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APPLICATION OF HAMILTONIAN APPROACH TO AN OSCILLATION OF A MASS ATTACHED TO A STRETCHED ELASTIC WIRE

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Abstract- This paper applies Hamiltonian approach to a nonlinear oscillation of a mass attached to a stretched wire. Comparison of the obtained results with those of the exact solution shows that the approximate solutions are accurate and valid for the whole solution domain.

Keywords- Variational principle; Periodic solution; Amplitude-frequency relationship

1. INTRODUCTION

With the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear oscillations. In particular, a nonlinear oscillation of a mass attached to a stretched wire is of much interest. Various kinds of analytical solutions methods and numerical solutions methods were used to handle the problem [1-5]. Hamiltonian approach is proven to be a very effective and convenient way for handling nonlinear problems [6, 7]. A conservative oscillator admits a Hamiltonian invariant [8], which can be used to determine an approximate solution [6-8].

In this paper we aim to apply Hamiltonian approach to a nonlinear oscillation of a mass attached to a stretched wire and handle the governing non-dimensional equation of motion [1]

$$u'' + u - \frac{\lambda u}{\sqrt{1+u^2}} = 0, 0 < \lambda \le 1, \tag{1}$$

with initial conditions

$$u(0) = A, \quad u'(0) = 0, \tag{2}$$

which is an example of a conservative nonlinear oscillatory system having an irrational elastic item. All the motions corresponding to Eq. (1) are periodic [1], the system will oscillate between symmetric bounds [-A, A], and its angular frequency and corresponding periodic solution are dependent on the amplitude A.

2. HAMILTONIAN APPROACH

We apply this method to the discussed system. Its Hamiltonian can be easily obtained, which reads [6]

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 - \lambda\sqrt{1+u^2}, \tag{3}$$

Integrating Eq.(3) with respect to t from 0 to T/4, we obtain

$$\bar{H}(u)=\int_0^{T/4}[\frac{1}{2}u'^2+\frac{1}{2}u^2-\lambda\sqrt{1+u^2}\,]dt\,, \tag{4}$$

Considering the initial conditions, we assume the solution of Eq.(1) can be expressed as $u = A\cos\omega t$. Substituting it into Eq.(4), we have

$$\begin{aligned}\bar{H}(u)&=\int_0^{T/4}[\frac{1}{2}A^2\omega^2\sin^2\omega t+\frac{1}{2}A^2\cos^2\omega t-\lambda\sqrt{1+A^2\cos^2\omega t}\,]dt\\&=\int_0^{\pi/2}\{\frac{1}{2}A^2\omega\sin^2\theta+\frac{1}{\omega}[\frac{1}{2}A^2\cos^2\theta-\lambda\sqrt{1+A^2\cos^2\theta}]\}d\theta\,,\\&=\frac{\pi}{8}A^2\omega+\frac{1}{\omega}[\frac{\pi}{8}A^2-\lambda\int_0^{\pi/2}\sqrt{1+A^2\cos^2\theta}d\theta]\end{aligned} \tag{5}$$

According to the Hamiltonian approach [6], we set

$$\frac{\partial}{\partial A}(\frac{\partial\bar{H}}{\partial(1/\omega)})=-\frac{\pi}{4}A\omega^2+\frac{\pi}{4}A-\lambda A\int_0^{\pi/2}\frac{\cos^2\theta}{\sqrt{1+A^2\cos^2\theta}}d\theta=0\,, \tag{6}$$

Therefore, we obtain

$$\begin{aligned}\omega^2&=1-\frac{4\lambda}{\pi}\int_0^{\pi/2}\frac{\cos^2\theta}{\sqrt{1+A^2\cos^2\theta}}d\theta\\&=1-\frac{4\lambda}{\pi A^2}\int_0^{\pi/2}\sqrt{1+A^2\cos^2\theta}d\theta+\frac{4\lambda}{\pi A^2}\int_0^{\pi/2}\frac{1}{\sqrt{1+A^2\cos^2\theta}}d\theta\,.\\&=1-\frac{4\lambda}{\pi A^2}[E(-A^2)-K(-A^2)]\end{aligned} \tag{7}$$

where K(m) and E(m) are the complete elliptic integrals of the first and second kind, respectively, defined as follows[9]

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m\cos^2\theta}}d\theta, E(m)=\int_0^{\pi/2}\sqrt{1-m\cos^2\theta}d\theta\,.$$

From Eq.(7), we can easily get the following approximate frequency-amplitude relationship

$$\omega=\sqrt{1-\frac{4\lambda}{\pi A^2}[E(-A^2)-K(-A^2)]}\,. \tag{8}$$

Hence, the approximate period is

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{\sqrt{1-\frac{4\lambda}{\pi A^2}[E(-A^2)-K(-A^2)]}}\,. \tag{9}$$

which is same as that obtained by the parameter-expansion method [2].

Its exact period [1] is

$$T_e = 4\int_0^{\pi/2}[1-2\lambda/(\sqrt{1+A^2\sin^2\theta}+\sqrt{1+A^2})]^{-1/2}d\theta\,. \tag{10}$$

In order to verify the correctness of the obtained periods, we consider some special cases.

For $\lambda = 0.1, 0.5, 0.75$ and 0.95, comparison of the approximate periods T with exact periods T_e is tabulated in Table 1.

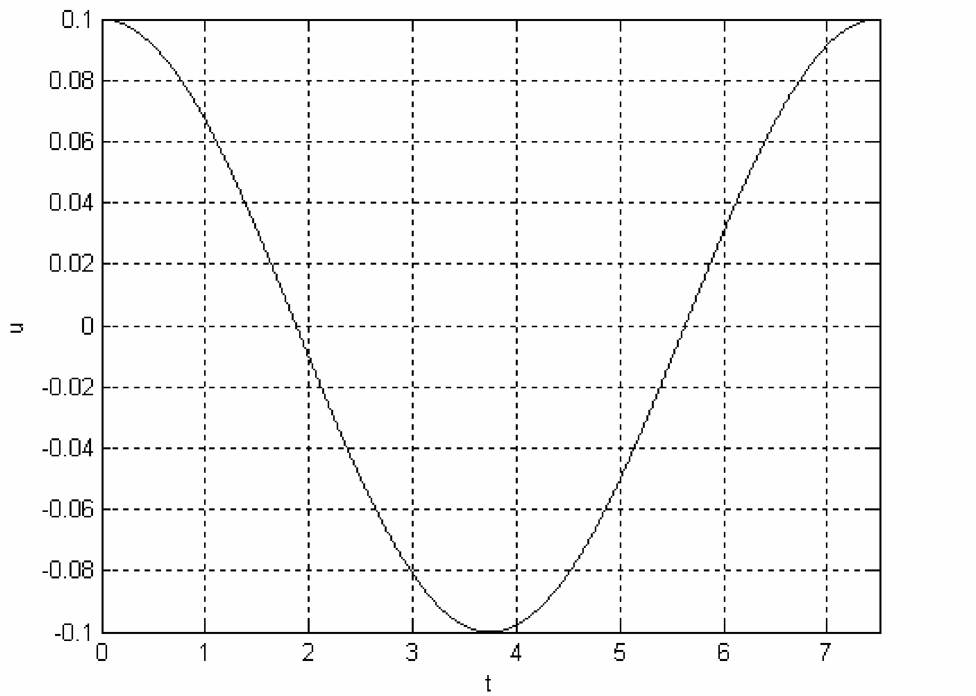
Table 1 Comparison of the approximate period with exact period for $\lambda = 0.1, 0.5, 0.75$ and 0.95

A	$\lambda=0.1$		$\lambda=0.5$		$\lambda=0.75$		$\lambda=0.95$	
	T	T_e	T	T_e	T	T_e	T	T_e
0.4	6.603286	6.603056	8.655950	8.653029	11.668481	11.65250	20.80578	19.78073
0.7	6.576824	6.571430	8.386613	8.316817	10.89667	10.53734	14.65771	14.73436
4	6.385624	6.378382	6.849978	6.853328	7.193035	7.201351	7.503453	7.517983
7	6.341117	6.339461	6.58965	6.590660	6.760005	6.748172	6.905591	6.891635
50	6.291201	6.291201	6.32357	6.323589	6.344056	6.344097	6.360587	6.360653
500	6.283985	6.283986	6.287189	6.287190	6.289193	6.289194	6.290799	6.290801

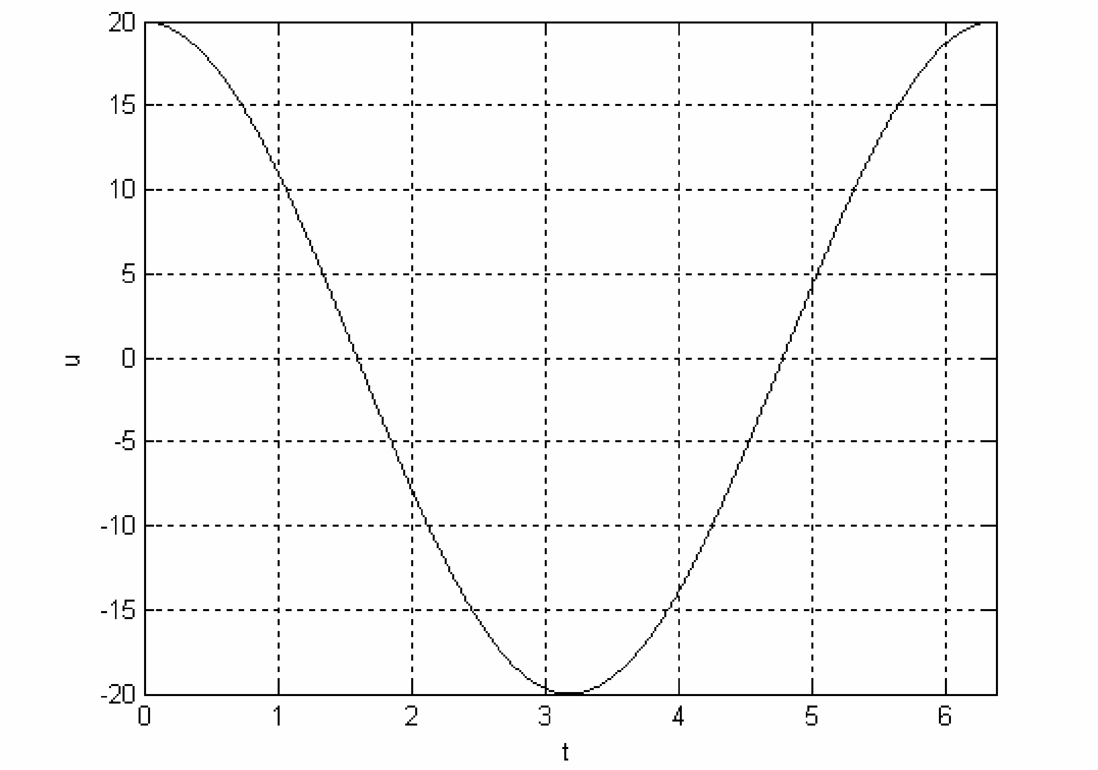
According to Eq.(8), we can obtain the following approximate solution

$$u = A\cos[(1-\frac{4\lambda}{\pi A^2}[E(-A^2)-K(-A^2)])^{1/2}t] \tag{11}$$

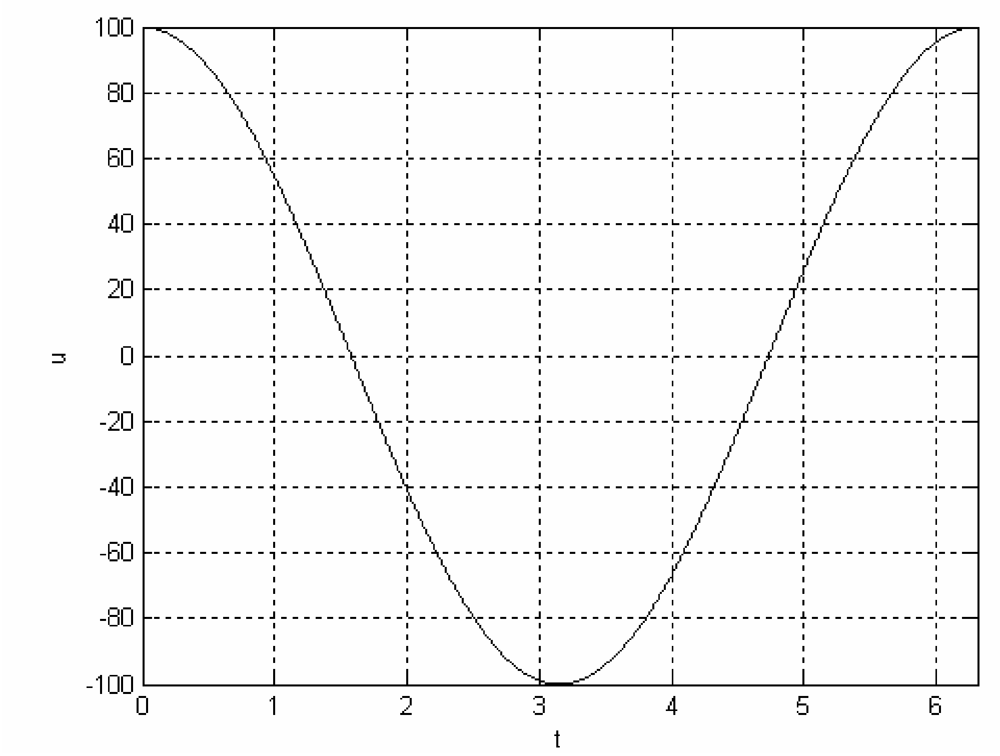
which agrees very well with the exact solution as illustrated in Fig.1.



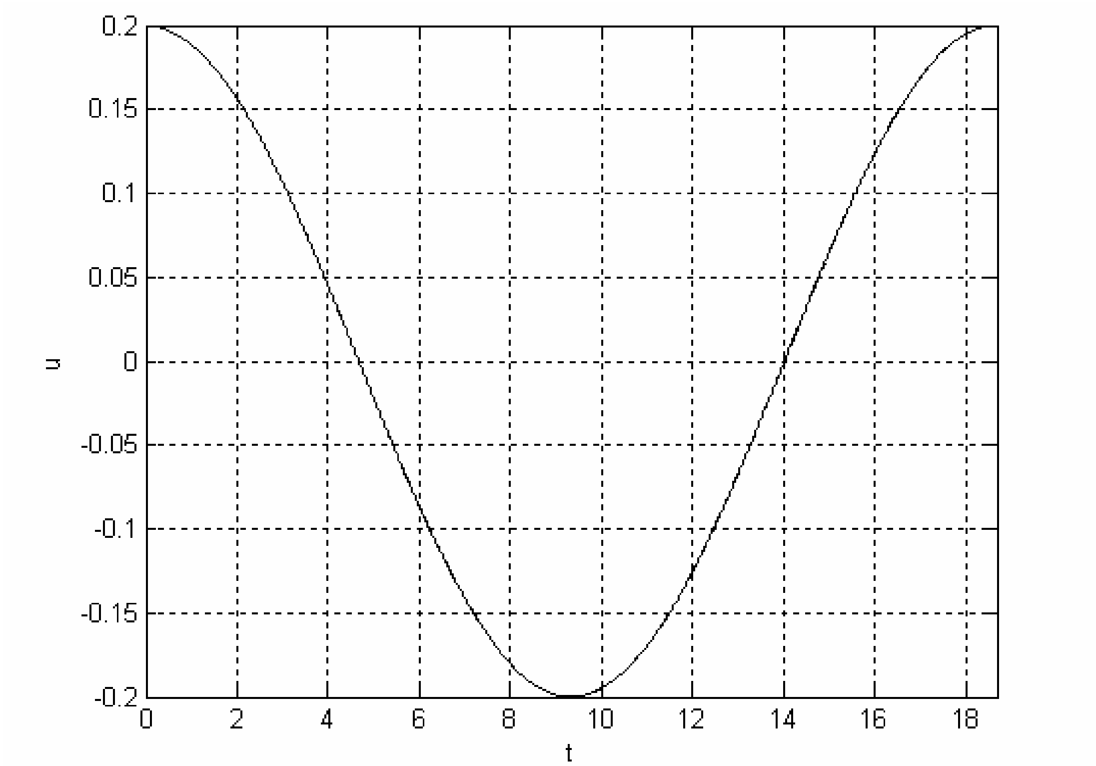
$\lambda = 0.3$, A=0.1



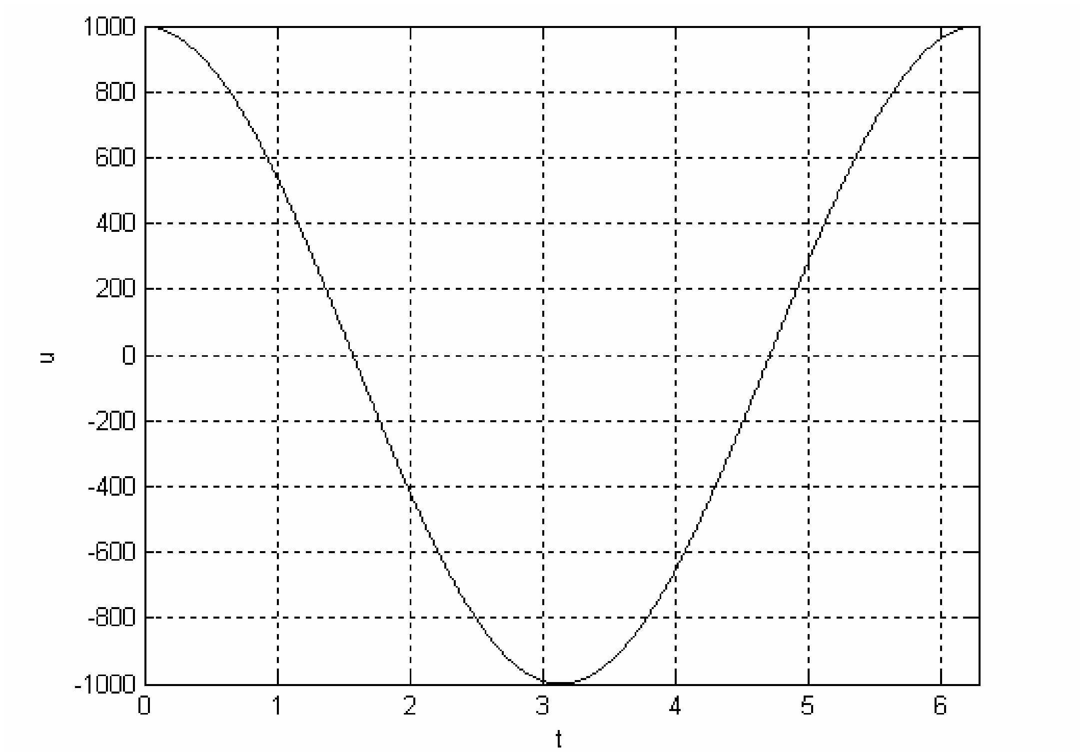
$\lambda = 0.45$, A=20



$\lambda = 0.65$, A=100



$\lambda = 0.9$, A=0.2



$\lambda = 0.9$, A=1000

Fig.1 Comparison of the approximate solution with the exact solution. Dashed line: the approximate solution, solid line: the exact solution.

3. CONCLUSION

In this paper Hamiltonian approach is proved to be a powerful mathematical tool to solving nonlinear oscillators, which can be easily extended to any conservative oscillators. The obtained solutions are in good agreement with exact ones for a wide range of values of oscillation amplitude. The results show that the solution procedure of Hamiltonian approach is of deceptive simplicity and high accuracy.

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