)}\sinh(\eta_3)$, $e^{\eta_1}\sinh(\eta_3)$, $e^{(-\eta_1)}\sin(\eta_2)$, $e^{\eta_1}\sin(\eta_2)$ and the constant term to zero, we can obtain a set of algebraic equations for H, K, L, a_i, b_i, c_i and d_i(i=1,2,3) as follows:

$$\cosh(\eta_3)\cos(\eta_2):\ (6a_2^2a_3^2+c_2^2-b_3^2+b_2^2-d_2^2-a_3^2+a_2^2-a_3^4-a_2^4+d_3^2-c_3^2)LH=0,$$
$$e^{(-\eta_1)}\cos(\eta_2):\ (-a_1^2+b_2^2-a_2^4-d_2^2+d_1^2-b_1^2+a_2^2-c_1^2+c_2^2-a_1^2-6a_1^2a_2^2)L=0,$$
$$e^{\eta_1}\cos(\eta_2):\ (-a_1^2+b_2^2-a_2^4-d_2^2+d_1^2-b_1^2+a_2^2-c_1^2+c_2^2-a_1^2+6a_1^2a_2^2)LK=0,$$
$$e^{(-\eta_1)}\cosh(\eta_3):\ (-a_3^4+d_1^2+d_3^2-6a_1^2a_3^2-a_1^2-b_3^2-c_1^2-a_3^2-b_1^2-a_1^4-c_3^2)H=0,$$
$$e^{\eta_1}\cosh(\eta_3):\ (-a_3^4+d_1^2+d_3^2-6a_1^2a_3^2-a_1^2-b_3^2-c_1^2-a_3^2-b_1^2-a_1^4-c_3^2)KH=0,$$
$$\sin(\eta_2)\sinh(\eta_3):\ (2d_2d_3-2c_2c_3-2a_2a_3+4a_2^3a_3-2b_2b_3-4a_3^3a_2)LH=0,$$
$$e^{(-\eta_1)}\sinh(\eta_3):(-4a_1^3a_3-2b_1b_3+2d_1d_3-2c_1c_3-4a_3^3a_1-2a_1a_3)H=0,$$
$$e^{\eta_1}\sinh(\eta_3):(2a_1a_3+4a_1^3a_3+2c_1c_3+4a_3^3a_1-2d_1d_3+2b_1b_3)KH=0,$$
$$e^{(-\eta_1)}\sin(\eta_2):(2b_1b_2-2d_1d_2+2a_1a_2+4a_1^3a_2+2c_1c_2-4a_3^2a_1)L=0,$$
$$e^{\eta_1}\sin(\eta_2):(-2a_1a_2+4a_2^3a_1+2d_1d_2-4a_1^3a_2-2b_1b_2-2c_1c_2)LK=0,$$

and constant term:

$$(-4a_3^4-b_3^2+d_3^2-a_3^2-c_3^2)H^2+(4d_1^2-4a_1^2-16a_1^4-4b_1^2-4c_1^2)K$$
$$+(-4a_2^4-d_2^2+a_2^2+b_2^2+c_2^2)L^2=0.$$

Solving the set of algebraic equations with the help of symbolic computation system, such as Maple, Mathematica, MatLab and so on, we obtain the following results.

Case 1.

$$H=0,\quad a_2=0,\quad b_1=0,\quad c_1=0,\quad d_1=0,$$
$$K=\frac{(1+a_1^2)L^2}{4(1+4a_1^2)},\ d_2=\sqrt{-a_1^2+b_2^2+c_2^2-a_1^4}\ .$$

Substituting these parameters into Eq. (4) and then (2), there exists following solution:

$$u(x,y,z,t)=\frac{4a_1^2(-2\tau_1\cosh(a_1x+\theta_1)^2-L\sqrt{\tau_1}\cosh(a_1x+\theta_1)\cos(b_2y+c_2z+\delta_1t)+2\tau\sinh(a_1x+\theta_1)^2)}{(2\sqrt{\tau_1}\cosh(a_1x+\theta_1)+L\cos(b_2y+c_2z+\delta_1t))^2},\tag{5}$$

where $\tau_1=\frac{(1+a_1^2)L^2}{4(1+4a_1^2)},\ \theta_1=\frac{1}{2}\ln(\frac{(1+a_1^2)L^2}{4(1+4a_1^2)}),\ \delta_1=\sqrt{-a_1^2+b_2^2+c_2^2-a_1^4}$, and L, a_1, b_2, c_2 are free parameters.

The solution given by Eq. (5) are periodic soliton solutions which is a periodic traveling wave on the y–z direction, meanwhile a soliton on the t-direction, see Fig. 1.

Case 2.

$$L=0,\quad a_3=0,\quad b_1=0,\quad c_1=0,\quad d_1=0,$$
$$K=\frac{(1+a_1^2)H^2}{4(1+4a_1^2)},\ d_3=\sqrt{a_1^2+b_3^2+c_3^2+a_1^4}\ .$$

Proceeding the same way as that for Case 1, we have the following solution

$$u(x,y,z,t)=\frac{4a_1^2(2\tau_2\cosh(a_1x+\theta_2)^2+H\sqrt{\tau_2}\cosh(a_1x+\theta_2)\cosh(b_3y+c_3z+\delta_2t)-2\tau_2\sinh(a_1x+\theta_2)^2)}{(2\sqrt{\tau_2}\cosh(a_1x+\theta_2)+H\cosh(b_3y+c_3z+\delta_2t))^2}, \quad (6)$$

where $\tau_2=\frac{(1+a_1^2)H^2}{4(1+4a_1^2)}$, $\theta_2=\frac{1}{2}\ln(\frac{(1+a_1^2)H^2}{4(1+4a_1^2)})$, $\delta_2=\sqrt{a_1^2+b_3^2+c_3^2+a_1^4}$ and H , a_1, b_3, c_3 are arbitrary real constants.

Obviously, the solutions given by (6) are cross-kink wave solutions which are periodic on the x–direction, meanwhile solitary on the x–t direction, see Fig. 2. In particular, by choosing different values of H , a_1, b_3, c_3 in (6), we can derive several classes of special solitary solutions of Eq. (1), here we omit them for simplicity.

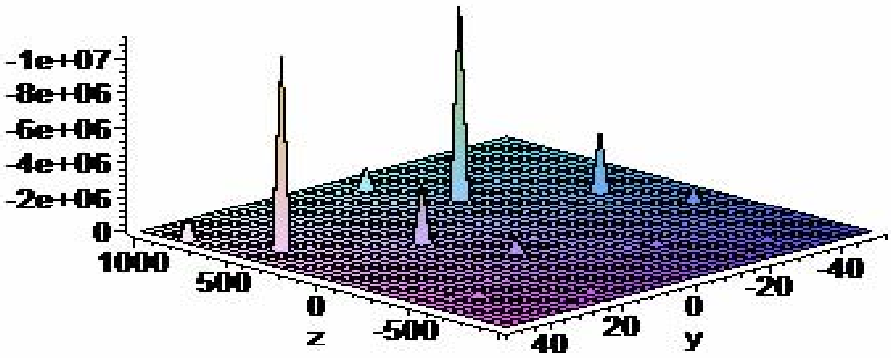

Fig. 1. The figure of u(x,y,z,t): L=$\sqrt{10}$, a_1=1, b_2= c_2=3, x=t=1

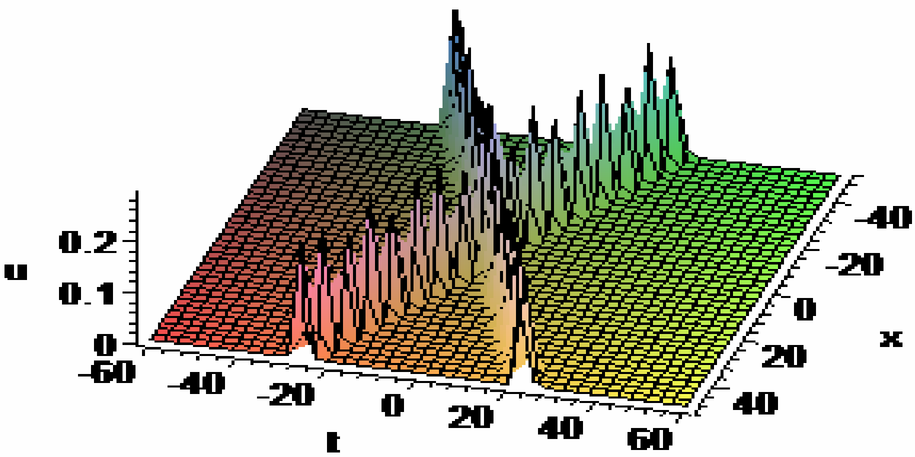

Fig. 2. The figure of u(x,y,z,t): H=$2\sqrt{5}$, a_1=b_3=c_3=1, y=z=1.

3. CONCLUSION

In this paper, the three-wave approach is applied to the (3+1)-dimensional Boussinesq equation. New three-wave solutions including periodic cross-kink wave solutions and cross-kink soliton solutions are obtained. Moreover, mechanical feature of wave is exhibited. All the presented solutions show the remarkable richness of the solution space of the (3+1)-dimensional Boussinesq equation (1). It is also shown that the three-wave method is direct, concise and effective; it can be used to treat many other types of nonlinear evolution equation.

Acknowledgements: The work is supported by NSF2010CD080, and the Science Research Foundation of Yunnan Educational Department under Grant No. 08Y0302, and the Science Research Foundation of Qujing Normal University Grant No. 2008MS018.

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