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VARIATIONAL ITERATION METHOD FOR DELAY DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract- Variational iteration method is applied to solve a class of delay differential-algebraic equations. The obtained sequence of iteration is based on the use of Lagrange multipliers. The corresponding convergence results are obtained and successfully confirmed by some numerical examples.

Keywords- Delay Differential-Algebraic Equations, Variational Iteration Method, Convergence

1. INTRODUCTION

The variational iteration method (VIM) was first proposed by He [1, 2], and has been extensively discussed by many authors [3-10]. Applications of this method have been enlarged due to its flexibility, convenience and efficiency. Some authors have applied VIM to delay differential equations [7] and differential-algebraic equations [3], but VIM for delay differential-algebraic equations (DDAEs) has not been considered. In fact, DDAEs are a very important class of mathematical models and often arise from the fields of computer aided design, circuit analysis, mechanical systems, etc. Some results in theoretical analysis and numerical solutions of DDAEs have been given, which include stability of Runge-Kutta methods for neutral delay integro-differential-algebraic equations [11], the classical convergence results of BDF methods and Runge-Kutta methods for index-2 DDAEs [12] and collocation methods for retarded differential-algebraic equations [13]. In this paper, we apply VIM to a class of DDAEs to obtain approximate analytical solutions. The convergence results of the VIM for DDAEs are obtained. Some illustrative examples confirm the theoretical results.

2. MAIN RESULTS

Consider the initial value problem of a DDAE

$$\begin{cases} x'(t) = f(x(t), x(\alpha(t)), y(t), y(\beta(t))), & 0 \le t \le T, \\ 0 = g(x(t), x(\alpha(t)), y(t)), & 0 \le t \le T, \\ x(t) = \varphi(t), \quad -\tau_1 \le t \le 0, \quad y(t) = \psi(t), & -\tau_2 \le t \le 0, \end{cases} \tag{1}$$

where the delay functions $\alpha(t)$ and $\beta(t)$ satisfy $-\tau_1 \le \alpha(t) \le t$, $-\tau_2 \le \beta(t) \le t$, $f : R^{n_1}$

$\times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_1}$, $g: R^{n_1} \times R^{n_1} \times R^{n_2} \to R^{n_2}$ are smooth vector functions on the real Euclidean spaces and have bounded derivatives, the initial value functions $\varphi: [-\tau_1, 0] \to R^{n_1}$ and $\psi: [-\tau_2, 0] \to R^{n_2}$ are continuous, $g_y(x(t), x(\alpha(t)), y(t))$ is invertible and bounded in a neighbourhood of the true solution. We assume that the problem (1) has a smooth solution $x(t)$, $y(t)$. Throughout this paper, $\|\cdot\|$ denotes the standard Euclidean norm, and the matrix norm is subordinate to $\|\cdot\|$.

According to the VIM, we can construct the correction functional as follows

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(s,t)(x_n'(s) - \tilde{f}(x_n(s), x_n(\alpha(s)), y_n(s), y_n(\beta(s))))ds, \tag{2a}$$

$$0 = g(x_{n+1}(t), x_{n+1}(\alpha(t)), y_{n+1}(t)), \tag{2b}$$

where $\lambda(s,t)$ is a general Lagrange multiplier, which can be defined optimally by variational theory, and $\tilde{f}$ denotes the restrictive variation, i.e., $\delta \tilde{f} = 0$. Thus, we have

$$\delta x_{n+1}(t) = \delta x_n(t) + \int_0^t \lambda(s,t)(\delta x_n'(s))ds,$$

and the stationary conditions are obtained as

$$1 + \lambda(s,t)\big|_{s=t} = 0, \qquad \frac{\partial \lambda(s,t)}{\partial s} = 0.$$

Moreover, the general Lagrange multiplier can be readily identified by $\lambda(s,t) = -1$. Therefore, the variational iteration formula can be written as

$$x_{n+1}(t) = x_n(0) + \int_0^t f(x_n(s), x_n(\alpha(s)), y_n(s), y_n(\beta(s)))ds, \tag{3a}$$

$$0 = g(x_{n+1}(t), x_{n+1}(\alpha(t)), y_{n+1}(t)). \tag{3b}$$

Theorem 1 *Let* $x(t)$, $x_i(t) \in (C^1[-\tau_1, T])^{n_1}$, $y(t)$, $y_i(t) \in (C^1[-\tau_2, T])^{n_2}$, $i = 0,1,\cdots$. *Then the sequences* $\{x_n(t)\}_{n=1}^{\infty}, \{y_n(t)\}_{n=1}^{\infty}$ *defined by* (3) *with* $x_0(t) = \varphi(t)$, $-\tau_1 \le t \le 0$, $y_0(t) = \psi(t)$, $-\tau_2 \le t \le 0$ *converge to the solution of* (1).

Proof. From the system (1), we obviously have

$$x(t) = x(0) + \int_0^t f(x(s), x(\alpha(s)), y(s), y(\beta(s)))ds, \tag{4a}$$

$$0 = g(x(t), x(\alpha(t)), y(t)). \tag{4b}$$

Introduce $E_i x(t) = x_i(t) - x(t)$, $E_i y(t) = y_i(t) - y(t)$, $i = 0,1,\cdots$, where $E_i x(t) = E_i y(t) = 0$, $t < 0$, $i = 0,1,\cdots$. From (3)-(4), we obtain

$$E_{n+1}x(t) = \int_0^t (f(x_n(s), x_n(\alpha(s)), y_n(s), y_n(\beta(s))) - f(x(s), x(\alpha(s)), y(s), y(\beta(s))))ds,$$

$$0 = g(x_{n+1}(t), x_{n+1}(\alpha(t)), y_{n+1}(t)) - g(x(t), x(\alpha(t)), y(t)).$$

Based on the fact that the functions f, g are smooth and the matrix g_y is invertible, we deduce

$$E_{n+1}x(t) = \int_0^t (f_1' E_n x(s) + f_2' E_n x(\alpha(s)) + f_3' E_n y(s) + f_4' E_n y(\beta(s)))ds, \tag{5a}$$

$$E_{n+1}y(t) = -(g_3')^{-1} g_1' E_{n+1}x(t) - (g_3')^{-1} g_2' E_{n+1}x(\alpha(t)), \tag{5b}$$

where $f_i'(i = 1,2,3,4)$ denotes the partial derivative of the function f to *ith* variable, $g_i'(i = 1,2,3)$ denotes the partial derivative of the function g to *ith* variable. We can derive

$$\begin{pmatrix} \|E_{n+1}x(t)\| \\ \|E_{n+1}y(t)\| \end{pmatrix} \le \begin{pmatrix} l_1 & l_3 \\ 2kl_1 & 2kl_3 \end{pmatrix} \begin{pmatrix} \int_0^t \|E_n x(s)\| ds \\ \int_0^t \|E_n y(s)\| ds \end{pmatrix} + \begin{pmatrix} l_2 & l_4 \\ 2kl_2 & 2kl_4 \end{pmatrix} \begin{pmatrix} \int_0^t \|E_n x(\alpha(s))\| ds \\ \int_0^t \|E_n y(\beta(s))\| ds \end{pmatrix},$$

where $l_i = \max\left\| f_i' \right\|, (i = 1,2,3,4),$ $k = \max(\|(g_3')^{-1} g_1'\|, \|(g_3')^{-1} g_2'\|)$. Therefore

$$\begin{pmatrix} \|E_1 x(t)\| \\ \|E_1 y(t)\| \end{pmatrix} \le \begin{pmatrix} l_1 & l_3 \\ 2kl_1 & 2kl_3 \end{pmatrix} \begin{pmatrix} \int_0^t \|E_0 x(s)\| ds \\ \int_0^t \|E_0 y(s)\| ds \end{pmatrix} + \begin{pmatrix} l_2 & l_4 \\ 2kl_2 & 2kl_4 \end{pmatrix} \begin{pmatrix} \int_0^t \|E_0 x(\alpha(s))\| ds \\ \int_0^t \|E_0 y(\beta(s))\| ds \end{pmatrix},$$

$$\le \begin{pmatrix} l_1 + l_2 & l_3 + l_4 \\ 2k(l_1 + l_2) & 2k(l_3 + l_4) \end{pmatrix} \begin{pmatrix} \max\limits_{-\tau_1 \le s \le T} \|E_0 x(s)\| t \\ \max\limits_{-\tau_2 \le s \le T} \|E_0 y(s)\| t \end{pmatrix}. \tag{6}$$

Moreover, we have

$$\begin{pmatrix} \|E_n x(t)\| \\ \|E_n y(t)\| \end{pmatrix} \le \frac{(\tau + T)^n \rho^n}{n!} \begin{pmatrix} \max\limits_{-\tau_1 \le s \le T} \|E_0 x(s)\| \\ \max\limits_{-\tau_2 \le s \le T} \|E_0 y(s)\| \end{pmatrix}. \tag{7}$$

where $\tau,\ T,\ \max\limits_{-\tau_1 \le s \le T} \|E_0 x(s)\|,\ \max\limits_{-\tau_2 \le s \le T} \|E_0 y(s)\|,\ k,\ l_i,\ (i = 1,2,3,4)$ are constants, ρ is the spectral radius of the last matrix in the above inequality (6). By using the Stirling's formula, we have

$$\begin{pmatrix} \|E_n x(t)\| \\ \|E_n y(t)\| \end{pmatrix} \le \frac{(\frac{(\tau+T)\rho e}{n})^n}{\sqrt{2n\pi}(1 + O(1/n))} \begin{pmatrix} \max\limits_{-\tau_1 \le s \le T} \|E_0 x(s)\| \\ \max\limits_{-\tau_2 \le s \le T} \|E_0 y(s)\| \end{pmatrix}, \tag{8}$$

thus, $(\|E_n x(t)\|, \|E_n y(t)\|)^T \to 0$ as $n \to \infty$.

3. ILLUSTRATIVE EXAMPLES

In this section, some illustrative examples are given to show the efficiency of the VIM for DDAEs.

Example 1 Consider the following initial value problem

$$\begin{cases} y'(t) = y^2(t) - 2y^2(\tfrac{t}{2}) + x^2(\tfrac{t}{2}), & t \geq 0, \\ 1 = 2y^2(\tfrac{t}{2}) + x(t), & t \geq 0, \\ x(0) = 1, \quad y(0) = 0. & \end{cases} \tag{9}$$

We apply the VIM to (9), and construct the correction functional

$$y_{n+1}(t) = y_n(0) + \int_0^t (y_n^2(s) - 2y_n^2(\tfrac{s}{2}) + x_n^2(\tfrac{s}{2}))ds, \tag{10a}$$

$$1 = 2y_{n+1}^2(\tfrac{t}{2}) + x_{n+1}(t). \tag{10b}$$

Moreover, the iteration sequence starts with the initial approximations $y_0(t) = t$, $x_0(t) = x(0) = 1$, and is obtained from (10) as follows

$$\begin{aligned}
&y_1(t) = t + \tfrac{1}{6}t^3, \\
&x_1(t) = 1 - 2t^2 - \tfrac{2}{3}t^4 - \tfrac{1}{18}t^6, \\
&y_2(t) = t - \tfrac{1}{6}t^3 + \tfrac{11}{120}t^5 + \tfrac{11}{1152}t^7 + o(t^7), \\
&x_2(t) = 1 - 2t^2 + \tfrac{2}{3}t^4 - \tfrac{19}{45}t^6 + \tfrac{11}{480}t^8 + o(t^8), \\
&y_3(t) = t - \tfrac{1}{6}t^3 + \tfrac{1}{120}t^5 + \tfrac{431}{20160}t^7 - \tfrac{13}{40960}t^9 + o(t^9), \\
&x_3(t) = 1 - 2t^2 + \tfrac{2}{3}t^4 - \tfrac{6}{45}t^6 - \tfrac{403}{5040}t^8 + \tfrac{148861}{9676800}t^{10} + o(t^{10}), \\
&\qquad \cdots
\end{aligned}$$

From the above iteration sequence, we can show that

$$\lim_{n\to\infty} y_n(t) = \sin(t), \qquad \lim_{n\to\infty} x_n(t) = \cos(2t). \tag{11}$$

Example 2 Consider the following initial value problem

$$\begin{cases} x'(t) = -2y(\tfrac{t}{2}) + 2z(t), & t \geq 0, \\ y'(t) = -2x(t)z^2(\tfrac{t}{2}) + z^2(t), & t \geq 0, \\ 0 = y(t) - x(t)z(t), & t \geq 0, \\ x(0) = 1, \quad y(0) = 0, \quad z(0) = 1. & \end{cases} \tag{12}$$

The exact solution of the system (12) is

$$\begin{aligned}
x(t) &= (1+t)e^{-t} = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{8}t^4 + \tfrac{1}{30}t^5 + o(t^5), \\
y(t) &= (1+t)e^{-2t} = 1 - t + \tfrac{2}{3}t^3 - \tfrac{2}{3}t^4 + \tfrac{2}{5}t^5 + o(t^5), \\
z(t) &= e^{-t} = 1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 - \tfrac{1}{120}t^5 + o(t^5).
\end{aligned}$$

We apply the VIM to (12), and construct the correction functional

$$x_{n+1}(t) = x_n(0) + \int_0^t (-2y_n(\tfrac{s}{2}) + 2z_n(s))ds, \tag{13a}$$

$$y_{n+1}(t) = y_n(0) + \int_0^t (-2x_n(s)z_n^2(\tfrac{s}{2}) + z_n^2(s))ds, \tag{13b}$$

$$0 = y_{n+1}(t) - x_{n+1}(t)z_{n+1}(t). \tag{13c}$$

Moreover, the iteration sequence starts with the initial approximations $x_0(t) = x(0) = 1,$ $y_0(t) = y(0) = 1,$ $z_0(t) = z(0) = 1,$ and is obtained from (13) as follows

$$\begin{aligned}
&x_1(t) = 1,\\
&y_1(t) = 1 - t,\\
&z_1(t) = 1 - t,\\
&x_2(t) = 1 - \tfrac{1}{2}t^2,\\
&y_2(t) = 1 - t + \tfrac{1}{6}t^3,\\
&z_2(t) = 1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{3}t^3 + \tfrac{1}{4}t^4 + o(t^4),\\
&x_3(t) = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \tfrac{1}{96}t^4,\\
&y_3(t) = 1 - t + \tfrac{2}{3}t^3 - \tfrac{7}{16}t^4 + \tfrac{23}{160}t^5 + o(t^5),\\
&z_3(t) = 1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{5}{32}t^4 - \tfrac{7}{60}t^5 + o(t^5),\\
&x_4(t) = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{8}t^4 + \tfrac{7}{640}t^5 - \tfrac{23}{15360}t^6 + o(t^6),\\
&y_4(t) = 1 - t + \tfrac{2}{3}t^3 - \tfrac{2}{3}t^4 + \tfrac{163}{480}t^5 + \tfrac{65}{576}t^6 + o(t^6),\\
&z_4(t) = 1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 - \tfrac{89}{1920}t^5 + \tfrac{197}{5120}t^6 + o(t^6),\\
&\qquad\cdots
\end{aligned}$$

The above iterate sequence shows that the VIM yields a very good approximation to the exact solution.

4. CONCLUSIONS

In this paper, we successfully apply VIM to a class of DDAEs and obtain highly accurate solutions with few iterations. VIM handles DDAEs without any especial assumption on the delay item, thus, it is a promising method for DDAEs.

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