Mathematical and Computational Applications, Vol. 15, No. 5, pp. 776-783, 2010.

THE MULTI-WAVE METHOD FOR NONLINEAR EVOLUTION EQUATIONS

Yeqiong Shi[1], Zhengde Dai[1,2], Song Han[1], Liwei Huang[1]

1. Department of Information and Computing Science, Guangxi University of Technology, 545006
Liuzhou, Guangxi, P.R. China, Shiyeqiong89@163.com
2. School of Mathematics and Statistics, Yunnan University, 650091
Kunming, P.R. China, zhddai2004@yahoo.com.cn

Abstract- The multi-wave method is proposed to find new exact solitary solutions of nonlinear evolution equations. The Caudrey-Dodd-Gibbon-Kaeada equation is employed as an example to illustrate the effectiveness of the suggested method and some new wave solutions with four different velocities and frequencies are obtained. Obviously, the method can be applied to solve other types of nonlinear evolution equations as well.

Keywords- The Multi-Wave Method, The Caudrey-Dodd-Gibbon-Kaeada Equation, Periodic Soliton Wave Solution, M-shape Solitary Solution

1. INTRODUCTION

Many phenomena in physics and in the nonlinear science can be modeled by a class of integrable nonlinear evolution equations. Consequently, construction of traveling wave solutions of nonlinear equations plays an important role in the study of nonlinear phenomena. Nowadays, with the rapid development of software technology, solving nonlinear evolution equations via symbolic computation is taking an increasing role due to its efficiency, accuracy and its easy use. Over the last few decades, directly searching for exact solutions of nonlinear partial differential equations，such as G'/G-expansion method, Exp-function method, Hirota bilinear forms, homoclinic test approach, extended three-soliton method [1-12], and so on. Recently, Dai and Wang has proposed a novel approach to study the result when three waves of different frequencies and different propagation velocity meet interaction, namely, extended three-soliton method[8].

In this work, the multi-wave method is proposed to find new exact solitary solution for nonlinear evolution equation. The Caudrey-Dodd-Gibbon-Kaeada equation is employed as an example to illustrate the effectiveness of the suggested method and some new wave solutions with four different velocities and frequencies are obtained, include periodic solitary wave solutions, bright soliton wave solutions, M-type wave solutions, etc.

2. THE BILINEAR FORM OF THE CAUDREY-DODD-GIBBON-KAEADA EQUATION

Let’s consider the Caudrey-Dodd-Gibbon-Kaeada(CDGK) equation

$$u_t + (60u^3 + 30uu_{xx} + u_{4x})_x = 0, \tag{1.1}$$

where $u = u(x,t)$ is an unknown real function. u_{4x} represents the fourth-order partial derivatives $\frac{\partial^4 u}{\partial x^4}$, u_{5x} represents the fifth-order partial derivatives $\frac{\partial^5 u}{\partial x^5}$, and so on.

Let's suppose

$$u(x,t) = \frac{\partial(w(x,t))}{\partial x}.$$

Substituting it into Eq. (1.1), we can get

$$w_{xt} + 180w_x^2 w_{xx} + 30w_{xx}w_{xxx} + 30w_x w_{xxxx} + w_{6x} = 0\,.$$

Integrate x one time, and let the integral constant is zero, we have

$$w_t + 60w_x^3 + 30w_x w_{xxx} + w_{5x} = 0. \tag{1.2}$$

Let

$$w(x,t) = \frac{G(x,t)}{F(x,t)}. \tag{1.3}$$

Substitution Eq. (1.3) into Eq. (1.2), the expansion is obtained as follows

$$\begin{aligned} &\frac{1}{F^2}[D_t(G\cdot F) + D_x^5(G\cdot F)] + \frac{1}{F^4}\{5D_x(G\cdot F)[2D_x^3(G\cdot f) - D_x^4(F\cdot F)] \\ &+10D_x^3(G\cdot F)[2D_x(G\cdot F) - D_x^2(F\cdot F)]\} \\ &+\frac{30}{F^6}D_x(G\cdot F)\{[D_x^2(F\cdot F) - D_x(G\cdot F)]\cdot[D_x^2(F\cdot F) \\ &-2D_x(G\cdot F)]\} = 0. \end{aligned} \tag{1.4}$$

From Eq. (1.4), we can get following Equations:

$$D_t(G\cdot F) + D_x^5(G\cdot F) = 0, \tag{1.5}$$

$$2D_x^3(G\cdot F) - D_x^4(F\cdot F) = 0, \tag{1.6}$$

$$2D_x(G\cdot F) - D_x^2(F\cdot F) = 0. \tag{1.7}$$

From Eq. (1.7), we can have

$$G = F_x. \tag{1.8}$$

From Eqs. (1.5)- (1.7), we can reduce Eq. (1.1) into the bilinear form as follows

$$(D_t D_x + D_x^6)(F\cdot F) = 0. \tag{1.9}$$

So, the solution of the Eq.(1.1) can be expressed as the following form

$$u = w_x = \left(\frac{G}{F}\right)_x = \left(\frac{F_x}{F}\right)_x = (\ln F)_{xx}. \tag{1.10}$$

3. THE APPLICATION OF THE FOUR-WAVE SOLUTIONS METHOD FOR SOLVING CDGK EQUATION

In this case we propose a novel test function of extended four-soliton method

$$F(x,t) = a_1 \cos\xi_1 + a_2 \sin\xi_2 + a_3 \cosh\xi_3 + \exp(-\xi_4) + a_4 \exp(\xi_4), \tag{2.1}$$

where $\xi_i = p_i x + q_i t,\quad i = 1,2,3,4$. Substituting (2.1) into (1.10), we can obtain the form of general solution of equation CDGK:

$$u = (\ln F)_{xx} =$$

$$\frac{-a_1 \sin(p_1 x + q_1 t) p_1^2 - a_2 \cos(p_2 x + q_2 t) p_2^2 + a_3 \cosh(p_3 x + q_3 t) p_3^2 + p_4^2 e^{-p_4 x - q_4 t} + a_4 p_4^2 e^{p_4 x + q_4 t}}{a_1 \sin(p_1 x + q_1 t) + a_2 \cos(p_2 x + q_2 t) + a_3 \cosh(p_3 x + q_3 t) + e^{-p_4 x - q_4 t} + a_4 e^{p_4 x + q_4 t}}$$

$$-\frac{\left(a_1 \cos(p_1 x + q_1 t) p_1 - a_2 \sin(p_2 x + q_2 t) p_2 + a_3 \sinh(p_3 x + q_3 t) p_3 - p_4 e^{-p_4 x - q_4 t} + a_4 p_4 e^{p_4 x + q_4 t}\right)^2}{\left(a_1 \sin(p_1 x + q_1 t) + a_2 \cos(p_2 x + q_2 t) + a_3 \cosh(p_3 x + q_3 t) + e^{-p_4 x - q_4 t} + a_4 e^{p_4 x + q_4 t}\right)^2} \tag{2.2}$$

Substituting Eq. (2.2) into Eq. (1.9), and equating the coefficients of all powers of $\sin\xi_1 \exp(\pm\xi_4)$, $\cos\xi_1 \exp(\pm\xi_4)$, $\sin\xi_2 \exp(\pm\xi_4)$, $\cos\xi_2 \exp(\pm\xi_4)$, $\sinh\xi_3 \exp(\pm\xi_4)$, $\cosh\xi_3 \exp(\pm\xi_4)$, $\sin\xi_1, \cos\xi_2, \cos\xi_1 \sin\xi_2, \sin\xi_1 \cosh\xi_3, \cos\xi_1 \sinh\xi_3$, $\sin\xi_2 \sinh\xi_3$, $\cos\xi_2 \cosh\xi_3$, we can obtain a set of algebraic equations for $a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$. Solving these algebraic equations with the aid of Maple, we obtain eight sets of solutions as follows:

Case 1

$$a_1 = 0, a_2 = 0, q_3 = -5p_4^4 p_3 - 10p_3^3 p_4^2 - p_3^4, q_4 = -p_4(10p_4^2 p_3^2 + p_4^4 + 5p_3^4), \tag{2.3}$$

where p_3, p_4, a_3, a_4 are free parameters. Substituting (2.3) into Eq. (2.2) yields the following bright solitary wave solutions of Eq. (1.1)

$$u = \frac{a_3 p_3^2 \cosh\xi_3 + p_4^2 e^{-\xi_4} + a_4 p_4^2 e^{\xi_4}}{a_3 \cosh\xi_3 + e^{-\xi_4} + a_4 e^{\xi_4}} - \frac{(a_3 p_3 \sinh\xi_3 - p_4 e^{-\xi_4} + a_4 p_4 e^{\xi_4})^2}{(a_3 \cosh\xi_3 + e^{-\xi_4} + a_4 e^{\xi_4})^2}, \tag{2.4}$$

where $\xi_i = p_i x + q_i t,\ i = 2,4$. Rewriting Eq. (2.4) as follows

$$u = \frac{-a_3^2 p_3^2 \operatorname{sech}(\xi_4 + \frac{1}{2}\ln|a_4|) + \left[a_3\sqrt{|a_4|}(p_4^2 - p_3^2)\cosh\xi_3 - 2a_3 p_3 p_4 \sqrt{|a_4|}\sinh\xi_3\right]}{\operatorname{sech}(\xi_4 + \frac{1}{2}\ln|a_4|)\left[a_3\cosh\xi_3 + \sqrt{|a_4|}\cosh(\xi_4 + \frac{1}{2}\ln|a_4|)\right]^2} \tag{2.5}$$

The shape of u_1 at $a_3 = 2, a_4 = 5, p_3 = -1, p_4 = 1$ is shown in Fig. 1, we can see clearly it is a bright soliton.

Case 2

$$a_2 = a_3 = 0\,, \quad q_1 = -5p_4^4 p_1 + 10p_1^3 p_4^2 - p_1^5, q_4 = -p_4(-10p_1^2 p_4^2 + 5p_1^4 + p_4^4) \tag{2.6}$$

Substituting (2.6) into Eq. (2.2) yields solution of Eq.(1.1) as follows

$$u = \frac{-a_1^2 p_2^2 \operatorname{sech}(\xi_4 + \frac{1}{2}\ln|a_4|) + \left[a_1\sqrt{|a_4|}(p_4^2 - p_1^2)\sin\xi_1 - 2a_1 p_1 p_4\sqrt{|a_4|}\cos\xi_1\right]}{\operatorname{sech}(\xi_4 + \frac{1}{2}\ln|a_4|)\left[a_1\sin\xi_1 + \sqrt{|a_4|}\cosh(\xi_4 + \frac{1}{2}\ln|a_4|)\right]^2} \tag{2.7}$$

where $\xi_i = p_i x + q_i t,\quad i = 1,4$, p_1, p_4 are free parameters. Taking $a_1 = 2, a_4 = 5,\ p_2 = 2$, $p_4 = 0.01$, we can see Eq. (2.7) is a periodic solitary wave solution, see Fig. 2.

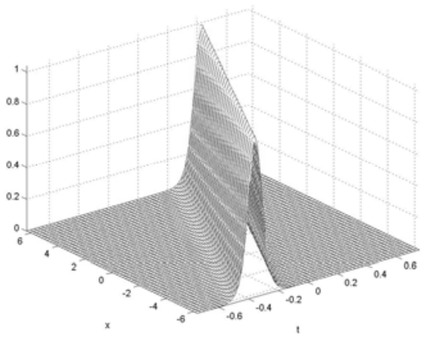

Fig.1 Bright soliton

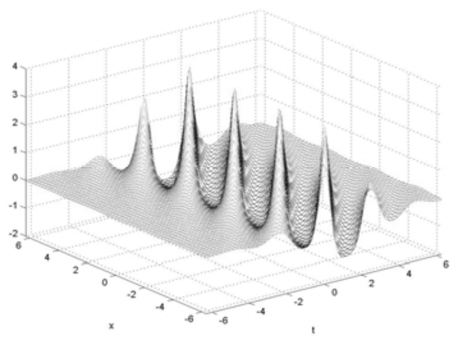

Fig.2 Periodic solitary wave

Case 3

$$a_2 = 0, p_3 = 0, p_4 = 0, q_3 = 0, q_4 = 0, q_1 = -p_1^5 \tag{2.8}$$

Substituting (2.9) into Eq. (2.2), the period wave solution can be obtained as follows (see Fig. 3)

$$u = \frac{(a_3 + 1 + a_4)a_1 p_1^2 \sin\xi_1 - a_1^2 p_1^2}{(a_1\sin\xi_1 + a_3 + 1 + a_4)^2} \tag{2.9}$$

where $\xi_1 = p_1 x + q_1 t$, a_1, a_3, a_4, p_1 are free parameters.

Case 4

$$a_2 = 0,\, p_3 = p_4 = \sqrt{1/3}\,p_1,\, q_1 = \frac{16}{9}p_1^5,\, q_3 = q_4 = -\frac{16}{9}\sqrt{1/3}\,p_1^5 \tag{2.10}$$

Substituting (2.10) into Eq. (2.2) yields periodic solitary wave solutions of Eq. (1.1) (see Fig.4)

$$u = \frac{-a_1p_1^2\sin\xi_1 + \frac{1}{3}a_3p_1^2\cosh\xi_3 + \frac{1}{3}p_1^2\sqrt{|a_4|}\cosh(\xi_4 + \ln(\sqrt{|a_4|}))}{a_1\sin\xi_1 + a_3\cosh\xi_3 + \sqrt{|a_4|}\cosh(\xi_4 + \ln(\sqrt{|a_4|}))} - \frac{(a_1p_1\cos\xi_1 + \frac{\sqrt{3}}{3}a_3p_1\sinh\xi_3 + \frac{\sqrt{3}}{3}p_1\sqrt{|a_4|}\cosh(\xi_4 + \ln(\sqrt{|a_4|})))^2}{(a_1\sin\xi_1 + a_3\cosh\xi_3 + \sqrt{|a_4|}\cosh(\xi_4 + \ln(\sqrt{|a_4|})))^2} \tag{2.11}$$

where $\xi_i = p_ix + q_it,\ i = 1,3,4$, a_1, a_3, a_4, p_1 are arbitrary constants.

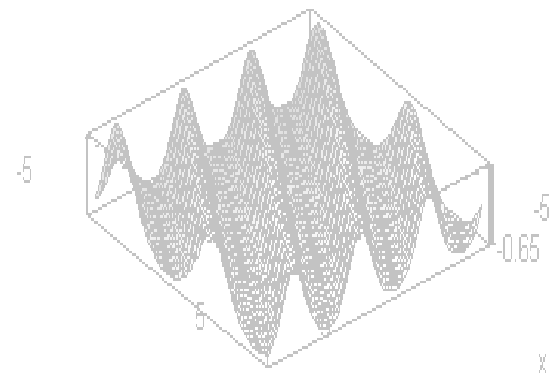

Fig.3 Triangular periodic wave

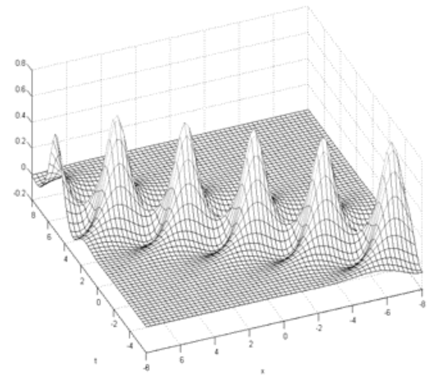

Fig. 4 periodic solitary wave

Case 5

$$a_1 = 0,\, p_2 = 0,\, p_3 = 0,\, q_2 = 0,\, q_3 = 0,\, q_4 = -p_4^5 \tag{2.12}$$

Inserting (2.12) into Eq. (2.2) yields M-type wave solutions of Eq. (1.1)

$$u = \frac{(a_2 + a_3)\sqrt{|a_4|}\,p_4^2\cosh(\xi_4 + \ln\sqrt{|a_4|}) + p_4^2a_4\cosh(2\xi_4 + \ln\sqrt{|a_4|})}{\left[(a_2 + a_3) + \sqrt{|a_4|}\cosh(\xi_4 + \ln\sqrt{|a_4|})\right]^2} \tag{2.13}$$

where $\xi_4 = p_4x + q_4t$, a_2, a_3, a_4, p_4 are arbitrary constants.

Case 6

$$a_2 = 0,\, p_1 = 0,\, p_3 = 0,\, q_1 = 0,\, q_3 = 0,\, q_4 = -p_4^5. \tag{2.14}$$

Inserting (2.14) into Eq. (2.2) yields bright soliton wave solution of Eq. (1.1) (see Fig. 5)

$$u=\frac{p_4^2 a_3\sqrt{a_4}\,\sec h(\xi_4+\frac{1}{2}\ln|a_4|)}{\left[a3\sec h(\xi_4+\frac{1}{2}\ln|a_4|)+\sqrt{a_4}\right]^2} \tag{2.15}$$

where $\xi_4 = p_4x+q_4t$, a_3,a_4,p_4 are arbitrary constants.

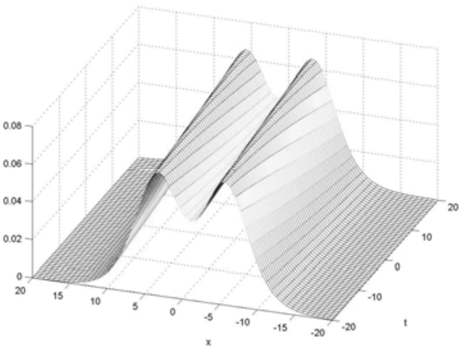

Fig. 5 M-type wave

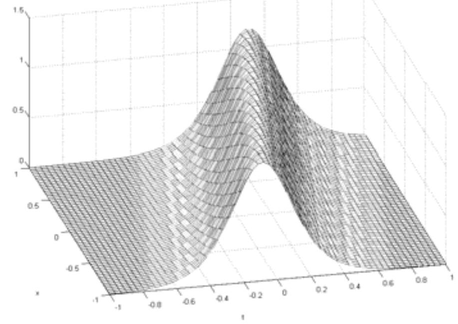

Fig. 6 bright soliton

Case 7

$$a_1=0, a_3=0, q_2=-5p_4^4p_2+10p_2^3p_4^2-p_2^5, q_4=-p_4(-10p_2^2p_4^2+5p_2^4+p_4^4) \tag{2.16}$$

Substituting (2.16) into Eq. (2.2) , we can obtain the periodic solitary wave solutions of Eq. (1.1) (see Fig.7)

$$u=\frac{-a_2^2p_2^2\,\mathrm{sech}(\xi_4+\frac{1}{2}\ln|a_4|)+\left[a_2\sqrt{|a_4|}(p_4^2-p_2^2)\cos\xi_2-2a_2p_2p_4\sqrt{|a_4|}\sin\xi_2\right]}{\mathrm{sech}(\xi_4+\frac{1}{2}\ln|a_4|)\left[a_2\cos\xi_2+\sqrt{|a_4|}\cosh(\xi_4+\frac{1}{2}\ln|a_4|)\right]^2} \tag{2.17}$$

where $\xi_i = p_ix+q_it,\ i=2,4$, a_2,a_4,p_2,p_4 are arbitrary constants.

Case 8

$$a_1=0, p_3=p_4=\sqrt{\frac{1}{3}}p_2, q_2=\frac{16}{9}p_2^5, q_3=q_4=-\frac{16}{9}\sqrt{\frac{1}{3}}p_2^5. \tag{2.18}$$

Inserting (2.18) into Eq. (2.2), the bright soliton solution of Eq. (1.1) is obtained as follows,

$$u = \frac{-a_2 p_2^2 \cos\xi_2 + \frac{1}{3} a_3 p_2^2 \cosh\xi_3 + \frac{1}{3} p_2^2 \sqrt{|a_4|} \cosh(\xi_4 + \frac{1}{2}\ln|a_4|))}{a_2 \cos\xi_2 + a_3 \cosh\xi_3 + \sqrt{|a_4|} \cosh(\xi_4 + \frac{1}{2}\ln|a_4|))} - \frac{\left[-a_2 p_2 \sin\xi_2 + \frac{\sqrt{3}}{3} a_3 p_2 \sinh\xi_3 + \frac{\sqrt{3}}{3} p_2 \sqrt{|a_4|} \cosh(\xi_4 + \frac{1}{2}\ln|a_4|)\right]^2}{\left[a_2 \cos\xi_2 + a_3 \cosh\xi_3 + \sqrt{|a_4|} \cosh(\xi_4 + \frac{1}{2}\ln|a_4|)\right]^2} \tag{2.19}$$

where $\xi_i = p_i x + q_i t,\ i = 2,3,4,$ a_2, a_3, a_4, p_2 are arbitrary constants. If take $a_2 = 1, a_3 = 15,\ a_4 = 12,\ p_2 = 1$, the graphics of Eq. (2.19) is drawn with the help of Matlab (see Fig. 8).

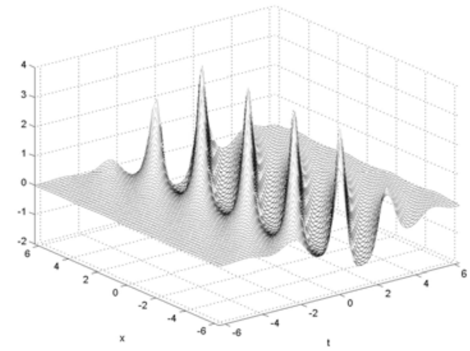

Fig. 7 Periodic solitary wave

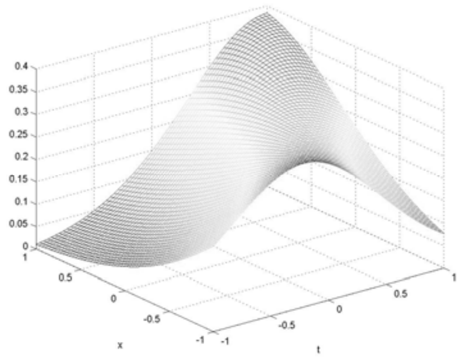

Fig.8 Bright soliton

4. CONCLUDING

In this work, a new test function is proposed to find new exact solitary solutions for nonlinear evolution equation, Caudrey-Dodd-Gibbon-Kaeada equation is employed as an example to illustrate the effectiveness of the suggested method, with the help of the mathematics software Maple and Matlab , some new solutions of CDGK equation are obtained , such as M-type wave solution, periodic solitary wave solution, triangular periodic wave solution, etc., which can be obtained by the exp-function method as well[13-15]. The results show that it is entirely possible for integrable equations or non-integrable equations to have periodic solitary waves, and their propagation is phase shifts of solitons.

5. REFERENCES

1. M. L. Wang, J. L. Zhang, X. Z. Li, The((G′)/(G))-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A* **372**(4), 417-423, 2008.

2. J. H. He, X.H. Wu, Exp-function method for nonlinear wave equations, *Chaos Solitons Fract,* **30**(3), 700-708, 2006.
3. S. Zhang, Exp-function method: solitary, periodic and rational wave solutions of nonlinear evolution equations, *Nonlinear Science Letters A* **1**, 143-146, 2010.
4. Y. Q. Shi, Z. D. Dai, D. L. Li, Application of Exp-function method for 2D cubic-quintic Ginzburg–Landau equation, *Appl. Math. Comput.* **210**, 269-275, 2009.
5. Z. D. Dai, J. Huang, M. R. Jiang, S. H. Wang, Homoclinic orbits and periodic solitons
for Boussinesq equation with even constraint, *Chaos, Solitons and Fractals* **26**, 1189-1194, 2005.
6. Z. D. Dai, J. Liu, D. L. Li, Applications of HTA and EHTA to the YTSF equation, *Appl. Math. Comput.* **207**, 360-364, 2009.
7. H. M. Fu, Z. D. Dai, Double Exp-function method and application, *Int. Journ. Non. Sci. Num. Simul.* **10**, 927-933, 2009.
8. Z. D. Dai, C. J. Wang, S. Q. Lin, D. L. Li, G. Mu, The three-wave method for nonlinear evolution equations , *Nonlinear Science Letters A* **1**, 77-82, 2010.
9. R. Hirota, Exact solutions of the Korteweg-de-Vries equation for multiple collisions of solitons, *Phys. Lett. A* **27**, 1192-1194, 1971.
10. M. R. Miurs, *Backlund Transformation*, Springer, Berlin, 1978.
11. K. Sawada, T. Kotera, A method for finding N-soliton solutions of the KdV equation and KdV-like equations, *Prog. Theor. Phys*. **51**, 1355-1362, 1974.
12. R. Hirota, *The Direct Methods in Soliton Theory*, Cambridge University Press, 2004.
13. H. Hosseini, M.M. Kabir , A. Khajeh, New Explicit Solutions for the Vakhnenko and a Generalized Form of the Nonlinear Heat Conduction Equations via Exp-Function Method, *Int. J. Nonlin. Sci. Num.,* **11**, 285-296 , 2010
14. M.M. Kabir, A. Khajeh. New Explicit Solutions for the Vakhnenko and a Generalized Form of the Nonlinear Heat Conduction Equations via Exp-Function Method , *Int. J. Nonlin. Sci. Num.,* **10,** 1307-1318 , 2009
15. A. Esen , S. Kutluay. Application of the Exp-function method to the two dimensional sine-Gordon equation , *Int. J. Nonlin. Sci. Num.,* **10**, 1355-1359, 2009