Mathematical and Computational Applications, Vol. 15, No. 5, pp. 762-767, 2010.

DYNAMICAL BEHAVIORS OF A DELAYED REACTION-DIFFUSION EQUATION

Zhihao Ge

Institute of Applied Mathematics
School of Mathematics and Information Sciences
Henan University, 475004
Kaifeng, Henan, P.R. China
zhihaoge@henu.edu.cn

Abstract- In this paper, we derive a delayed reaction-diffusion equation to describe a two-species predator-prey system with diffusion terms and stage structure. By coupling the uniformly approximate approach with the method of upper and lower solutions, we prove that the traveling wave fronts exist, which connect the zero solution with the positive steady state. Finally, we draw a conclusion that the existence of traveling wave fronts for the delayed reaction-diffusion equation is an interesting and difficult problem.

Keywords- Reaction-Diffusion Equations, Asymptotical Stability, Traveling Wave fronts

1. INTRODUCTION

Delay ordinary differential equations (also called by retarded functional differential equations) have been extensively studied by many authors, such as [2], [3], [4], [7], [10] and so on. Recently, a two-species predator-prey system described by a delayed ordinary differential equation was considered in [9], where the delay means the stage for the prey population. We remark that the above mentioned models did not consider the effect of diffusion on the stability of the equilibrium and traveling wave fronts. However, the specie's diffusion is a natural tendency to move into areas of smaller population density. So we follow the normal technique to handle with the diffusion (see [3], [5], [6] and [12]) to give the following delayed reaction-diffusion equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D_1 \Delta u_1 = \alpha u_2(x,t) - \gamma u_1(x,t) - \alpha e^{-\gamma\tau} u_2(x,t-\tau), \\ \dfrac{\partial u_2}{\partial t} - D_2 \Delta u_2 = -\beta u_2^2(x,t) - a_1 u_2(x,t) v(x,t) + \alpha e^{-\gamma\tau} u_2(x,t-\tau) - h u_2(x,t), \\ \dfrac{\partial v}{\partial t} - D_3 \Delta v = v(x,t)(-r_1 + a_2 u_2(x,t) - b v(x,t)), \ \ x \in \Omega, t > 0, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \ \ x \in \partial\Omega, t > 0, \\ u_1(x,t) = \varphi_1(x,t), u_2(x,t) = \varphi_2(x,t), v(x,t) = \varphi_3(x,t), \ x \in \overline{\Omega}, t \in [-\tau, 0], \end{cases} \tag{1}$$

where $\Omega \in R^N$ is open and bounded with smooth boundary $\partial\Omega$, $\partial / \partial n$ is differentiation in the direction of the outward unit normal, $u_1(x,t), u_2(x,t)$ represent the immature and mature prey population densities at x-space and t-time, respectively; $v(x,t)$ represents the density of predator population at x-space and t-time; a_1 is the transformation coefficient of mature predator population; τ represents the transformation of immature to mature; $\alpha > 0$ is the birth rate of the immature prey population; $h \geq 0$ is the harvesting effort of the prey species; $\beta > 0$ represents the death and overcrowding rate of the mature prey population. And the constants $r_1 > 0, a_2 > 0, b > 0$. And the constant D_i $(i = 1,2,3)$ is positive, and the initial functions $\varphi_1(x,0), \varphi_3(x,0)$ are continuous in $\overline{\Omega}$ and $\varphi_2(x,t)$ is continuous in $\overline{\Omega} \times [-\tau, 0]$.

In this paper, we aim to study the dynamical behaviors of the system (1). Note that $u_2(x,t)$ and $v(x,t)$ of the system (1) are independent of $u_1(x,t)$, so we obtain the dynamical behaviors of the system (1) by studying the following system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - D_2 \Delta u_1 = -\beta u_1^2(x,t) - a_1 u_1(x,t) u_2(x,t) + \alpha e^{-\gamma\tau} u_1(x, t-\tau) - h u_1(x,t), \\ \dfrac{\partial u_2}{\partial t} - D_3 \Delta u_2 = u_2(x,t)(-r_1 + a_2 u_1(x,t) - b u_2(x,t)), \quad x \in \Omega, t > 0, \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0, \quad x \in \partial\Omega, t > 0, \\ u_1(x,t) = \varphi_1(x,t), u_2(x,0) = \varphi_2(x,0), \; x \in \overline{\Omega}, t \in [-\tau, 0], \end{cases} \tag{2}$$

where $u_1(x,t), u_2(x,t)$ and $\varphi_1(x,t), \varphi_2(x,t)$ denote $u_2(x,t), v(x,t)$ and $\varphi_2(x,t), \varphi_3(x,t)$ of the system (1), respectively. For this single specie model of [10], S.A. Gourley and Y. Kuang pointed out that the existence of wave front solutions is an open question. Motivated by the results of [10], we study the existence of traveling wave fronts of the two-species delayed system (2). The key idea is to couple the uniformly approximated approach introduced by J. Canosa in [1] with the method of upper and lower solutions. The method to construct the upper and lower solutions of the system (2) is derived from the idea of [11]. The remaining parts of the paper are organized as follows. In section 2, we study the locally asymptotical stability of the constant equilibrium and the existence of traveling wave fronts of the system (2). Finally, we draw a conclusion.

2. DYNAMICAL BEHAVIORS OF THE SYTEM (2)

It is easy to check that the system (2) has only three nonnegative constant solutions: $E_1(0,0), E_2((\alpha e^{-\gamma\tau} - h)/\beta, 0)$ and the positive equilibrium $E_3(c_1^*, c_2^*)$ as $\alpha e^{-\gamma\tau} - h > \beta r_1$, where $c_1^* = \frac{b(\alpha e^{-\gamma\tau} - h) + a_1 r_1}{a_1 a_2 + b\beta}, c_2^* = \frac{(\alpha e^{-\gamma\tau} - h) - \beta r_1}{a_1 a_2 + b\beta}$. Using the linearization technique ([8] or [12]) and omitting the detailed derivation ([12]), we have

Theorem 2.1 The equilibrium $E_1(0,0)$ of the system (2) is unstable; if $a_2\alpha e^{-\gamma\tau}-h\ge\beta r_1$, then the equilibrium $E_2((\alpha e^{-\gamma\tau}-h)/\beta,0)$ is unstable; if $a_2\alpha e^{-\gamma\tau}-h>\beta r_1$, the positive equilibrium $E_3(c_1^*,c_2^*)$ is locally asymptotically stable.

Next, we study the existence of traveling wave solution for the infinite spatial $x\in(-\infty,+\infty)$. To seek a pair of traveling wave fronts of the system (2), we set $u_1(x,t)=\phi_1(s)$ and $u_2(x,t)=\phi_2(s)$, where $s=x+ct$ and c is the wave speed. Substituting $\phi_1(s)$ and $\phi_2(s)$ into the system (2), we have

$$\begin{cases} D_2\phi_1''-c\phi_1'-\beta\phi_1^2-h\phi_1-a_1\phi_1(s)\phi_2(s)+\alpha e^{-\gamma\tau}\phi_1(s-c\tau)=0, \\ D_3\phi_2''-c\phi_2'-r_1\phi_2+a_2\phi_1\phi_2-b\phi_2^2=0, \\ \phi_1(-\infty)=0,\phi_1(+\infty)=c_1^*,\phi_2(-\infty)=0,\phi_2(+\infty)=c_2^*. \end{cases} \tag{3}$$

Let $\theta=1/c^2$, for the large values of the wave speed c, then θ is a small parameter. Denote $\eta=\sqrt{\theta}s=s/c$, under the transformation $\phi_i(s)=\psi_i(\eta)$ $(i=1,2)$, then the system (3) becomes

$$\begin{cases} \theta D_2\psi_1''-\psi_1'-\beta\psi_1^2-a_1\psi_1\psi_2+\alpha e^{-\gamma\tau}\psi_1(\eta-\tau)-h\psi_1=0, \\ \theta D_3\psi_2''-\psi_2'-r_1\psi_2+a_2\psi_1\psi_2-b\psi_2^2=0, \\ \psi_1(-\infty)=0,\psi_1(+\infty)=c_1^*,\psi_2(-\infty)=0,\psi_2(+\infty)=c_2^*. \end{cases} \tag{4}$$

Let $\psi_1(\eta,\theta)=\psi_{10}+\theta\psi_{11}+\cdots,\psi_2(\eta,\theta)=\psi_{20}+\theta\psi_{21}+\cdots,$ and substitute into (4) and group the same powers of θ, denote $\psi_{i0}(\eta)$ by $\psi_i(\eta)$ $(i=1,2)$, respectively, then we have

$$\begin{cases} \psi_1'=-\beta\psi_1^2-a_1\psi_1\psi_2+\alpha e^{-\gamma\tau}\psi_1(\eta-\tau)-h\psi_1, \\ \psi_2'=-r_1\psi_2+a_2\psi_1\psi_2-b\psi_2^2, \\ \psi_1(-\infty)=0,\psi_1(+\infty)=c_1^*,\psi_2(-\infty)=0,\psi_2(+\infty)=c_2^*. \end{cases} \tag{5}$$

Theorem 2.2 If $a_2\alpha e^{-\gamma\tau}-h>\beta r_1$, then the system (5) has at least one non-decreasing positive solution $\psi=(\psi_1(\eta),\psi_1(\eta))^T\in C^1(R,R^2)$.

Proof. To prove the theorem, we need to check that a quasi-monotone condition (see [7] or [11]) holds and show that there exists a pair of upper and lower solutions $(\overline{\psi}_1(\eta),\overline{\psi}_2(\eta))^T$ and $(\underline{\psi}_1(\eta),\underline{\psi}_2(\eta))^T$. To do that, we define the functional $f_c(\psi)=(f_{c1}(\psi),f_{c2}(\psi))^T$ by

$$\begin{cases} f_{c1}(\psi)=-\beta\psi_1^2(0)-a_1\psi_1(0)\psi_2(0)+\alpha e^{-\gamma\tau}\psi_1(-\tau)-h\psi_1(0), \\ f_{c2}(\psi)=-r_1\psi_2(0)+a_2\psi_1(0)\psi_2(0)-b\psi_2^2(0). \end{cases} \tag{6}$$

Letting $\delta=(\delta_1,\delta_2)^T$, for arbitrary $\phi,\psi\in C([-\tau,0];R^2)$ satisfying $0\le\psi(\eta)\le\phi(\eta)$, we easily obtain

$$f_c(\phi)-f_c(\psi)+\delta(\phi(0)-\psi(0))\ge(\delta I-\mathrm{B})(\phi(0)-\psi(0))\ge 0, \tag{7}$$

where I is a 2×2 identity matrix, $B=diag(2\beta c_1^*+a_1c_2^*+hc_1^*, r_1+2bc_2^*)$, $\delta_1\ge\alpha e^{-\gamma\tau}+\beta c_1^*$ and $\delta_2\ge r_1+2bc_2^*$.

Next, we show that there exists a pair of upper and lower solutions. To do that, we introduce the following set

$$\Gamma=\left\{\psi=\begin{pmatrix}\psi_1(\eta)\\ \psi_2(\eta)\end{pmatrix}\middle| \begin{array}{l}(1)\ \psi \textit{ is piecewise continuous and nondecreasing in } R\\ (2)\ \lim_{\eta\to-\infty}\psi=0,\ \lim_{\eta\to+\infty}\psi=(c_1^*,c_2^*)^T\end{array}\right\}.$$

Define

$$\overline{\psi}_1(\eta)=\begin{cases}\frac{c_1^*e^{\lambda\eta}}{2}, \eta\le 0,\\ c_1^*-\frac{c_1^*e^{-\lambda\eta}}{2}, \eta>0,\end{cases}\qquad \overline{\psi}_2(\eta)=\begin{cases}\frac{c_2^*e^{\lambda\eta}}{2}, \eta\le 0,\\ c_2^*-\frac{c_2^*e^{-\lambda\eta}}{2}, \eta>0,\end{cases} \tag{8}$$

where

$$\lambda>2\alpha e^{-\gamma\tau}. \tag{9}$$

And it is easy to check that $\overline{\psi}=\begin{pmatrix}\overline{\psi}_1(\eta)\\ \overline{\psi}_2(\eta)\end{pmatrix}\in\Gamma$. Next, we check that $\overline{\psi}$ is a pair of upper solutions to (5). To do that, we have two cases

Case i: $\eta>0$. Then, we have

$$\overline{\psi}_2{}'(\eta)+r_1\overline{\psi}_2(\eta)-a_2\overline{\psi}_1(\eta)\overline{\psi}_2(\eta)+b\overline{\psi}_2^{\,2}(\eta)\ge\frac{(\lambda+r_1)c_2^*e^{-\lambda\eta}}{2}-\frac{r_1c_2^*e^{-\lambda\eta}}{4}>0. \tag{10}$$

From (8) and (9), for the case $\eta>\tau$ it follows that

$$\begin{aligned}&\overline{\psi}_1{}'(\eta)-\alpha e^{-\gamma\tau}\overline{\psi}_1(\eta-\tau)+\beta\overline{\psi}_1^{\,2}(\eta)+a_1\overline{\psi}_1(\eta)\overline{\psi}_2(\eta)+h\overline{\psi}_1(\eta)\\ &\ge\frac{\lambda c_1^*e^{-\lambda\eta}}{2}+\frac{\alpha e^{-\gamma\tau}c_1^*e^{-\lambda\eta}}{2}-\beta c_1^{*2}e^{-\lambda\eta}+\frac{\beta c_1^{*2}e^{-2\lambda\eta}}{4c}-\frac{a_1c_1^*c_2^*e^{-\lambda\eta}}{c}+\frac{a_1c_1^*c_2^*e^{-2\lambda\eta}}{4}\\ &\ge\frac{c_1^*e^{-\lambda\eta}}{2}(\lambda+\alpha e^{-\gamma\tau}-\frac{2(\beta c_1^*+a_1c_2^*)}{c})=\frac{c_1^*e^{-\lambda\eta}}{2c}[c\lambda+(1-\tfrac{2}{c})\alpha e^{-\gamma\tau}]>0,\end{aligned} \tag{11}$$

and for the case $0<\eta\le\tau$, we get

$$\begin{aligned}&\overline{\psi}_1{}'(\eta)-\alpha e^{-\gamma\tau}\overline{\psi}_1(\eta-\tau)+\beta\overline{\psi}_1^{\,2}(\eta)+a_1\overline{\psi}_1(\eta)\overline{\psi}_2(\eta)+h\overline{\psi}_1(\eta)\\ &\ge\frac{\lambda c_1^*e^{-\lambda\eta}}{2}-\beta c_1^{*2}e^{-\lambda\eta}+\frac{\beta c_1^{*2}e^{-2\lambda\eta}}{4}-a_1c_1^*c_2^*e^{-\lambda\eta}+\frac{a_1c_1^*c_2^*e^{-2\lambda\eta}}{4}\\ &\ge\frac{c_1^*e^{-\lambda\eta}}{2}(\lambda-2\alpha e^{-\gamma\tau})>0.\end{aligned} \tag{12}$$

Case ii: $\eta < 0$. Using (8) and (9), we have

$$\begin{aligned}&\overline{\psi}_1\,'(\eta) - \alpha e^{-\gamma\tau}\overline{\psi}_1(\eta-\tau) + \beta\overline{\psi}_1^{\,2}(\eta) + a_1\overline{\psi}_1(\eta)\overline{\psi}_2(\eta) + h\overline{\psi}_1(\eta)\\ &\geq \tfrac{c_1^* e^{\lambda\eta}}{2}(\lambda - \alpha e^{-\gamma\tau} + \tfrac{\beta c_1^* e^{\lambda\eta}}{2} + \tfrac{a_1 c_2^* e^{\lambda\eta}}{2}) > 0,\end{aligned}$$

$$\begin{aligned}&\overline{\psi}_2\,'(\eta) + r_1\overline{\psi}_2(\eta) - a_2\overline{\psi}_1(\eta)\overline{\psi}_2(\eta) + b\overline{\psi}_2^{\,2}(\eta)\\ &= \tfrac{c_2^* e^{\lambda\eta}}{2}(\lambda + r_1 - \tfrac{r_1^* e^{\lambda\eta}}{2} + \tfrac{r_1^* e^{\lambda\eta}}{2} - \tfrac{a_2 c_1^* e^{\lambda\eta}}{2} + \tfrac{b c_2^* e^{\lambda\eta}}{2}) > 0.\end{aligned}$$

From Case i and Case ii, we know that $\overline{\psi}$ is an upper solution to (5).

Define

$$\underline{\psi}_1(\eta) = \begin{cases}\xi\varepsilon e^{\lambda_1\eta}, \eta < 0,\\ \varepsilon - \xi\varepsilon e^{-\lambda_1\eta}, \eta \geq 0,\end{cases} \quad \overline{\psi}_2(\eta) = 0, \tag{13}$$

where $0 < \varepsilon < \frac{\alpha e^{-\gamma\tau}\xi e^{-\lambda_1\tau} - \lambda_1\xi}{\beta + h}$, ξ is small enough, $0 < \lambda_1 < \alpha e^{-\gamma\tau}e^{-\lambda_1\tau}$.

From (13), we get

$$\underline{\psi}_1\,'(\eta) = \begin{cases}\lambda_1\xi\varepsilon e^{\lambda_1\eta}, \eta < 0,\\ \lambda_1\xi\varepsilon e^{-\lambda_1\eta}, \eta \geq 0,\end{cases} \quad \underline{\psi}_1\,''(\eta) = \begin{cases}\lambda_1^{\,2}\xi\varepsilon e^{\lambda_1\eta}, \eta < 0,\\ -\lambda_1^{\,2}\xi\varepsilon e^{-\lambda_1\eta}, \eta \geq 0.\end{cases} \tag{14}$$

Using (13) and (14), for $\eta \geq \tau$ we have

$$\begin{aligned}&\underline{\psi}_1\,'(\eta) - \alpha e^{-\gamma\tau}\underline{\psi}_1(\eta-\tau) + \beta\underline{\psi}_1^{\,2}(\eta) + a_1\underline{\psi}_1(\eta)\underline{\psi}_2(\eta) + h\underline{\psi}_1(\eta)\\ &= \lambda_1\xi\varepsilon e^{-\lambda_1\eta} - \alpha e^{-\gamma\tau}(\varepsilon - \xi\varepsilon e^{-\lambda_1(\eta-\tau)}) + \beta(\varepsilon - \xi\varepsilon e^{-\lambda_1\eta})^2 + h(\varepsilon - \xi\varepsilon e^{-\lambda_1\eta}) < 0,\end{aligned} \tag{15}$$

for $0 < \eta < \tau$ we obtain

$$\begin{aligned}&\underline{\psi}_1\,'(\eta) - \alpha e^{-\gamma\tau}\underline{\psi}_1(\eta-\tau) + \beta\underline{\psi}_1^{\,2}(\eta) + a_1\underline{\psi}_1(\eta)\underline{\psi}_2(\eta) + h\underline{\psi}_1(\eta)\\ &= \lambda_1\xi\varepsilon e^{-\lambda_1\eta} - \alpha e^{-\gamma\tau}\xi\varepsilon e^{\lambda_1(\eta-\tau)} + \beta(\varepsilon - \xi\varepsilon e^{-\lambda_1\eta})^2 + h(\varepsilon - \xi\varepsilon e^{-\lambda_1\eta}) < 0\end{aligned} \tag{16}$$

if $0 < \varepsilon \leq \frac{\alpha e^{-\gamma\tau}\xi e^{-\lambda_1\tau} - \lambda_1\xi}{\beta + h}$; for $\eta < 0$ we have

$$\begin{aligned}&\underline{\psi}_1\,'(\eta) - \alpha e^{-\gamma\tau}\underline{\psi}_1(\eta-\tau) + \beta\underline{\psi}_1^{\,2}(\eta) + a_1\underline{\psi}_1(\eta)\underline{\psi}_2(\eta) + h\underline{\psi}_1(\eta)\\ &= \lambda_1\xi\varepsilon e^{\lambda_1\eta} - \alpha e^{-\gamma\tau}\xi\varepsilon e^{\lambda_1(\eta-\tau)} + \beta(\xi\varepsilon e^{\lambda_1\eta})^2 + h\xi\varepsilon e^{\lambda_1\eta}\\ &\leq \xi\varepsilon e^{\lambda_1\eta}[\lambda_1 - \alpha e^{-\gamma\tau}e^{-\lambda_1\tau} + (\beta + h)\varepsilon] < 0.\end{aligned} \tag{17}$$

So, $\underline{\psi} = \begin{pmatrix}\underline{\psi}_1(\eta)\\ \underline{\psi}_2(\eta)\end{pmatrix}$ is a pair of lower solutions.

Therefore, if $a_2\alpha e^{-\gamma\tau} > \beta r_1$, from [11] we know that there exists at least one solution in the set Γ. The proof of the theorem is completed.

3. CONCLUSION

In our work, we prove the existence of traveling wave fronts for the two-species model for large values of the wave speed c. The system (2) is a new model and the method to prove the existence of traveling wave fronts is also novel, and it is effective to deal with the case of large wave speeds, which is deserved future study.

Acknowledgement: The present work is supported by National Natural Science Foundation of China under Grant No.10901047.

4. REFERENCES

1. J. Canosa, On a nonlinear diffusion equation describing population growth, *IBM Journal of Research and Development* **17**, 307-313, 1973.
2. H. Freedman, *Deterministic Mathematical Models in Population Ecology,* Marcel Dekker, New York, 1980.
3. W. Aiello and H. Freedman, A time-delay model of single-species growth with stage structure, *Mathematical Biosciences* **101**, 139-153, 1990.
4. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, London, 1993.
5. C. V. Pao, Dynamics of nonlinear parabolic systems with time delays, *Journal of Mathematical Analysis and Applications* **198**, 751-779, 1996.
6. C. V. Pao, Systems of parabolic equations with continuous and discrete delays, *Journal of Mathematical Analysis and Applications* **205**, 157-185, 1997.
7. T. Faria, Stability and bifurcation for a delayed predator-prey model and the effect of diffusion, *Journal of Mathematical Analysis and Applications* **254**, 433-463, 2001.
8. W. Joseph, J. H. Wu and X. F. Zou, A reaction-diffusion model for a single species with age structure. I. travelling fronts on unbounded domains, *Proc. R. Soc. London A* **457**, 1-13, 2001.
9. X. Y. Song and L. S. Chen, Optimal harvesting and stability for a predator-prey system with stage structure, *Acta Mathematicae Applicatae Sinica (English Series)* **18**, 307-314, 2002.
10. S. A. Gourley, Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, *Journal of Mathematical Biology* **49**, 188-200, 2004.
11. A. Boumenir, N. V. Minh, Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations, *Journal of Differential Equations* **244**, 1551-1570, 2008.
12. Z. H. Ge and Y. N. He, Diffusion effect and stability analysis of a predator-prey system described by a delayed reaction diffusion equations, *Journal of Mathematical Analysis and Applications* **339**, 1432-1450, 2008.