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COUPLING DRIFT-FLUX MODELS WITH UNEQUAL SONIC SPEEDS

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Abstract- The well-posedness of a Riemann problem at a junction in a pipeline network is discussed. In addition computational results on the dynamics of the flow of a multi-component gas at such network junctions are presented. The work presented here is a generalisation of [M. K. Banda, M. Herty, and J. M. T. Ngnotchouye, Towards a mathematical analysis of multiphase drift-flux model in networks, *SIAM J. Sci. Comput.*, 31(6): 4633 – 4653, 2010] to models in which the equation of state has different compressibility factors or sonic speeds.

Keywords- Drift-flux, multi-component, compressible flow, Riemann Problem, flow networks, coupling conditions, compressibility factor.

1. INTRODUCTION

We consider pipe networks through which multi-component fluids flow. The pipes in the network intersect at joints or junctions. The idea is to propose coupling conditions for the case in which the individual fluid components have unequal sonic speeds or compressibility factors. The analysis presented in this paper is an extension of the work in [4]. In [4] multi-component flow in which sonic speeds of fluid components are equal were considered.

An isothermal no-slip drift-flux model for multi-component flows is considered. In this model the closure law, the so called slip condition, has a vanishing slip function. The drift-flux model obtained is presented in Section 2. The same section also discusses a mathematical analysis of the model. These kinds of models have many applications in the chemical, petroleum and nuclear industries.

The mathematical analysis of networked flow has recently become an active area of research. Some examples include [2, 3, 4, 8, 10]. The well-posedness of the coupling of flow at junctions will be presented in Section 3. In Section 4 results of numerical tests of such coupling conditions will be presented.

2. MODEL FORMULATION AND THE RIEMANN PROBLEM

The conservative isothermal no-slip drift-flux model takes the form

$$\begin{aligned} \partial_t \rho_1 \quad &+ \partial_x \frac{\rho_1 I}{\hat{\rho}} = 0; \\ \partial_t \rho_2 \quad &+ \partial_x \frac{\rho_2 I}{\hat{\rho}} = 0; \\ \partial_t I \quad &+ \partial_x \left(\frac{I^2}{\hat{\rho}} + p(\rho_1, \rho_2) \right) = 0; \end{aligned} \tag{1}$$

where we denote $I = \hat{\rho} u$, $\hat{\rho} = \rho_1 + \rho_2$. In the above, components of an immiscible mixture of two fluids are denoted by the subscripts $i \in \{1, 2\}$. Hence the density, volume fraction, velocity and pressure are denoted by ϱ_i, α_i, u_i, p_i, respectively, see also [4] and references therein. The density of each component is denoted as $\rho_1 = \alpha_1 \varrho_1$ and $\rho_2 = \alpha_2 \varrho_2$. It is also assumed that $u = u_1 = u_2$. Since each component is isothermal, the equation of state is of the form $p \doteq p_i(\rho_i) = a_i^2 \rho_i, \ i \in \{1, 2\}$, where the positive constants a_i are the compressibility factors or sonic speeds of phase i. From the relation $\alpha_1 + \alpha_2 = 1$, we obtain an equation of state:

$$p(\rho_1, \rho_2) = a_1^2 \rho_1 + a_2^2 \rho_2. \tag{2}$$

In the pipe network, at each junction where pipes intersect, we need to solve a Riemann problem [9, 11]. This consists of solving the model (1) in each pipe with constant data in addition to some coupling conditions at the intesection.
We note that the eigenvalues $\lambda_{1,2,3}$ and the eigenvectors $r_{1,2,3}$ of the drift-flux model (1) are given by

$$\lambda_{1,3}(w) = \frac{I}{\hat{\rho}} \mp \sqrt{\frac{\rho_1 a_1^2 + \rho_2 a_2^2}{\hat{\rho}}} = \frac{I}{\hat{\rho}} \mp \sqrt{\frac{p}{\hat{\rho}}}, \quad \lambda_2(w) = \frac{I}{\hat{\rho}};$$

$$r_{1,3}(w) = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \hat{\rho}\lambda_{1,3}(w) \end{bmatrix}, \quad r_2(w) = \begin{bmatrix} a_2^2 \\ -a_1^2 \\ (a_2^2 - a_1^2)\lambda_2(w) \end{bmatrix},$$

The field 2 is always linearly degenerate while the fields 1 and 3 are genuinely non-linear [9, 11]. Moreover since $p \geq 0, \ \forall \rho_1, \rho_2 > 0$ model (1) is hyperbolic and strictly hyperbolic if $\sqrt{p} \neq \frac{I}{\sqrt{\hat{\rho}}}$.
We now discuss the Lax-curves as a pre-requisite to presenting the solution of the standard Riemann problem.

2.1. Shock Curves

The Lax shock curves are derived from the Rankine-Hugoniot jump conditions (3). Let w be a given state and assume that another state $\bar{w}$ is connected to w by a 1-, 3-shock wave of shock speed s. Then w and $\bar{w}$ satisfy

$$f(w) - f(\bar{w}) = s(w - \bar{w}). \tag{3}$$

For the equation of state given in (2), the shock curves are given by

$$S_{1,3}(\xi; w) = \begin{bmatrix} \rho_1 \xi \\ \rho_2 \xi \\ I\xi \mp (\xi - 1)\sqrt{\xi}\sqrt{\hat{\rho} p} \end{bmatrix} \tag{4a}$$

with shock speed

$$s_{1,3}(\xi; w) = \frac{I}{\hat{\rho}} \mp \sqrt{\xi}\sqrt{\frac{p}{\hat{\rho}}}. \tag{4b}$$

The forward (respectively, backward) admissible 1-shock curves are obtained using the Lax admissibility conditions [9] denoted as $S_1(\xi; w)$ given by (4) with $\xi \geq 1$, (respectively, $\xi \leq 1$). Similarly, the forward (respectively, backward) 3-shock curves are given by $S_3(\xi; w)$ in (4) with $\xi \leq 1$, (respectively $\xi \geq 1$).

2.2. Contact Discontinuity

Let w and $\bar{w}$ be two given states. Using the linear degeneracy of the 2-field and the Rankine-Hugoniot jump condition (3), we declare that w belongs to the 2-curve emanating from $\bar{w}$ if $w - \bar{w} = \xi r_2(\bar{w}), \quad \xi \in \mathbb{R}$. Eliminating ξ from this system and applying suitable scaling, one obtains the contact discontinuity wave emanating from any state w given by the curve

$$L_2(\xi; w) = \frac{1}{a_2^2} \begin{bmatrix} a_2^2 \rho_1 \xi \\ a_2^2 \rho_2 + a_1^2 (1 - \xi)\rho_1 \\ \frac{I}{\hat{\rho}} \left(a_1^2 \rho_1 + a_2^2 \rho_2 + (a_2^2 - a_1^2)\rho_1 \xi \right) \end{bmatrix}.$$

Note that we have continuity of the pressure along the contact discontinuity.

2.3. Rarefaction Curves

We first note that rarefaction waves are given as integral curves of the eigenvectors of the flux function

$$\frac{dw}{d\xi} = \frac{r_{1,3}(w(\xi))}{\nabla \lambda_{\mp}(w(\xi)) \cdot r_{1,3}(w(\xi))}, \quad \xi \geq \xi_{1,3},$$

with $\xi_{1,3} = \lambda_{1,3}(w)$. This yields

$$R_{1,3}(\xi; w) = \begin{bmatrix} \rho_1 \xi \\ \rho_2 \xi \\ I\xi \mp \xi \log(\xi)\sqrt{\hat{\rho}(a_1^2\rho_1 + a_2^2\rho_2)} \end{bmatrix}. \tag{5}$$

The forward (respectively, backward) admissible 1-rarefaction curves for (5) are obtained using the Lax admissibility condition as $R_1(\xi; w)$ with $\xi < 1$, (respectively, $\xi > 1$). Similarly, the forward (respectively, backward) 3-rarefaction curves are given by $R_3(\xi; w)$ with $\xi > 1$, (respectively, $\xi < 1$).
In summary the Lax-curves for model (1) with the equation of state (2) are given by

$$L_1^+(\xi; w) = \begin{cases} S_1(\xi; w), & \xi \geq 1; \\ R_1(\xi; w), & \xi < 1; \end{cases} \qquad L_3^+(\xi; w) = \begin{cases} S_3(\xi; w), & \xi \leq 1; \\ R_3(\xi; w), & \xi > 1; \end{cases}$$

$$L_1^-(\xi; w) = \begin{cases} S_1(\xi; w), & \xi \leq 1; \\ R_1(\xi; w), & \xi > 1; \end{cases} \qquad L_3^-(\xi; w) = \begin{cases} S_3(\xi; w), & \xi \geq 1; \\ R_3(\xi; w), & \xi < 1. \end{cases}$$

The shock speeds are given by $s_{1,3}(\xi, w) = \dfrac{I}{\hat{\rho}} \mp \sqrt{\xi}\sqrt{\dfrac{p}{\hat{\rho}}}$.

2.4. Solution to the Standard Riemann Problem for (2)

Proposition 0.1 *[9] We consider the Riemann problem for* (1) *with initial data*

$$w(x, 0) = \begin{cases} w^- & \text{if } x < 0; \\ w^+ & \text{if } x > 0. \end{cases}$$

For $|w^+ - w^-|$ sufficiently small, there exists a unique weak self-similar solution to this Riemann problem with small total variation. This solution comprises four constant states $w_0 = w^-, w_1, w_2, w_3 = w^+$. When the i-th characteristic family is genuinely nonlinear w_i is joined to w_{i-1} by either an i-rarefaction wave or an i-shock, while when the i-characteristic family is linearly degenerate, w_i is joined to w_{i-1} by an i-contact discontinuity.

Assuming the states w^- and w^+ are given and satisfy the conditions of Proposition 0.1, then the intermediary states, w_1 and w_2, could be:

$$w_1 = L_1^+(\xi_1; w^-), \quad w_2 = L_2(\xi_2; w_1), \quad \text{and} \quad w_2 = L_3^-(\xi_3; w^+).$$

Denoting the momentum components of w_1 and w_2 as $I_1(\xi_1; w^-)$ and $I_3(\xi_3; w^+)$, respectively. The solution for the Riemann problem is found if we can solve for

ξ_1, ξ_2 and ξ_3 the system

$$\begin{aligned}
&\rho\,{}^{+}_{1}\xi_3 = \rho_1^-\xi_1\xi_2,\\
&\rho\,{}^{+}_{2}\xi_3 = \rho_2^-\xi_1 + \frac{a_2^2}{a_1^2}(1-\xi_2)\rho_1^-\xi_1,\\
&I_{\,3}(\xi_3; w^+) = \frac{I_1(\xi_1; w^-)}{a_2^2(\rho_1^- + \rho_2^-)\xi_1}\left((a_1^2\rho_1^- + a_2^2\rho_2^-)\xi_1 + (a_2^2 - a_1^2)\rho_1^-\xi_1\xi_2\right).
\end{aligned}$$

Proposition 0.2 *Assume that we have a multi-component fluid described with the equation of state* (2) *with* a_1^2 *and* a_2^2 *given. Let* w *be a state that satisfies the following condition* $\dfrac{\rho_1\,I}{\hat\rho}\dfrac{a_1^2 - a_2^2}{a_2^2} + 2\sqrt{\hat\rho p} \neq 0$*. Then, for* w^- *and* w^+ *close to* w*, the standard Riemann problem with data* (w^-, w^+) *admits a solution.*

3. PIPE-TO-PIPE INTERSECTIONS

We assume two pipes are joined at a junction located at $x = 0$. The flow in the pipes is defined by (1) and at the junction the flow must also satisfy the coupling conditions:

$$\Psi\left(w^-(t, 0-); w^+(t, 0+)\right) = 0.$$

The variables w^- and w^+ denote the flow variables in the left and right pipes, respectively. The notations $\mathbb{R}^+ =]0, +\infty[$ and $\overset{\circ}{\mathbb{R}}{}^{+} = [0, +\infty[$ will also be used. We consider solutions of the Riemann problem at the junction as perturbations of stationary solutions of (1). Consider the subsonic region defined as

$$A_0 = \{w \in \overset{\circ}{\mathbb{R}}{}^{+} \times \overset{\circ}{\mathbb{R}}{}^{+} \times \mathbb{R} : \lambda_1(w) < 0 < \lambda_2(w) < \lambda_3(w)\}.$$

Further define *flow of the density of phase 1* as $M(w) = \rho_1 I/\hat\rho$, *flow of the density of phase 2* as $N(w) = \rho_2 I/\hat\rho$, *flow of the linear momentum* as $P(w) = I^2/\hat\rho + p(\rho_1, \rho_2)$. In general, we are interested in Ψ-solutions, that is, weak solutions depending on the coupling conditions map Ψ. We consider a map $\hat w$ defined as

$$\hat w(x) = \begin{cases} \hat w^- & \text{if } x < 0 \\ \hat w^+ & \text{if } x > 0 \end{cases} \quad \text{with} \quad \begin{array}{l} \Psi(\hat w^-; \hat w^+) = 0, \\ \hat w^-,\ \hat w^+ \in A_0. \end{array}$$

The existence of $\hat w^+$ for a given $\hat w^-$ is guaranteed by Lemma 0.1. In a neighborhood of a junction between two pipes, one can integrate the stationary model from (1) to obtain the coupling conditions map [5, 6, 10]

$$\Psi\left(w^-; w^+\right) = \begin{bmatrix} M(w^+) - M(w^-) \\ N(w^+) - N(w^-) \\ P(w^+) - P(w^-) \end{bmatrix}. \tag{7}$$

Note that the conservation of mass of each phase and the equality of the dynamic pressure at the junction are preserved. We can prove that when a stationary flow w^- is prescribed in the ingoing pipe, we can solve for the flow in the outgoing pipe in general which is also stationary.

Lemma 0.1 *Let $\bar{w} \in A_0$. Then there exists $\bar{\delta} > 0$ and a Lipschitz map $T : B(\bar{w}; \bar{\delta}) \to A_0$ where $B(\bar{w}; \bar{\delta})$ is a ball centred at $\bar{w}$ with radius $\bar{\delta}$, such that $\Psi(w^-; w^+) = 0; \quad w^-, w^+ \in B(\bar{w}; \bar{\delta})$ iff $w^+ = T(w^-)$.*

This result is similar to a result presented in [7] in the context of the p-system. Moreover, Lemma 0.1 ensures the "additivity" property [5, 9] for the Riemann problem at a junction. For the well-posedness of the Riemann problem at a junction with more than two pipes, we present the following:

Proposition 0.3 *Let $\hat{w}_1, \hat{w}_2 \in A_0$ be the data in the ingoing and outgoing pipe connected at $x = 0$. Assume that the following condition is satisfied*

$$\frac{1}{a_1^2}\lambda_1(\hat{w}_1)\,\lambda_2(\hat{w}_2)\,\lambda_3(\hat{w}_2)\left(\lambda_2(\hat{w}_2)\rho_1^2(a_2^2 - a_1^2) - \lambda_3(\hat{w}_2)p(\hat{\rho}_1^2, \hat{\rho}_2^2)\right)\left(\hat{\rho}_1^1\hat{\rho}_2^2 - \hat{\rho}_2^1\hat{\rho}_1^2\right) \neq 0 \quad (8)$$

Then, there exists a constant $\delta > 0$ and for any states $\bar{w}_1$ and $\bar{w}_2$ such that $|\bar{w}_i - \hat{w}_i| < \delta$, for $i = 1, 2$, the Riemann problem at the junction with data $(\bar{w}_1, \bar{w}_2)$ has a unique solution.

The argument in the proof is standard and has also been developed in [2, 3, 6, 8, 10] for other models.

4. NUMERICAL EXPERIMENTS

The numerical schemes used to test the homogeneous no-slip drift-flux multiphase flow model as defined in (1) are the second-order relaxed schemes which were discussed in [1] and references therein. For a comparison with other schemes applied to similar problems refer to [4]. Initial conditions are some perturbation of stationary solutions, see Proposition 0.3. Newton's method is used to solve the system in equation (7) combined with the Lax curves which gives the boundary conditions at the junctions of the network. For the external (inlet to network or outlet from network) boundary conditions, the transparent boundary conditions are imposed.

4.1. Effect of the compressibility of the different phases on the flow

Here we consider model equation (1) with the following data, refer to [4] and references therein, in the primitive variables $v = (\rho_1, \rho_2, u) : v^-(x, 0) = (500/9, 0.95/18, 10)$;

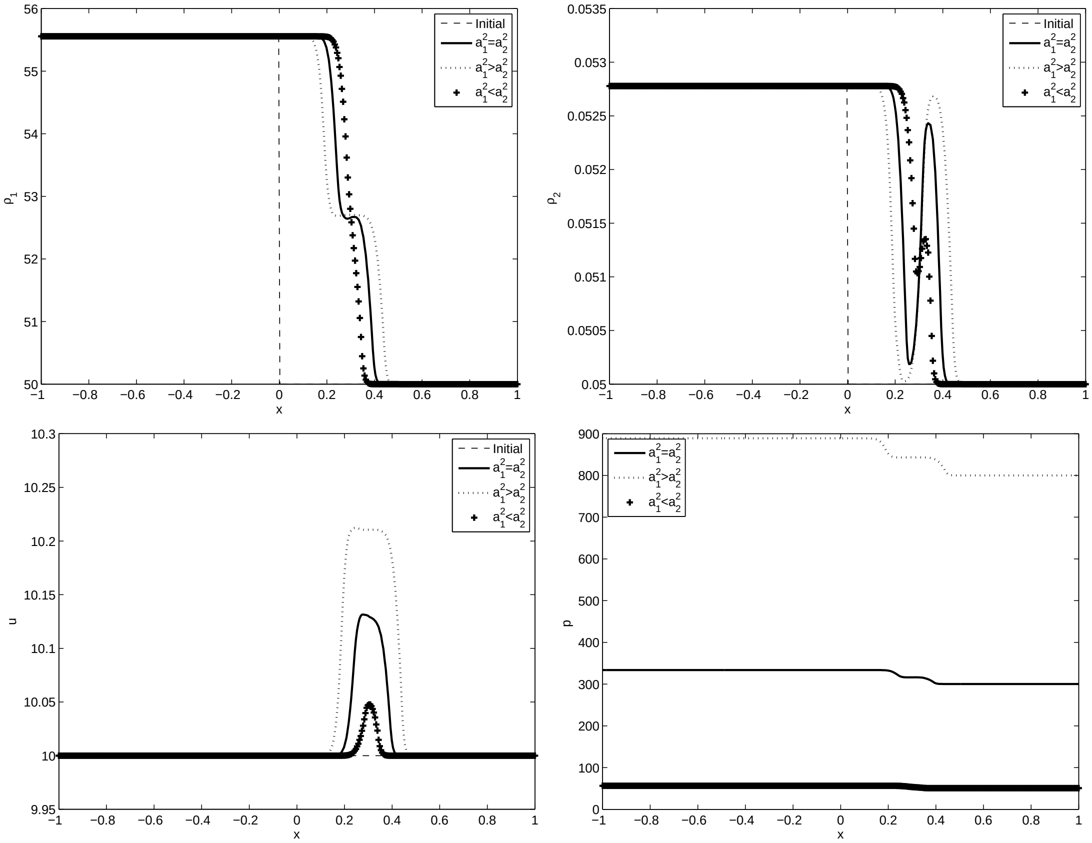

Figure 1: Profiles of the densities, the common velocity and the pressure for the Riemann problem for the drift flux model with different compressibility factors.

$v^+(x,0) = (500/10, 1/20, 10)$. In Figure 1 the plots of the densities, the velocity and the pressure at time $t = 0.8$ are shown. We also present results for the case $\frac{a_2^2}{a_1^2} = 1$, see also [4].

4.2. A junction connecting two pipes

Here we will firstly verify the qualitative behavior of the coupling conditions. Secondly, we will use it to present the effect of the variation of the sound speed of each phase at the junction. The compressibility factors are taken as $a_1^2 = 16.0$ and $a_2^2 = 1.0$ with the Riemann data $w^- = (3.17123, 3.38324, 3.71816)$, $w^+ = (2.70708, 4.0434, 3.5629)$. The mesh size of $N = 400$ was employed on a single pipe on which the standard Riemann solver was applied and $N = 200$ was applied on each coupled pipe. In Figure 2 computational results at $t = 0.05$ are shown. Qualitatively, there is very good agreement between the two. The results in Figure 3 show the effects of the sonic speeds. Three cases were considered: $a_1^2 = a_2^2 = 6.0$, $16 = a_1^2 > a_2^2 = 1$ and $1 = a_1^2 < a_2^2 = 16.0$. We observe that higher values of the

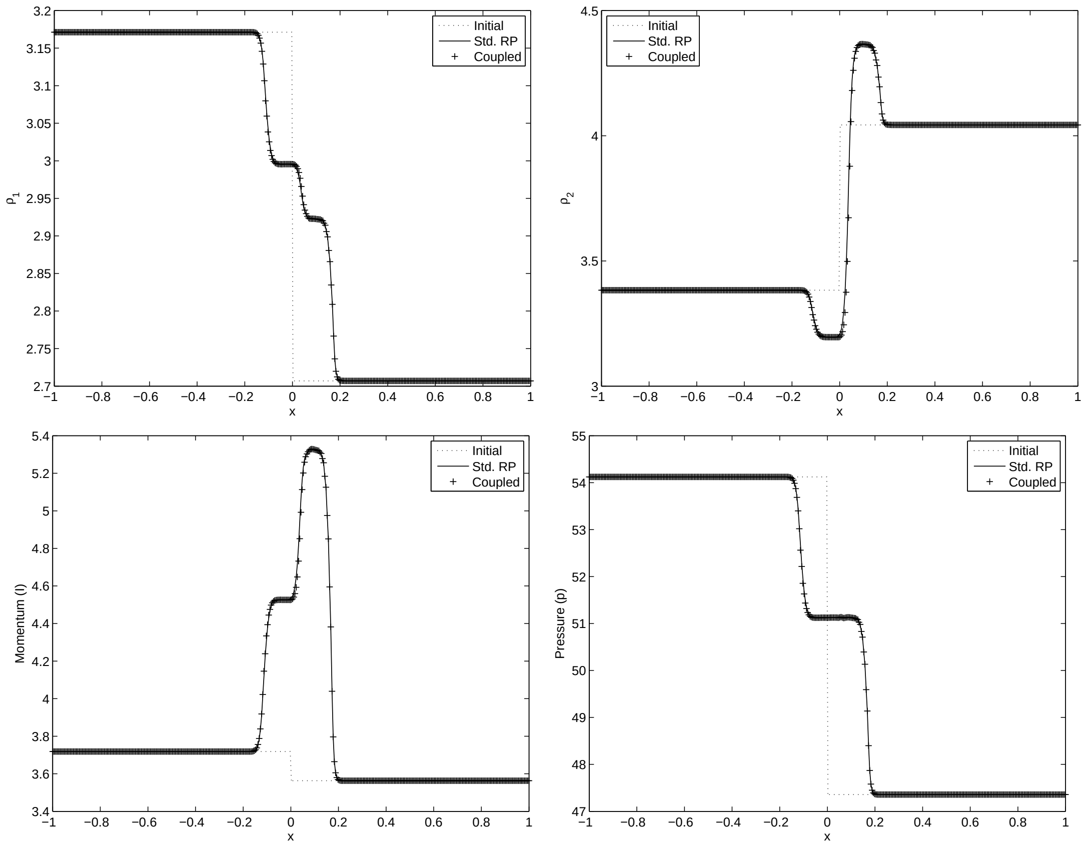

Figure 2: Profiles of densities, momentum, and the common pressure p for the standard Riemann problem (continuous line) and two coupled pipes (dashed line).

pressure are obtained with smaller values of a_1^2.

4.3. A Junction with three connected pipes.

We apply coupling conditions proposed in (8). We consider the case of one ingoing and two outgoing pipes, case A, and two ingoing and one outgoing pipes, case B. For case A the initial data in each pipe is $w_1 = (6.4500000, 12.8050000, 31.9713732)$; $w_2 = (10.3300000, 3.3578000, 2.4903271)$; $w_3 = (1.9534000, 4.5682760, 29.4810461)$; and for case B: $w_1 = (5.5000000, 6.6050000, 17.2150344)$;
$w_2 = (7.11300000, 4.8110000, -2.3628774)$; $w_3 = (5.9534000, 7.8359377, 14.8521570)$. The results in Figure 4, show the snapshots of the densities for case A with $0 \leq t \leq 0.1$.

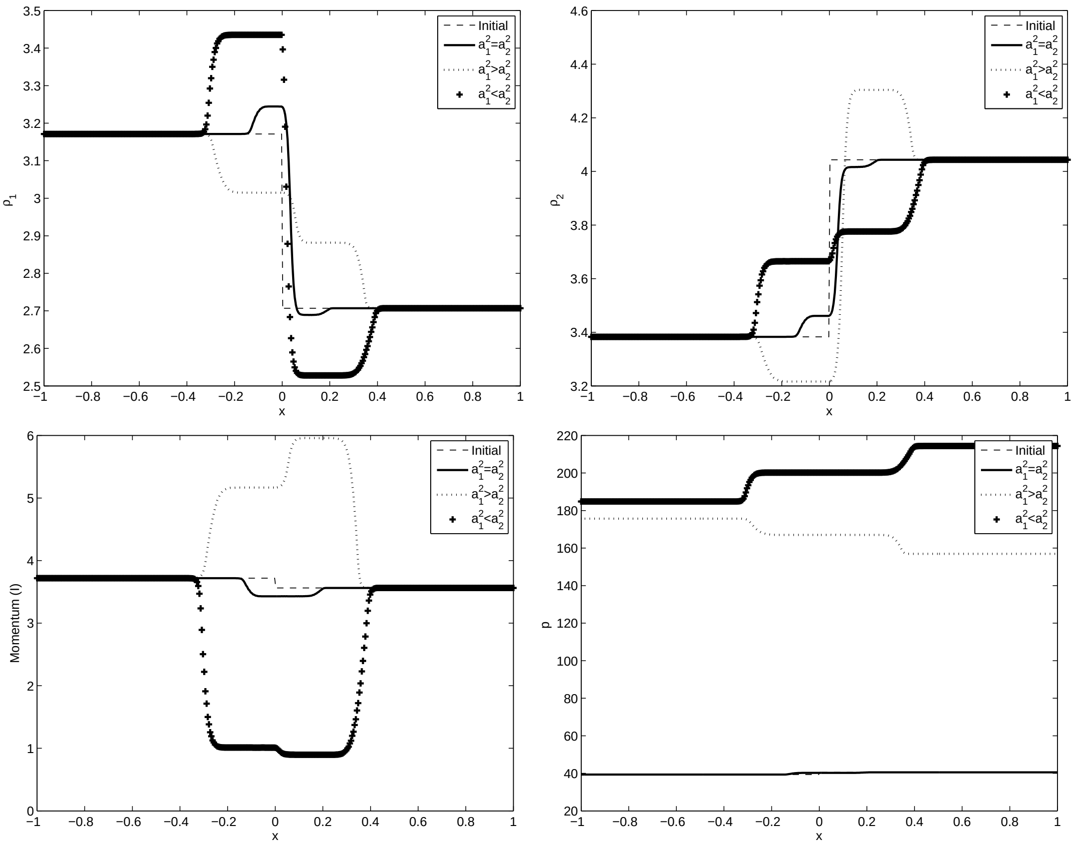

Figure 3: Profiles of densities, common velocity and pressure for coupled models with different compressibility factors at time $t = 0.06$.

In Figure 5 we present the snapshot of momentum for the case of two ingoing and one outgoing pipes.

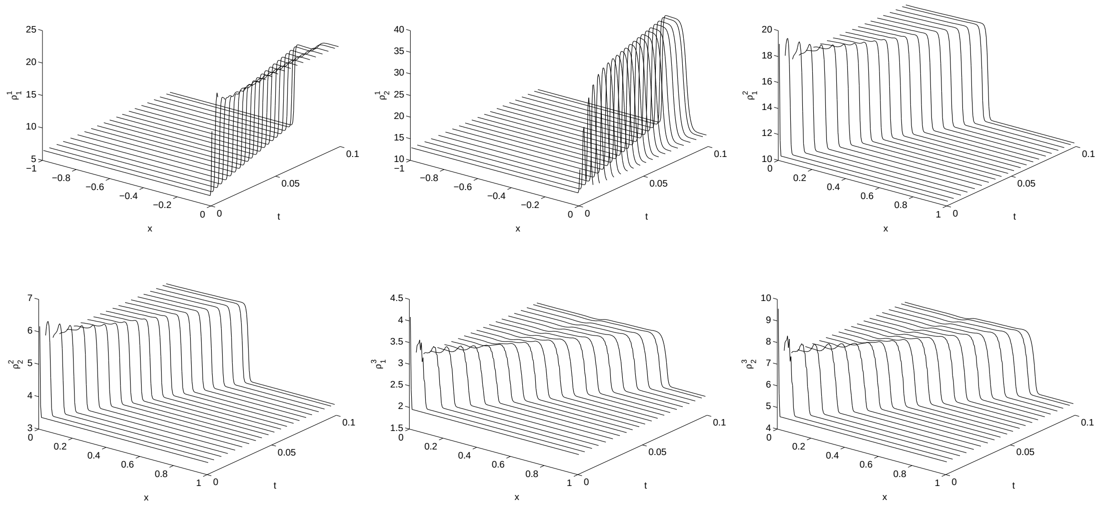

Figure 4: Snapshots of the densities for the solution of the Riemann problem at the junction with one ingoing and two outgoing pipes.

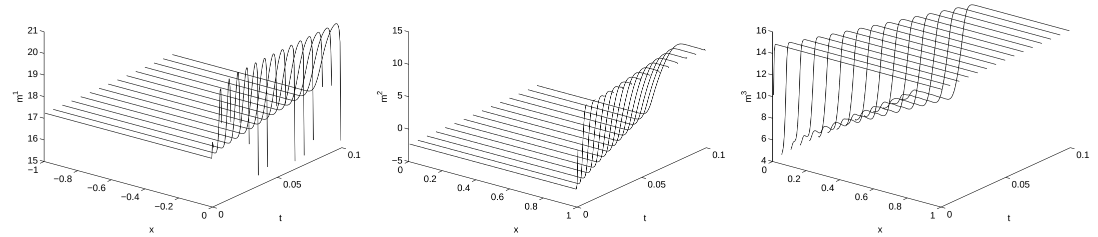

Figure 5: Snapshots of the momentum for the solution of the Riemann problem at the junction with two ingoing and one outgoing pipes.

5. CONCLUSION

We have solved the standard Riemann problem for the multi-phase model (1) with the pressure law (2). For the case of two connected pipes at a junction, the Riemann problem at the junction have been proved to have a unique solution under some conditions. In general we have proven that when the inflow is given and the coupling conditions are defined in a suitable way, one can always solve for the outflow in the outgoing pipe. We have also presented some numerical results that demonstrate the applicability of our model and its possible extension to more general networks.

6. REFERENCES

[1] M. K. Banda. Variants of relaxed schemes and two-dimensional gas dynamics. *J. Comput. and Appl. Math.*, 175(1): 41 – 62, 2005.

[2] M. K. Banda, M. Herty, and A. Klar. Gas flow in pipeline networks. *Networks and Heterogeneous media*, 1(1):41–56, 2006.

[3] M. K. Banda, and M. Herty. Multiscale modelling of gas flow in pipe networks. *Math. Models in Appl. Sci.*, 31(8):915–936, 2008.

[4] M. K. Banda, M. Herty, and J. M. T. Ngnotchouye. Towards a mathematical analysis of multiphase drift-flux model in networks, *SIAM J. Sci. Comput.*, 31(6): 4633 – 4653, 2010.

[5] R. M. Colombo and M. Garavello. On the Cauchy problem for the p-system at a junction. *SIAM J. of Math. Anal.*, 39(5): 1456 – 1471, 2008.

[6] R. M. Colombo and M. Garavello. A well posed Riemann problem for the p-system at a junction. *Networks and Heterogeneous Media*, 1(3):495–511, 2006.

[7] R. M. Colombo and F. Marcellini. Smooth and discontinuous junctions in the p-system. *Journal of Mathematical Analysis and Applications*, 361(2): 440 – 456, 2010.

[8] R. M. Colombo and C. Mauri. Euler system for compressible fluids at a junction. *Journal of Hyperbolic Differential equations*, 5(3):547–568, 2008.

[9] C. M. Dafermos. *Hyperbolic conservation laws in continum physics*, volume 325 of *Grundlehren der mathematischen Wissenschaften.* Springer, 2 edition, 2005.

[10] M. Herty. Coupling conditions for networked systems of Euler equations. *SIAM J. Sci. Comp.*, 30(3):1596–1612, 2007.

[11] R. J. LeVeque. *Finite volume methods for hyperbolic problems.* Cambridge University Press, 2002.