



GROUP CLASSIFICATION OF COUPLED DIFFUSION SYSTEM WITH APPLICATIONS IN SOIL SCIENCE

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Abstract- We perform a complete group classification of a coupled system of diffusion equations with applications in soil science. The canonical forms of the low-dimensional Lie algebras and the Lie algebras of higher dimension provide a means to specify the diffusion coefficients completely.

Keywords- Group classification, Coupled diffusion, Lie algebra.

Dedicated to Professor David P. Mason on the occasion of his 65th birthday.

1. INTRODUCTION

A group classification for a general second-order system of diffusion equations based on Lie algebras of low dimension was performed in [1]. We follow exactly the procedure used in this classification for the current problem. Briefly, the classification procedure involves the utilization of the structure of the low-dimensional Lie algebras and the Lie algebras of higher dimension to find the symmetry operators admitted by the underlying equation or system. The procedure is continued until the functional forms of the unknown functions are completely specified. That is, the equivalence group is used to obtain the canonical forms of the symmetry operators which satisfy the model under consideration. Even though this procedure was suggested in [2, 3] for partial differential equations (PDEs), a much earlier work on ordinary differential equations (ODEs) using these ideas was done in [4]. We use the results on classification of solvable Lie algebras by Mubarakzyanov [5] reported in Basarab-Horwarth [3].

The theory of coupled diffusion appears in the works of Philip and De Vries [6], De Vries [7] in the late fifties and later by many authors, such as Jury et al [8].

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The coupled diffusion is described by the partial differential equations [9, 10, 11], i.e.,

$$\begin{aligned} u_t &= [f^1(u, v)u_x + f^2(u, v)v_x]_x, \\ v_t &= [f^3(u, v)u_x + f^4(u, v)v_x]_x, \end{aligned} \quad (1)$$

where $u(t, x)$ and $v(t, x)$ are the soil temperature and volumetric water content respectively. The variable x measures the depth of the soil and t is the time. The arbitrary functions f^1, \dots, f^4 are the diffusion coefficients.

Wiltshire et al [11, 13] investigated the Lie symmetries of a simplified model of the coupled diffusion system (1) written in the form

$$\mathbf{y}_t = [\Lambda(\mathbf{y})\mathbf{y}_x]_x, \quad \mathbf{y} = \{y_i\} \quad (i = 1, 2, \dots, n), \quad (2)$$

where $\Lambda(\mathbf{y})$ is a square matrix of diffusion coefficients. These investigations involve rewriting a system of coupled diffusion equations into a single equation in which the diffusion coefficients are written in a matrix form. Thus the resulting determining equations to be solved for symmetries are in terms of the systems of matrix equations. The matrix diffusion equation (2) is a generalized form of the extensively studied one-dimensional nonlinear heat conduction equation the group properties of which were first considered in [12]. Therefore the group classification procedure on Eq. (2) is analogous to the analysis performed in the cited reference and those that followed.

In this work, our goal is to derive the Lie symmetries of system (1) as it is and see whether a comparison with those of the simplified model (2) can be established. The non classical (potential) symmetries of the coupled system (1) were generated in [13]. The generation of the determining equations and the manipulation of them are with the aid of the YaLie software package [14].

2. EQUIVALENCE GROUP AND GENERATOR OF SYMMETRY

The equivalence group for system (1) was obtained in [10] using the infinitesimal approach. It can shown that the direct method yields the same result. However, instead of using these equivalence transformations we choose the equivalence group of the form

$$\bar{t} = T(t), \quad \bar{x} = X(x), \quad \bar{u} = U(u, v), \quad \bar{v} = V(u, v), \quad (3)$$

where the functions T , X , U and V satisfy the conditions

$$T_t \neq 0, \quad X_x \neq 0, \quad \frac{D(U, V)}{D(u, v)} \neq 0.$$

Using the basic Lie technique [15, 16, 17] we seek a symmetry generator of the form

$$\Gamma = \xi^1(t, x, u, v)\partial_t + \xi^2(t, x, u, v)\partial_x + \eta^1(t, x, u, v)\partial_u + \eta^2(t, x, u, v)\partial_v. \tag{4}$$

The operator Γ is a generator of symmetry group of system (1) if and only if

$$\Gamma^{[2]} (u_t - f^1 u_{xx} - f^2 v_{xx} - u_x [f^1_u u_x + f^1_v v_x] - v_x [f^2_u u_x + f^2_v v_x]) = 0, \tag{5}$$

$$\Gamma^{[2]} (v_t - f^3 u_{xx} - f^4 v_{xx} - u_x [f^3_u u_x + f^3_v v_x] - v_x [f^4_u u_x + f^4_v v_x]) = 0$$

whenever system (1) is satisfied and where

$$\Gamma^{[2]} = \Gamma + \zeta_1^1 \partial_{u_t} + \zeta_2^1 \partial_{u_x} + \zeta_1^2 \partial_{v_t} + \zeta_2^2 \partial_{v_x} + \zeta_{22}^1 \partial_{u_{xx}} + \zeta_{22}^2 \partial_{v_{xx}}. \tag{6}$$

The variables ζ_j^i are given by the prolongation formulae

$$\begin{aligned} \zeta_1^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_2^1 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_1^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \\ \zeta_2^2 &= D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2), \\ \zeta_{22}^1 &= D_x(\zeta_2^1) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \\ \zeta_{22}^2 &= D_x(\zeta_2^2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2), \end{aligned} \tag{7}$$

where D_t and D_x are the total derivative operators defined as follows

$$D_t = \partial_t + u_t \partial_u + v_t \partial_v + \dots, \quad D_x = \partial_x + u_x \partial_u + v_x \partial_v + \dots. \tag{8}$$

With the help of program YaLie the generator of symmetry group for system (1) is of the form

$$\Gamma = a(t)\partial_t + b(t, x)\partial_x + c(t, x, u, v)\partial_u + d(t, x, u, v)\partial_v, \tag{9}$$

where a, \dots, d are the smooth differentiable functions satisfying determining equations resulting from the invariance conditions (5).

It is not easy to employ the group classification procedure discussed in [17] to solve completely these determining equations. Therefore we opt for the approach mentioned earlier in the previous section to perform the complete group classification up to equivalence group (3).

3. CLASSIFICATION WITH RESPECT TO LOW-DIMENSIONAL LIE ALGEBRAS

3.1. One-dimensional Lie algebras

A symmetry generator (9) can be transformed by the change of variables (3) into the form

$$\bar{\Gamma} = a(t)\dot{T}(t)\partial_{\bar{t}} + bX'(x)\partial_{\bar{x}} + (cU_u + dU_v)\partial_{\bar{u}} + (cV_u + dV_v)\partial_{\bar{v}}, \quad (10)$$

where the overdot and prime denote differentiation with respect to respective arguments and subscripts represent partial differentiation.

We follow an argument similar to that used in [3]. We consider the cases $a(t) \neq 0$ and $a(t) = 0$. If $a(t) \neq 0$, then the choice in (3) of a function $T(t)$ such that $a(t)\dot{T}(t) = 1$ and functions X, U, V of the fundamental solutions of the system of PDEs

$$bX' = 0, \quad cU_u + dU_v = 0, \quad cV_u + dV_v = 0,$$

gives rise to $\Gamma \rightarrow \bar{\Gamma} = \partial_{\bar{t}}$.

If $a(t) = 0$, then the choice in (3) of a particular solution of the PDE $bX' = 1$ as a function of X and fundamental solutions of the PDEs $cU_u + dU_v = 0, cV_u + dV_v = 0$ as functions of U and V respectively, we have $\Gamma \rightarrow \bar{\Gamma} = \partial_{\bar{x}}$. Therefore, when we drop the bars, the symmetry operator (9) reduces to one of the canonical operators

$$\Gamma = \partial_t, \quad \Gamma = \partial_x, \quad \Gamma = \partial_u, \quad \Gamma = \partial_v.$$

Consequently there are four inequivalent realizations of one-dimensional Lie algebras denoted: $A_1^1 = \langle \partial_t \rangle$, $A_1^2 = \langle \partial_x \rangle$, $A_1^3 = \langle \partial_u \rangle$ and $A_1^4 = \langle \partial_v \rangle$. The superscript distinguishes one realization from the other while the subscript denotes the dimension of the Lie algebra.

3.2. Two-dimensional Lie algebras

There are two inequivalent two-dimensional solvable Lie algebras: $A_{2,1} : [e_1, e_2] = 0$, $A_{2,2} : [e_1, e_2] = e_2$. These algebras contain the 1D Lie algebras $A_1^1 = \langle \partial_t \rangle$, $A_1^2 = \langle \partial_x \rangle$, $A_1^3 = \langle \partial_u \rangle$ and $A_1^4 = \langle \partial_v \rangle$. Hence in search for realizations of the 2D Lie algebras there are four cases to consider: $\langle \partial_t, e_2 \rangle$, $\langle \partial_x, e_2 \rangle$, $\langle \partial_u, e_2 \rangle$ and $\langle \partial_v, e_2 \rangle$ where e_2 is of the form (9).

Consider firstly the Lie algebra $A_{2,1}$.

(a) $\langle \partial_t, e_2 \rangle$: The Lie Bracket $[e_1, e_2] = 0$ yields

$$e_2 = b(x)\partial_x + c(x, u, v)\partial_u + d(x, u, v)\partial_v.$$

We use the equivalence transformations (3) to simplify e_2 . Also we need the equivalence group of $A_1^1 = \langle \partial_t \rangle$: i.e.,

$$\partial_{\bar{t}} = \partial_t(\bar{t})\partial_{\bar{t}} + \partial_t(\bar{x})\partial_{\bar{x}} + \partial_t(\bar{u})\partial_{\bar{u}} + \partial_t(\bar{v})\partial_{\bar{v}} = \dot{T}\partial_{\bar{t}}.$$

Thus the equivalence transformations for A_1^1 are given by

$$\mathcal{E}(\partial_t) = \{\bar{t} = t + a_1, \bar{x} = X(x), \bar{u} = U(u, v), \bar{v} = V(u, v)\}. \tag{11}$$

By the use of these transformations e_2 is transformed into

$$\bar{e}_2 = bX_x\partial_{\bar{x}} + (cU_u + dU_v)\partial_{\bar{u}} + (cV_u + dV_v)\partial_{\bar{v}}. \tag{12}$$

According to (12) we have the following realizations of 2D Lie algebras: $A_{2,1}^1 = \langle \partial_t, \partial_x \rangle$, $A_{2,1}^2 = \langle \partial_t, \partial_u \rangle$, $A_{2,1}^3 = \langle \partial_t, \partial_v \rangle$.

(b) Likewise for the case $\langle \partial_x, e_2 \rangle$ we obtain the realizations: $A_{2,1}^4 = \langle \partial_x, \partial_u \rangle$, $A_{2,1}^5 = \langle \partial_x, \partial_v \rangle$. The cases $\langle \partial_u, e_2 \rangle$ and $\langle \partial_v, e_2 \rangle$ lead to the realization $A_{2,1}^6 = \langle \partial_u, \partial_v \rangle$. This realization yields a trivial case (i.e. all the arbitrary functions are constants) hence it is not utilized for further classification.

Therefore we have six inequivalent realizations for the 2D Lie algebra $A_{2,1}$. The realizations of Lie algebras $A_{2,1}^2, A_{2,1}^4$ and $A_{2,1}^3, A_{2,1}^5$ imply that the arbitrary functions f^1, \dots, f^4 are respectively independent of u and v .

If we proceed in the same manner, the 2D Lie algebra $A_{2,2}$ has the realizations: $A_{2,2}^3 = \langle -x\partial_x - u\partial_u, \partial_x \rangle$ and $A_{2,2}^4 = \langle -x\partial_x - v\partial_v, \partial_x \rangle$. Their corresponding functional forms for the arbitrary functions are given by

$$\begin{aligned} A_{2,2}^3: f^1 &= u^2F(v), & f^2 &= u^3F(v), & f^3 &= uF(v), & f^4 &= u^2F(v). \\ A_{2,2}^4: f^1 &= v^2F(u), & f^2 &= vF(u), & f^3 &= v^3F(u), & f^4 &= v^2F(u). \end{aligned}$$

In general the arbitrary functions of their respective arguments are different.

When we look at the above functional forms of the diffusion coefficients, the cases $A_{2,2}^3$ and $A_{2,2}^4$ can be regarded as one case for a choice of equivalence transformations of the form

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = v, \quad \bar{v} = u. \tag{13}$$

3.3. Three-dimensional solvable Lie algebras

The solvable Lie algebras can be classified into decomposable and nondecomposable Lie algebras (see Appendix in [3]). There are two realizations for the 3D decomposable Lie algebra $A_{3,1}$ obtained from the realization of the 2D Lie algebra $A_{2,1}^1$. They are $A_{3,1}^1 = \langle \partial_t, \partial_x, \partial_u \rangle$ and $\tilde{A}_{3,1}^1 = \langle \partial_t, \partial_x, \partial_v \rangle$. These realizations satisfy the consistency conditions for system (1) provided the diffusion coefficients are functions of v and u respectively. There are also two realizations for the 3D Lie algebra

$A_{3,2}$. This Lie algebra contains the 2D Lie algebras $A_{2,2}^3$ and $A_{2,2}^4$ from which we obtain $\tilde{A}_{3,2}^3 = \langle -x\partial_x - u\partial_u, \partial_x, u\partial_u + v\partial_v \rangle$ and $\tilde{A}_{3,2}^4 = \langle -x\partial_x - v\partial_v, \partial_x, u\partial_u + v\partial_v \rangle$ respectively. Therefore the respective forms of the diffusion coefficients are

$$\begin{aligned} \tilde{A}_{3,2}^3: f^1 &= k_1 \left(\frac{u}{v}\right)^2, & f^2 &= k_2 \left(\frac{u}{v}\right)^3, & f^3 &= k_3 \left(\frac{u}{v}\right), & f^4 &= k_4 \left(\frac{u}{v}\right)^2, \\ \tilde{A}_{3,2}^4: f^1 &= \bar{k}_1 \left(\frac{v}{u}\right)^2, & f^2 &= \bar{k}_2 \left(\frac{v}{u}\right), & f^3 &= \bar{k}_3 \left(\frac{v}{u}\right)^3, & f^4 &= \bar{k}_4 \left(\frac{v}{u}\right)^2, \end{aligned}$$

where k_1, \dots, k_4 and $\bar{k}_1, \dots, \bar{k}_4$ are the nonzero arbitrary constants. Likewise, these cases can be taken as one case if we make use of the change of variables (13).

Secondly we look at the 3D nondecomposable solvable Lie algebras. The results are summarized in Table 1. The Lie algebras that do not feature in this table of results imply that there are no realizations for such Lie algebras which satisfy the consistency conditions for system (1).

Table 1: 3D nondecomposable solvable Lie algebras.

Algebra	f^1	f^2	f^3	f^4	Realization
$A_{3,5}^1$	$uF(v)$	$u^2F(v)$	$F(v)$	$uF(v)$	$\langle \partial_t, \partial_x, t\partial_t + x\partial_x + u\partial_u \rangle$
$\tilde{A}_{3,5}^2$	K_1v^{-1}	K_2v^{-1}	K_3v^{-1}	K_4v^{-1}	$\langle \partial_t, \partial_u, t\partial_t + u\partial_u + v\partial_v \rangle$
$\tilde{A}_{3,5}^4$	K_1v^2	K_2v^2	K_3v^2	K_4v^2	$\langle \partial_x, \partial_u, x\partial_x + u\partial_u + v\partial_v \rangle$
$A_{3,6}^1$	$u^{-3}F(v)$	$u^{-2}F(v)$	$u^{-4}F(v)$	$u^{-3}F(v)$	$\langle \partial_t, \partial_x, t\partial_t - x\partial_x + u\partial_u \rangle$
$\tilde{A}_{3,6}^2$	K_1v^{-1}	K_2v^{-3}	K_3v	K_4v^{-1}	$\langle \partial_t, \partial_u, t\partial_t - u\partial_u + v\partial_v \rangle$
$\tilde{A}_{3,6}^4$	K_1v^2	K_2	K_3v^4	K_4v^2	$\langle \partial_x, \partial_u, x\partial_x - u\partial_u + v\partial_v \rangle$
$A_{3,7}^1$	$u^{2q-1}F(v)$	$u^{2q}F(v)$	$u^{2(q-1)}F(v)$	$u^{2q-1}F(v)$	$\langle \partial_t, \partial_x, t\partial_t + qx\partial_x + u\partial_u \rangle$
$\hat{A}_{3,7}^1$	$u^{2-q}F(v)$	$u^{3-q}F(v)$	$u^{1-q}F(v)$	$u^{2-q}F(v)$	$\langle \partial_x, \partial_t, qt\partial_t + x\partial_x + u\partial_u \rangle$
$\tilde{A}_{3,7}^2$	K_1v^{-1}	K_2v^{q-2}	K_3v^{-q}	K_4v^{-1}	$\langle \partial_t, \partial_u, t\partial_t + qu\partial_u + v\partial_v \rangle$
$\tilde{A}_{3,7}^2$	K_1v^{-q}	K_2v^{-q}	K_3v^{-q}	K_4v^{-q}	$\langle \partial_u, \partial_t, qt\partial_t + u\partial_u + v\partial_v \rangle$
$\tilde{A}_{3,7}^4$	K_1v^2	K_2v^{q+1}	K_3v^{3-q}	K_4v^2	$\langle \partial_x, \partial_u, x\partial_x + qu\partial_u + v\partial_v \rangle$
$\tilde{A}_{3,7}^4$	K_1v^{2q}	K_2v^{2q}	K_3v^{2q}	K_4v^{2q}	$\langle \partial_u, \partial_x, qx\partial_x + u\partial_u + v\partial_v \rangle$

The arbitrary functions $F(v)$ are in general different and K_i are (generally different) arbitrary constants.

It should be noted that making use of the equivalence transformations (13), the arbitrary functions depend upon u and the cases for which the diffusion coefficients are completely specified are in terms of u . This note applies to all the summarized results in the sequel.

4. COMPLETE GROUP CLASSIFICATION

We proceed to the classification with respect to four-dimensional solvable Lie algebras, that is, we extend the realizations of the 3D Lie algebras obtained in the previous section. After this classification the functional forms of the arbitrary functions are in terms of the arbitrary constants only (c.f. Table 1). However, there is a need to investigate whether or not the realizations of both the 3D and 4D Lie algebras give the maximal symmetry Lie algebra.

Firstly we consider the decomposable 4D solvable Lie algebras. The extension of the realizations is possible for the 4D Lie algebra $4A_1$ in the following cases:

$$4A_1^1 = \langle \partial_t, \partial_x, \partial_u, v\partial_v \rangle: f^1 = \ell_1, \quad f^2 = \ell_2 v^{-1}, \quad f^3 = \ell_3 v, \quad f^4 = \ell_4,$$

$$4A_1^2 = \langle \partial_t, \partial_x, u\partial_u, \partial_v \rangle: f^1 = \bar{\ell}_1, \quad f^2 = \bar{\ell}_2 u, \quad f^3 = \bar{\ell}_3 u^{-1}, \quad f^4 = \bar{\ell}_4,$$

for the nonzero arbitrary constants, ℓ_1, \dots, ℓ_4 and $\bar{\ell}_1, \dots, \bar{\ell}_4$, are arbitrary.

Now in Table 2 we present the realizations of 4D nondecomposable solvable Lie algebras which contain the 3D realizations and their corresponding forms of the diffusion coefficients.

Table 2: 4D non decomposable solvable Lie algebras.

Algebra	f^1	f^2	f^3	f^4	Realization
$A_{4,5}^1$	$\lambda_1 v^{2q-1}$	$\lambda_2 v^{p+2q-2}$	$\lambda_3 v^{2q-p}$	$\lambda_4 v^{2q-1}$	$\langle \partial_t, \partial_x, \partial_u, t\partial_t + qx\partial_x + pu\partial_u + v\partial_v \rangle$
$\tilde{A}_{4,5}^1$	$\lambda_1 v^{2p-1}$	$\lambda_2 v^{2p+q-2}$	$\lambda_3 v^{2p-q}$	$\lambda_4 v^{2p-1}$	$\langle \partial_t, \partial_u, \partial_x, t\partial_t + px\partial_x + qu\partial_u + v\partial_v \rangle$
$\hat{A}_{4,5}^1$	$\lambda_1 v^{2-q}$	$\lambda_2 v^{p-q+1}$	$\lambda_3 v^{3-p-q}$	$\lambda_4 v^{2-q}$	$\langle \partial_x, \partial_t, \partial_u, qt\partial_t + x\partial_x + pu\partial_u + v\partial_v \rangle$
$\bar{A}_{4,5}^1$	$\lambda_1 v^{2-p}$	$\lambda_2 v^{1-p+q}$	$\lambda_3 v^{3-p-q}$	$\lambda_4 v^{2-p}$	$\langle \partial_x, \partial_u, \partial_t, pt\partial_t + x\partial_x + qu\partial_u + v\partial_v \rangle$
$\check{A}_{4,5}^1$	$\lambda_1 v^{2p-q}$	$\lambda_2 v^{2p-q}$	$\lambda_3 v^{2p-q}$	$\lambda_4 v^{2p-q}$	$\langle \partial_u, \partial_t, \partial_x, qt\partial_t + px\partial_x + u\partial_u + v\partial_v \rangle$
$\hat{A}_{4,5}^1$	$\lambda_1 v^{2q-p}$	$\lambda_2 v^{2q-p}$	$\lambda_3 v^{2q-p}$	$\lambda_4 v^{2q-p}$	$\langle \partial_u, \partial_x, \partial_t, pt\partial_t + qx\partial_x + u\partial_u + v\partial_v \rangle$

The λ_i are arbitrary constants which are in general different.

5. FURTHER ANALYSIS

In order to investigate the maximal symmetry Lie algebra, we perform symmetry analysis of all the cases in Tables 1 and 2 for which the arbitrary functions are completely specified. The cases arising from the decomposable solvable Lie algebras are also taken into account.

We show the detailed calculations of one case for both the classification with respect to 3D and 4D Lie algebras and present the results only for the rest.

Firstly we consider the 3D realization $\tilde{A}_{3,7}^2 = \langle \partial_t, \partial_u, t\partial_t + qu\partial_u + v\partial_v \rangle$ from Table 1. We proceed by substituting the corresponding forms of the arbitrary functions into the system (1) under consideration. Lie's algorithm yields the symmetry generator (9) the coordinates of which are of the form $a = a(t)$, $b = b(t, x)$, $c = e(t, x)u + f(t, x)$, $d = d(t, x, v)$ for arbitrary functions $e(t, x)$ and $f(t, x)$. The smooth differentiable functions, a , b , d , e and f , satisfy the determining equations

$$K_1 v (ue_x + f_x) + v^q K_2 [vb_{xx} - 2((q-2)d_x + vd_{xv})] = 0, \quad (14)$$

$$K_1 (ue_{xx} + f_{xx}) + K_2 v^{q-1} d_{xx} - v(ue_t + f_t) = 0, \quad (15)$$

$$v(\dot{a} - 2b_x + d_v) - 2d = 0, \quad (16)$$

$$d - v\dot{a} + 2vb_x = 0, \quad (17)$$

$$(q-2)d + v(-e + \dot{a} - 2b_x + d_v) = 0, \quad (18)$$

$$v[(q-2)\dot{a} - (q-2)e + 2(2-q)b_x + 2(q-2)d_v + vd_{vv}] + (6-5q+q^2)d = 0, \quad (19)$$

$$K_1 (vb_{xx} + d_x - 2ve_x) - v^2 b_t = 0, \quad (20)$$

$$vb_{xx} + qd_x - 2ve_x = 0, \quad (21)$$

$$v^{-(q+1)} [K_3 v (ue_{xx} + f_{xx}) + K_4 v^q d_{xx}] - d_t = 0, \quad (22)$$

$$v(\dot{a} - 2b_x + d_v - vd_{vv}) - 2d = 0, \quad (23)$$

$$(1+q)d - v(e + \dot{a} - 2b_x) = 0, \quad (24)$$

$$v(e + \dot{a} - 2b_x - d_v) - qd = 0, \quad (25)$$

$$K_3 q v^{1-q} (ue_x + f_x) + K_4 (vb_{xx} + 2d_x - 2vd_{xv}) - v^2 b_t = 0. \quad (26)$$

In the above equations and throughout the overdot represents total derivative with respect to t and the subscripts denote partial differentiation. The solution of Eqs. (14)–(26) is given by

$$a = C_3 t + C_4, \quad b = C_1 x + C_2, \quad c = q(C_3 - 2C_1)u + C_5, \quad d = (C_3 - 2C_1)v, \quad (27)$$

where C_1, \dots, C_5 are the arbitrary constants of integration.

Thus the symmetry Lie algebra for this case is spanned by the operators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_u, \quad \Gamma_4 = t\partial_t + qu\partial_u + v\partial_v, \quad \Gamma_5 = x\partial_x - 2qu\partial_u - 2v\partial_v.$$

The full symmetry Lie algebra for the rest of the cases that arise from symmetry analysis are presented below.

$$\tilde{A}_{3,5}^2: \Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_u, \quad \Gamma_4 = t\partial_t + u\partial_u + v\partial_v, \quad \Gamma_5 = x\partial_x - 2u\partial_u - 2v\partial_v.$$

$$\tilde{A}_{3,5}^4: \Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_u, \quad \Gamma_4 = 2t\partial_t - u\partial_u - v\partial_v, \quad \Gamma_5 = x\partial_x + u\partial_u + v\partial_v.$$

$$\begin{aligned} \tilde{A}_{3,6}^2: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = t\partial_t - u\partial_u + v\partial_v, \Gamma_5 = x\partial_x + 2u\partial_u - 2v\partial_v. \\ \tilde{A}_{3,6}^4: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = 2t\partial_t + u\partial_u - v\partial_v, \Gamma_5 = x\partial_x - u\partial_u + v\partial_v. \\ \tilde{A}_{3,7}^2: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = t\partial_t + qu\partial_u + v\partial_v, \Gamma_5 = x\partial_x - 2qu\partial_u - 2v\partial_v. \\ \tilde{A}_{3,7}^2: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = qt\partial_t + u\partial_u + v\partial_v, \Gamma_5 = qx\partial_x - 2u\partial_u - 2v\partial_v. \\ \tilde{A}_{3,7}^4: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = 2t\partial_t - qu\partial_u - v\partial_v, \Gamma_5 = x\partial_x + qu\partial_u + v\partial_v. \\ \tilde{A}_{3,7}^4: & \Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = 2qt\partial_t - u\partial_u - v\partial_v, \Gamma_5 = qx\partial_x + u\partial_u + v\partial_v. \end{aligned}$$

For every case above the symmetry Lie algebra is spanned by the three operators from the 3D realization and two additional operators that arise from symmetry analysis.

Next we show the details of symmetry analysis for the case of 4D decomposable Lie algebra $4A_1$ with the realization: $4A_1^1 = \langle \partial_t, \partial_x, \partial_u, v\partial_v \rangle$.

We proceed as in the previous symmetry analysis. Following the usual Lie's algorithm the coefficients of the symmetry generator (9) are of the form

$$a = a(t), \quad b = \frac{1}{2}\dot{a}x + \bar{e}(t), \quad c = \bar{f}(t, x)u + \bar{g}(t, x), \quad d = d(t, x, v). \tag{28}$$

The arbitrary smooth functions a, d, \bar{e}, \bar{f} and \bar{g} from (28) satisfy the determining equations

$$vd_{xv} - d_x = 0, \tag{29}$$

$$\ell_1 v (u\bar{f}_{xx} + \bar{g}_{xx}) + \ell_2 d_{xx} - v (u\bar{f}_t + \bar{g}_t) = 0, \tag{30}$$

$$2d + v (\bar{f} - 2d_v + vd_{vv}) = 0, \tag{31}$$

$$d + v (\bar{f} - d_v) = 0, \tag{32}$$

$$2\bar{e}_t + x\ddot{a} + 4\ell_1 \bar{f}_x = 0, \tag{33}$$

$$\ell_4 d_{vv} = 0, \tag{34}$$

$$2v\bar{f}_x + d_x = 0, \tag{35}$$

$$\ell_3 v (u\bar{f}_{xx} + \bar{g}_{xx}) + \ell_4 d_{xx} - d_t = 0, \tag{36}$$

$$\ell_3 \bar{f} = 0, \tag{37}$$

$$2\bar{e}_t + x\ddot{a} + 2\ell_3 (u\bar{f}_x + \bar{g}_x) + 4\ell_4 d_{xv} = 0. \tag{38}$$

Eventually the solution of the determining equations (29)–(38) is given by

$$a = 2\bar{C}_1 t + \bar{C}_2, \quad b = \bar{C}_1 x + \bar{C}_3, \quad c = \bar{C}_4, \quad d = \bar{C}_5 v, \tag{39}$$

where $\bar{C}_1, \dots, \bar{C}_5$ are arbitrary constants of integration.

Therefore for this case the symmetry Lie algebra is spanned by the operators

$$\Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = \partial_u, \Gamma_4 = v\partial_v, \Gamma_5 = 2t\partial_t + x\partial_x.$$

The symmetry Lie algebra for the other case of the 4D decomposable Lie algebra $4A_1^2$ is spanned by

$$\Gamma_1 = \partial_t, \Gamma_2 = \partial_x, \Gamma_3 = u\partial_u, \Gamma_4 = \partial_v, \Gamma_5 = 2t\partial_t + x\partial_x.$$

The maximal symmetry Lie algebras for every case in Table 2 of the four-dimensional nondecomposable Lie algebras are summarized as follows:

$$A_{4,5}^1 \cup \langle (2q-1)t\partial_t - pu\partial_u - v\partial_v, (2q-1)x\partial_x + 2pu\partial_u + 2v\partial_v \rangle.$$

$$\tilde{A}_{4,5}^1 \cup \langle (2p-1)t\partial_t - qu\partial_u - v\partial_v, (2p-1)x\partial_x + 2qu\partial_u + 2v\partial_v \rangle.$$

$$\hat{A}_{4,5}^1 \cup \langle (q-2)t\partial_t + pu\partial_u + v\partial_v, (q-2)x\partial_x - 2pu\partial_u - 2v\partial_v \rangle.$$

$$\bar{A}_{4,5}^1 \cup \langle (p-2)t\partial_t + qu\partial_u + v\partial_v, (p-2)x\partial_x - 2qu\partial_u - 2v\partial_v \rangle.$$

$$\check{A}_{4,5}^1 \cup \langle (2p-q)t\partial_t - u\partial_u - v\partial_v, (2p-q)x\partial_x + 2u\partial_u + 2v\partial_v \rangle.$$

$$\hat{A}_{4,5}^1 \cup \langle (p-2q)t\partial_t + u\partial_u + v\partial_v, (p-2q)x\partial_x - 2u\partial_u - 2v\partial_v \rangle.$$

In all the above cases the symmetry Lie algebra is six-dimensional, four operators from the classification with respect to 4D Lie algebras and the additional two operators obtained via symmetry analysis. Moreover it can be seen that, if $p = q$, then a pair of similar cases is reduced to one case.

6. CONCLUDING REMARKS

We have managed to specify completely the functional forms of the diffusion coefficients through the classification with respect to the solvable Lie algebras of dimension up to four. In order to determine the maximal symmetry Lie algebra, we performed the symmetry analysis of individual cases for the realizations of the 3D and 4D Lie algebras in which the functional forms of the unknown functions are in terms of the arbitrary constants. In short the full classification has been achieved. The comparison with the previous studies of the model considered in [9, 11] can hardly be established. It should be noted that group classification using the other types of Lie algebras has not been considered in this work. These are semisimple Lie algebras and the algebras which are semidirect sum of semisimple algebras and solvable Lie algebras (c.f. Appendix [3]). The next step will be to find both the analytical and numerical solutions of the submodels.

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