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REDUCED DIFFERENTIAL TRANSFORM METHOD FOR GENERALIZED KDV EQUATIONS

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Abstract- In this paper, a general framework of the reduced differential transform method is presented for solving the generalized Korteweg–de Vries equations. This technique doesn't require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computation. Comparing the methodology with some known techniques shows that the present approach is effective and powerful. In addition, three test problems of mathematical physics are discussed to illustrate the effectiveness and the performance of the reduced differential transform method.

Key Words- Reduced differential transform method, Adomian decomposition method, Variational iteration method, Homotopy perturbation method, Korteweg–de Vries equations

1.INTRODUCTION

Partial differential equations (PDEs) have numerous essential applications in various fields of science and engineering such as fluid mechanic, thermodynamic, heat transfer, physics [1]. One of the most attractive and surprising wave phenomenon is the creation of solitary waves or solitons. Most of these equations are nonlinear partial differential equations. It is difficult to handle nonlinear part of these equations. Although most of scientists applied numerical methods to find the solution of these equations, solving such equations analytically is of fundamental importance since the existent numerical methods which approximate the solution of partial differential equations don't result in such an exact and analytical solution which is obtained by analytical methods. Hirota's bilinear method [2], the balance method [3], inverse scattering method [4], sine–cosine method [5], the homotopy analysis method [6], the homotopy perturbation method (HPM) [7-8], the differential transform method (DTM) [9-11], the variational iteration method (VIM) [12,14,15], the Adomian's decomposition method (ADM) [16-19] are some examples of analytical methods. It was approximately hundred years ago that an adequate theory for solitary waves was developed, in the form of a modified wave equation known as the Korteweg–de Vries equation (KdV) [17].

In this paper we will apply the reduced differential transform method (RDTM) [20-22] to the generalized Korteweg–de Vries equation defined in [23] as follows:

$$u_t + (p+1)(p+2)u^p u_x + u_{xxx} = g(x,t) \tag{1}$$

where $g(x,t)$ is a given function and $p=1,2,...$ with $u,u_x,u_{xxx}\to 0$ as $|x|\to\infty$. If $p=0, p=1$ and $p=2$, Equation (1) becomes linearized KdV, nonlinear KdV, and modified KdV equation, respectively [17,24].

2. ANALYSIS OF THE METHOD

The basic definitions of reduced differential transform method are introduced as follows:

Definition 2.1. If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \tag{2}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k\,. \tag{3}$$

Then combining equation (2) and (3) we write

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k\,. \tag{4}$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion.

For illustration of the methodology of the proposed method, we write the generalized Korteweg–de Vries equation in the standard operator form

$$L(u(x,t))+R(u(x,t))+N(u(x,t))=g(x,t) \tag{5}$$

with initial condition

$$u(x,0)=f(x) \tag{6}$$

where $L=\frac{\partial}{\partial t}$ and $R=\frac{\partial^3}{\partial x^3}$, are linear operators which have partial derivatives, $N(u(x,t))=(p+1)(p+2)u^p u_x$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.

Table 1. Reduced differential transformation [20-22]

Functional Form	**Transformed Form**
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t)=u(x,t)\pm v(x,t)$	$W_k(x)=U_k(x)\pm V_k(x)$
$w(x,t)=\alpha u(x,t)$	$W_k(x)=\alpha U_k(x)$ (α is a constant)
$w(x,y)=x^m t^n$	$W_k(x)=x^m\delta(k-n),\ \delta(k)=\begin{cases}1, k=0\\ 0, k\neq 0\end{cases}$
$w(x,y)=x^m t^n u(x,t)$	$W_k(x)=x^m U_{k-n}(x)$
$w(x,t)=u(x,t)v(x,t)$	$W_k(x)=\sum_{r=0}^{k}V_r(x)U_{k-r}(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$w(x,t)=\frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x)=(k+1)...(k+r)U_{k+1}(x)=\frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W_k(x)=\frac{\partial}{\partial x}U_k(x)$

$Nu(x,t)$	**Maple Code for** Nonlinear Function restart; NF:=Nu(x,t):#Nonlinear Function m:=5: # Order u[t]:=sum(u[b]*t^b,b=0..m): NF[t]:=subs(Nu(x,t)=u[t],NF): s:=expand(NF[t],t): dt:=unapply(s,t): for i from 0 to m do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); #Transform Function od:

According to the RDTM and Table 1, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - R\left(U_k(x)\right) - N\left(U_k(x)\right) \tag{7}$$

where $R\left(U_k(x)\right), N\left(U_k(x)\right)$ and $G_k(x)$ are the transformations of the functions $R\left(u(x,t)\right)$, $N\left(u(x,t)\right)$ and $g(x,t)$ respectively.

For the easy to follow of the reader, we can give the first few nonlinear term are

$$N_0 = (p+1)(p+2)U_0^p(x)\frac{\partial}{\partial x}U_0(x)$$

$$N_1 = (p+1)(p+2)\left(pU_0^{p-1}(x)U_1(x)\frac{\partial}{\partial x}U_0(x) + U_0^p(x)\frac{\partial}{\partial x}U_1(x)\right)$$

$$N_2 = (p+1)(p+2)\left(\begin{array}{l} pU_0^{p-1}(x)U_2(x)\frac{\partial}{\partial x}U_0(x) + \frac{p(p-1)}{2}U_0^{p-2}(x)U_1^2(x)\frac{\partial}{\partial x}U_1(x) + \\ pU_0^{p-1}(x)U_1(x)\frac{\partial}{\partial x}U_1(x) + U_0^p(x)\frac{\partial}{\partial x}U_2(x) \end{array}\right)$$

From initial condition (6), we write

$$U_0(x) = f(x) \tag{8}$$

Substituting (8) into (7) and by a straight forward iterative calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^n$ gives approximation solution as,

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k \tag{9}$$

where n is order of approximation solution.

Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t)\,. \tag{10}$$

3.APPLICATIONS

To illustrate the effectiveness of the present method, several test examples are considered in this section. The accuracy of the method is assessed by comparison with the exact solutions.

Example 1: We consider the inhomogeneous KdV equations [17]

$$u_t-uu_x+u_{xxx}=-e^x-t^2e^{2x} \tag{11}$$

with initial condition

$$u(x,0)=1 \tag{12}$$

Taking differential transform of (11) and the initial condition (12) respectively, we obtain

$$(k+1)U_{k+1}(x)=\sum_{r=0}^{k}U_r(x)\frac{\partial}{\partial x}U_{k-r}(x)-\frac{\partial^3}{\partial x^3}U_k(x)-\delta(k)e^x-\delta(k-2)e^{2x} \tag{13}$$

where the t-dimensional spectrum function $U_k(x)$ are the transformed function.

From the initial condition (12) we write

$$U_0(x)=1 \qquad (14).$$

Now, substituting (14) into (13), we obtain the following $U_k(x)$ values successively

$$U_1(x)=-e^x,\; U_k(x)=0,\; k=2,3,...$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=1-te^x$$

which is exactly the same as that obtained by ADM [17].

Example 2. Consider the KdV equation which takes the form [13,17]:

$$u_t+6uu_x+u_{xxx}=0, x\in R, \tag{15}$$

and initial condition

$$u(x,0)=\frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) \tag{16}$$

where $u=u(x,t)$ is a function of the variables x and t.

Then, by using the basic properties of the reduced differential transformation, we can find the transformed form of equation (15) as

$$(k+1)U_{k+1}(x)=-6\sum_{r=0}^{k}U_{k-r}(x)\frac{\partial}{\partial x}U_r(x)-\frac{\partial^3}{\partial x^3}U_k(x), \text{ for } k=1,2,... \tag{17}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

From the initial condition (16) we write

$$U_0(x)=\frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) \tag{18}$$

Substituting (18) into (17), we obtain the following $U_k(x)$ values successively

$$U_1(x)=\frac{1}{2}\frac{\sinh\left(\frac{x}{2}\right)}{\cosh\left(\frac{x}{2}\right)^3},\quad U_2(x)=\frac{1}{8}\frac{\left(2\cosh\left(\frac{x}{2}\right)^2-3\right)}{\cosh\left(\frac{x}{2}\right)^4}$$

$$U_3(x)=\frac{1}{12}\frac{\sinh\left(\frac{x}{2}\right)\left(\cosh\left(\frac{x}{2}\right)^2-3\right)}{\cosh\left(\frac{x}{2}\right)^5},$$

$$U_4(x)=\frac{1}{96}\frac{\left(2\cosh\left(\frac{x}{2}\right)^4+15-15\cosh\left(\frac{x}{2}\right)^2\right)}{\cosh\left(\frac{x}{2}\right)^6}$$

$$\vdots$$

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\frac{1}{2}\operatorname{sech}^2\left(\frac{x-t}{2}\right)\right]_{t=0}$$

and so on.

Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives n-terms approximation solution as

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k=\frac{1}{2}\frac{1}{\cosh\left(\frac{x}{2}\right)^2}+\frac{1}{2}\frac{\sinh\left(\frac{x}{2}\right)}{\cosh\left(\frac{x}{2}\right)^3}t$$
$$+\frac{1}{96}\frac{\left(24\cosh\left(\frac{x}{2}\right)^2-36\right)}{\cosh\left(\frac{x}{2}\right)^6}t^2+\ldots+\frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}\frac{1}{2}\operatorname{sech}^2\left(\frac{x-t}{2}\right)\right]_{t=0}t^n \tag{19}$$

Therefore, the exact solution of problem is given by

$$u(x,y)=\lim_{n\to\infty}\tilde{u}_n(x,y).$$

The approximate solution $\tilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k$ is convergent to the exact solution as in [17] and it is also analogous to the approximate solution obtained by variational iteration method in [25]. (see Figure 1). Therefore the solution is obtained as

$$u(x,t)=\frac{1}{2}\operatorname{sech}^2\left(\frac{x-t}{2}\right) \tag{20}$$

which is the exact solutions of (14)–(15).

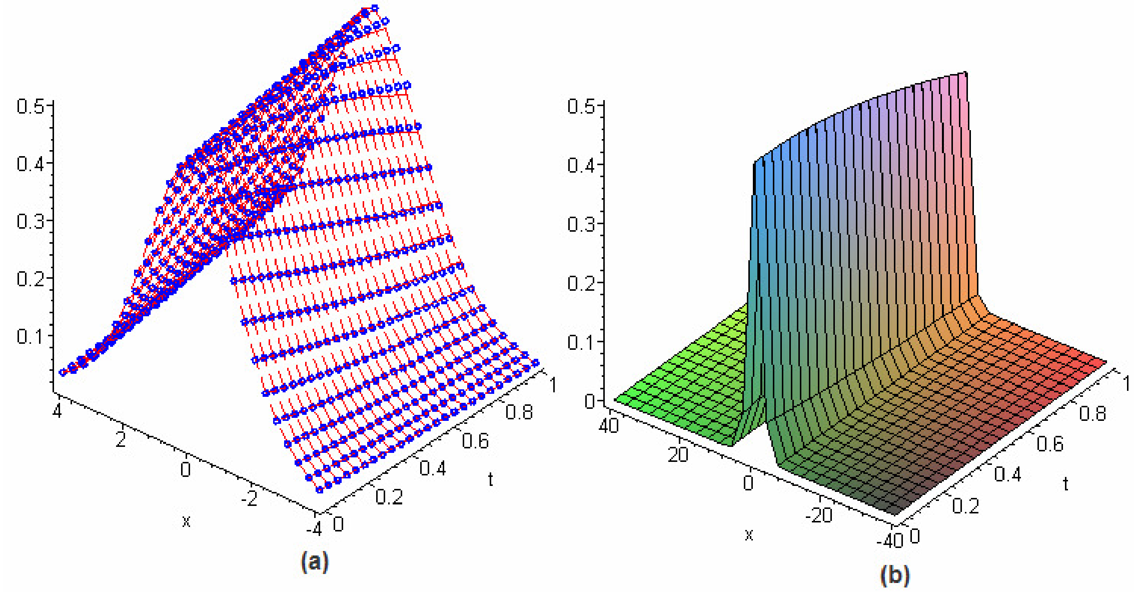

Figure 1. The numerical results for $u_4(x,y)$: (a) in comparison with the analytical solutions Eq.(20), (b) for the solitary wave solution with the initial condition of Example 2.

Figure 1 shows the comparison of the approximate solution by RDTM of order four, the (a) exact solution Eq.(20) the solid line represents the solution by the reduced differential transform method, while the circle represents the exact solution. From the figure 1, it is clearly seen that the RDTM approximation and the exact solution are in good agreement.

Example 3. Lastly, Consider the mKdV equation [13]

$$u_t+6u^2u_x+u_{xxx}=0 \tag{21}$$

and initial conditions

$$u(x,0)=\operatorname{sech}(x) \tag{22}$$

Taking differential transform of (21) and the initial condition (22) respectively, we obtain

$$(k+1)U_{k+1}(x)=-N_k(x)-\frac{\partial^3}{\partial x^3}U_k(x) \tag{23}$$

and the transformed initial condition

$$U_0(x) = \operatorname{sech}(x) \tag{24}$$

Now, substituting (24) into (25), we obtain the following $U_k(x)$ values successively

$$U_1(x) = \frac{\sinh(x)}{\cosh(x)^2}, \quad U_2(x) = \frac{1}{2}\frac{\cosh(x)^2 - 2}{\cosh(x)^3},$$

$$U_3(x) = \frac{1}{6}\frac{\sinh(x)(-6+\cosh(x)^2)}{\cosh(x)^4}$$

$$U_4(x) = \frac{1}{24}\frac{\cosh(x)^4 - 20\cosh(x)^2 + 24}{\cosh(x)^5},$$

$$U_5(x) = \frac{1}{120}\frac{\sinh(x)(\cosh(x)^4 - 60\cosh(x)^2 + 120)}{\cosh(x)^6}$$

$$\vdots$$

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\operatorname{sech}(x-t)\right]_{t=0}$$

and so on.

Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives n-terms approximate solution as

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k = \frac{1}{\cosh(x)} + \frac{\sinh(x)}{\cosh(x)^2}t + \frac{\cosh(x)^2 - 2}{2\cosh(x)^3}t^2 + \frac{\sinh(x)(-6+\cosh(x)^2)}{6\cosh(x)^4}t^3 + \ldots + \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n}\operatorname{sech}(x-t)\right]_{t=0} t^n$$

Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\to\infty}\tilde{u}_n(x,t).$$

This solution is convergent to the exact solution as in [25] and the same as approximate solution of the variational iteration method [25]. (see Table 2)

$$u(x,t) = \operatorname{sech}(x-t)$$

which is the exact solutions of (21)–(22)

Table 2. Comparison of the approximate solutions with exact solution $u(x,t)=\operatorname{sech}(x-t)$

$x-t$	Exact	RDTM	VIM	Absolute error: exact solution and RDTM (three iteration)	Absolute error: exact solution and VIM (three iteration)
0-0	1	1	1	0	0
0.1-0.1	1	0.99998000	0.99848722	0.00001999	0.00151277
0.2-0.2	1	0.99971792	0.99009788	0.00028207	0.00990211
0.3-0.3	1	0.99885767	1.03152422	0.00114232	0.03152422
0.4-0.4	1	0.99745757	1.28189069	0.00254242	0.28189069
0.5-0.5	1	0.99646993	1.81976294	0.00353006	0.81976294
0.6-0.6	1	0.99779444	2.37421865	0.00220555	1.37421865
0.7-0.7	1	1.00385275	2.31840591	0.00385275	1.31840591
0.8-0.8	1	1.01689213	1.06357992	0.01689213	0.06357992
0.9-0.9	1	1.03834358	-1.39360993	0.03834358	2.39360997
1.0-1.0	1	1.06844832	-4.17836604	0.06844831	5.17836604

4. CONCLUSION

The main concern of this article is to construct an approximate analytical solution for the generalized Korteweg–de Vries equation. We have achieved this goal by applying reduced differential transform method. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms. Analytical solutions enable researchers to study the effect of different variables or parameters on the function under study easily. Its small size of computation in comparison with the computational size required in other numerical methods, and its rapid convergence show that the method is reliable and introduces a significant improvement in solving the generalized Korteweg–de Vries equation over existing methods. As the method is usually tedious to use by hand, we have to use the Maple Package to calculate the series obtained from the RDTM.

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