Mathematical and Computational Applications, Vol. 15, No. 3, pp. 354-363, 2010.

MHD FLOW OF A SECOND ORDER/GRADE FLUID DUE TO NON-COAXIAL ROTATION OF A POROUS DISK AND THE FLUID AT INFINITY

H. Volkan Ersoy
Department of Mechanical Engineering
Yildiz Technical University, 34349
Yildiz, İstanbul, Turkey
ersoyhv@yahoo.com

Abstract- The magnetohydrodynamic (MHD) flow of an electrically conducting second order/grade fluid past a porous disk is studied when the disk and the fluid at infinity rotate with the same angular velocity about non-coincident axes. It is found that the existence of solutions is in connection with the sign of the material modulus α_1 for both suction and blowing cases. The effects of all the parameters on the flow are carefully examined.

Key Words- fluid rotating at infinity, non-coaxial rotation, second order/grade fluid, magnetohydrodynamics.

1. INTRODUCTION

The flow induced by non-coaxial rotation of a disk and a fluid at infinity has attracted the interest of many investigators. Following Coirier [1], Erdoğan [2] examined the flow produced by the rotation non-coaxially of a porous disk and a Newtonian fluid at infinity with the same angular velocity. Murthy and Ram [3] extended the flow in [2] to the magnetohydrodynamic flow and studied the effect of heat transfer. Ersoy [4] analysed the flow of an Oldroyd-B fluid for a porous disk in the presence of a uniform magnetic field. The case of the flow of a second order/grade fluid past a porous disk was studied by Ersoy and Barış [5]. Hayat *et al.* [6] examined the flow of a second grade fluid past a porous disk under the influence of an applied magnetic field, depending on the restrictions $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$, and solved the problem using a perturbation method. They also studied the MHD flow between two porous disks rotating about a common axis. Chakraborti *et al.* [7] reconsidered the flow in [3] and analysed the flow in detail. Apart from steady flows, for unsteady flows in the same geometry, we refer the reader to [8-18] for a Newtonian fluid and to [19-27] for various non-Newtonian fluids. In addition, the reader may consult [28-32] for the studies that deal with the flow of non-Newtonian fluids between two disks rotating about non-coincident axes in the presence of a magnetic field.

In this paper, we are concerned with the flow of an incompressible and electrically conducting second order/grade fluid caused by the non-coaxial rotation of a porous disk and the fluid at infinity with the common angular velocity under the application of a uniform magnetic field. It should be pointed out clearly that our main purpose is to examine the problem depending on the sign of the material modulus α_1. The flow is characterized by non-dimensional parameters β (the elastic parameter), e

(the suction-blowing parameter) and N (the magnetic parameter). All results we have found are drawn in the figures.

2. BASIC EQUATIONS AND SOLUTION

In a Cartesian coordinate system, let us consider a porous disk in the xy-plane rotating counterclockwise at a constant rate of Ω about the z- axis perpendicular to the disk. A second order/grade fluid is present in the upper half-space $z \geq 0$. The axis of rotation of the fluid at infinity which rotates at equal angular velocity with the disk is parallel to Oz axis and passes through the point O' (x=0, $y=\ell$). A uniform magnetic induction $\mathbf{B}_0$ acts normal to the insulated disk, i. e. along z-direction. We assume that the induced magnetic field is negligible in comparison with the applied magnetic field.

The Cauchy stress $\boldsymbol{T}$ in an incompressible and homogeneous second order/grade fluid is given by Rivlin and Ericksen [33]

$$\boldsymbol{T} = -p\,\boldsymbol{I} + \mu\,\boldsymbol{A}_1 + \alpha_1 \boldsymbol{A}_2 + \alpha_2 \boldsymbol{A}_1^2 \tag{1}$$

where p is the pressure, μ the dynamic viscosity of the fluid, α_1 and α_2 the material moduli which are usually referred to as the normal stress coefficients. In the above representation, $\boldsymbol{I}$ is the identity tensor, and the kinematical tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are defined through

$$\boldsymbol{A}_1 = (\mathrm{grad}\,\mathbf{v}) + (\mathrm{grad}\,\mathbf{v})^T, \qquad \boldsymbol{A}_2 = \frac{D\boldsymbol{A}_1}{Dt} + \boldsymbol{A}_1\,(\mathrm{grad}\,\mathbf{v}) + (\mathrm{grad}\,\mathbf{v})^T\,\boldsymbol{A}_1 \tag{2}$$

where $\mathbf{v}$ is the velocity vector and D/Dt the material time derivative. We notice that if $\alpha_1 = \alpha_2 = 0$ the model Eq.(1) reduces to the classical linearly viscous fluid model.

The thermodynamical principles impose some restrictions on α_1 and α_2 [34]. In particular, the Clasius-Duhem inequality implies that

$$\mu \geq 0, \qquad \alpha_1 + \alpha_2 = 0 \tag{3a}$$

and the requirement that the specific Helmholtz free energy be a minimum in equilibrium implies that

$$\alpha_1 \geq 0 \tag{3b}$$

The fluids characterized by above restrictions are called the second grade fluids in the literature. On the other hand, the model Eq.(1) is called a second order fluid model ($\alpha_1 < 0$ and $\alpha_1 + \alpha_2 \neq 0$), which is in good agreement with experimental results, if it is not required to be compatible with thermodynamics [35]. Therefore, it is clear that the results established for the case $\alpha_1 > 0$ have more value than the solution $\alpha_1 < 0$. In this study, we consider both positive and negative values of α_1. Moreover, another result that emerges from this analysis is that there is no effect of the material modulus α_2 on the velocity field.

The governing equations are

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{T} + \mathbf{J} \times \mathbf{B}, \qquad \nabla \cdot \mathbf{v} = 0,$$
$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \qquad \nabla \times \mathbf{E} = 0, \qquad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{4a-f}$$

where ρ is the density, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic induction, μ_m the magnetic permeability, $\mathbf{E}$ the electric field, and σ is the electrical conductivity of the fluid.

The boundary conditions for the velocity field are taken to be

$$\begin{aligned} &u = -\Omega y, && v = \Omega x, && w = \text{constant} && \text{at } z = 0 \\ &u = -\Omega(y - \ell), && v = \Omega x, && w = \text{constant} && \text{at } z \to \infty \end{aligned} \tag{5}$$

where u, v, w denote the x, y, z components of the velocity, respectively.

We seek a solution, compatible with the continuity equation (4b), such as to have the following form:

$$u = -\Omega y + f(z), \qquad v = \Omega x + g(z), \qquad w = \text{constant} \tag{6}$$

The appropriate boundary conditions for $f(z)$ and $g(z)$ from Eqs.(5) and (6) are

$$f(0) = 0, \qquad g(0) = 0, \qquad f(\infty) = \Omega \ell, \qquad g(\infty) = 0 \tag{7}$$

From Eqs.(1), (2), (4a) and (6), one has

$$\frac{\partial p}{\partial x} = \rho\Omega(\Omega x + g) - \rho w f' + \mu f'' + \alpha_1(\Omega g'' + w f''') + J_y B_0 \tag{8a}$$

$$\frac{\partial p}{\partial y} = -\rho\Omega(-\Omega y + f) - \rho w g' + \mu g'' + \alpha_1(-\Omega f'' + w g''') - J_x B_0 \tag{8b}$$

$$\frac{\partial p}{\partial z} = 2(2\alpha_1 + \alpha_2)(f' f'' + g' g'') \tag{8c}$$

where a prime denotes differentiation with respect to z. Using Eq.(4f), we obtain

$$J_x = \sigma(E_x + v B_0), \qquad J_y = \sigma(E_y - u B_0), \qquad J_z = \sigma E_z \tag{9}$$

Bearing in mind that the disk is non-conducting, when we use the current conservation equation $\nabla \cdot \mathbf{J} = 0$ which is a consequence of Eq.(4d) with Eqs.(4e), (8a-c) and (9), we have

$$\rho\left(\Omega g - wf'\right) + \mu f'' + \alpha_1\left(\Omega g'' + wf'''\right) - \sigma B_0^2 f = \text{constant} \tag{10a}$$

$$-\rho\left(\Omega f + wg'\right) + \mu g'' - \alpha_1\left(\Omega f'' - wg'''\right) - \sigma B_0^2 g = \text{constant} \tag{10b}$$

Defining $F(z) = f + ig$, Eqs. (10a-b) reduce to the following equation

$$\alpha_1 wF''' + (\mu - i\alpha_1\Omega)F'' - \rho wF' - (\sigma B_0^2 + i\rho\Omega)F = \text{constant} \tag{11}$$

with the conditions

$$F(0) = 0, \qquad F(\infty) = \Omega\ell \tag{12}$$

Furthermore, all derivatives of $F(z)$ go to zero as $z \to \infty$ because the fluid at infinity is free of shear stress. Thus, we find that the constant in (11) is equal to $-\Omega\ell(\sigma B_0^2 + i\rho\Omega)$.

Let us make the variables non-dimensional by the following substitutions:

$$\Gamma = \frac{F}{\Omega\ell}, \quad \zeta = \sqrt{\frac{\Omega}{2\nu}}z, \quad \beta = \frac{\alpha_1\Omega}{\mu}, \quad e = \frac{w}{2\sqrt{\Omega\nu}}, \quad N = \frac{\sigma B_0^2}{\rho\Omega} \tag{13}$$

Here ν denotes the kinematic viscosity of the fluid, β the elastic parameter, e the suction-blowing parameter, and N is the magnetic parameter. As seen from $e = w/(2\sqrt{\Omega\nu})$, the case of suction corresponds to e<0 and the case of blowing to e>0. The non-dimensional equation becomes

$$\sqrt{2}\beta e\Gamma''' + (1 - i\beta)\Gamma'' - 2\sqrt{2}e\Gamma' - 2(N + i)\Gamma = -2(N + i) \tag{14}$$

with the conditions as follows

$$\Gamma(0) = 0, \qquad \Gamma(\infty) = 1, \qquad \Gamma'(\infty) = \Gamma''(\infty) = \Gamma'''(\infty) = \ldots = 0 \tag{15}$$

It is noticed that Eq.(14) is one order higher than the Navier-Stokes equations due to the viscoelasticity of the fluid. It would thus appear that the additional boundary condition must be imposed to determine the solution completely. The issue of difficulties with regard to prescribing boundary conditions is discussed in detail by Rajagopal [36]. Since the flow under consideration takes place in unbounded domain, we are able to overcome this difficulty by using asymptotic conditions and boundedness of solutions, as in the study of Rajagopal and Gupta [37]. They examined the existence of solutions that is tied in with the sign of material modulus α_1 for the flow of a second order/grade fluid past an infinite porous plate subjected to either suction or blowing at the plate. They found that if the material modulus $\alpha_1 > 0$ it is possible to exhibit an exact solution which is asymptotic in nature for both suction and blowing at the plate. However, in the case of $\alpha_1 < 0$, they found that such solutions cannot exist for the blowing case.

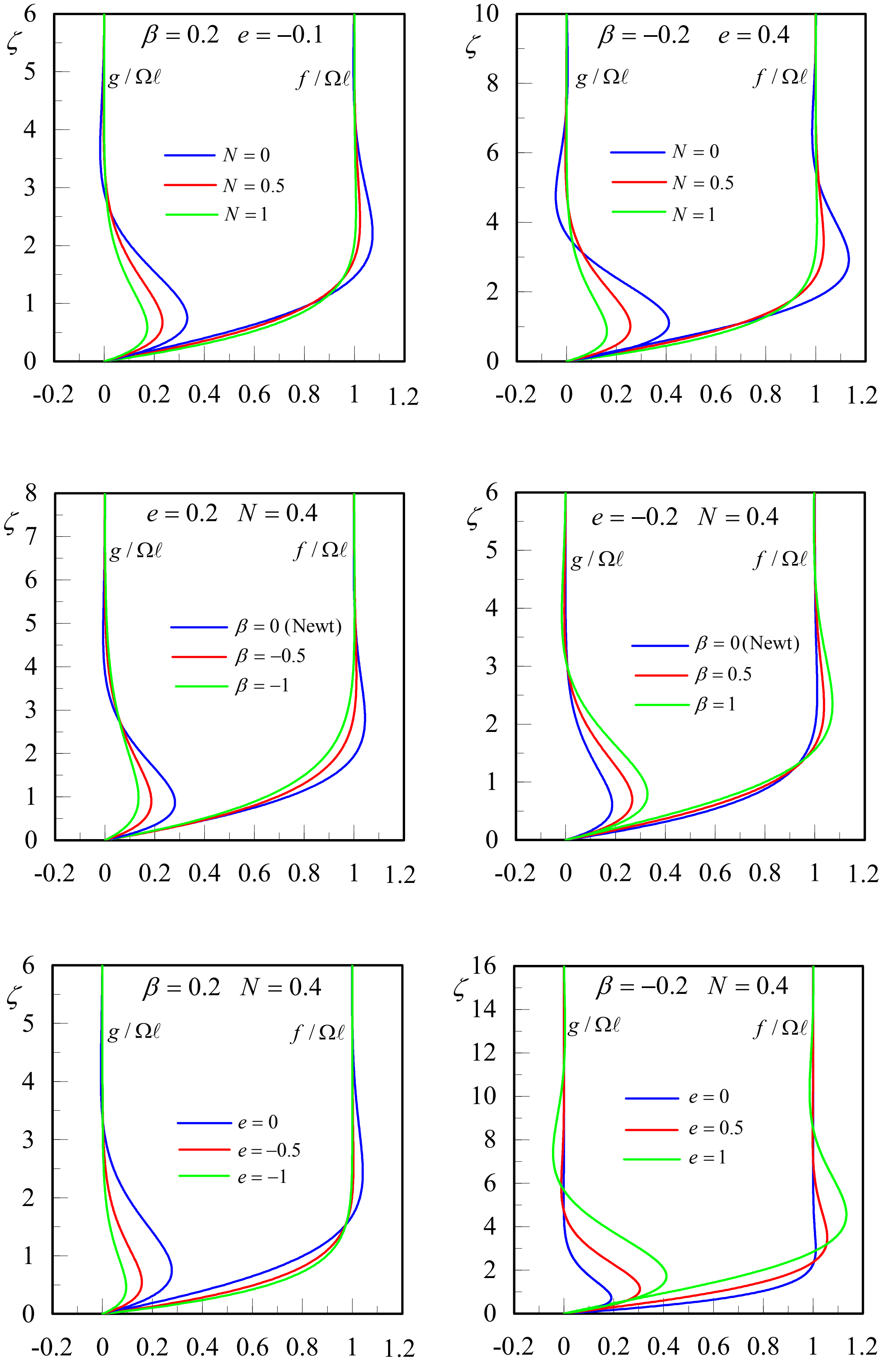

Figure 1- Profiles of $f/\Omega\ell$ and $g/\Omega\ell$ for various values of β, e and N.

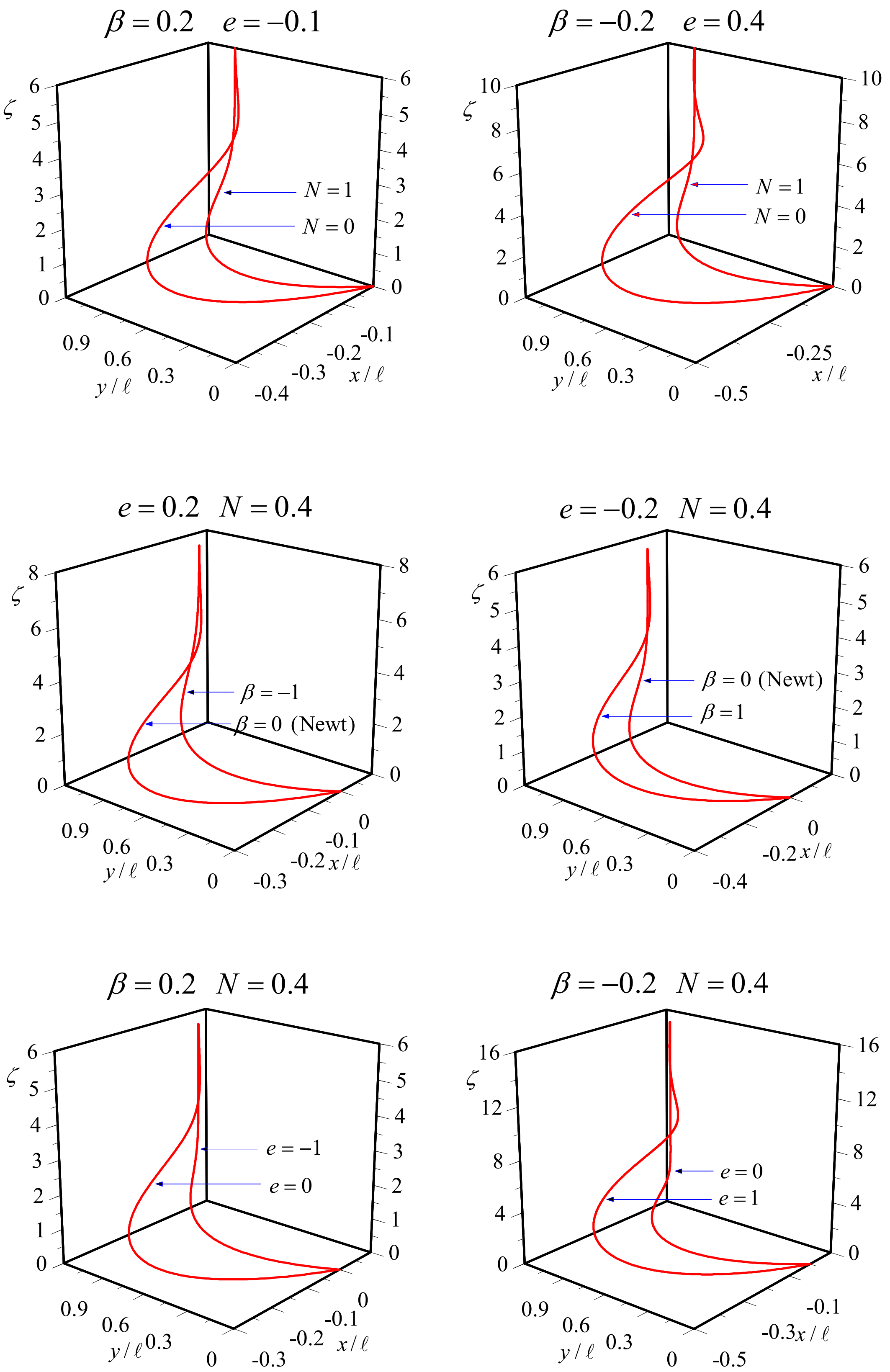

Figure 2- Space curves consisting of the points about which the fluid layers rotate as a rigid body with a constant angular velocity Ω.

The characteristic equation of Eq.(14) is in the form of a cubic equation and has three roots. In order to obtain physically acceptable solutions to Eq.(14) under the conditions (15) this characteristic equation must have only one complex root with negative real part. Otherwise, the conditions Eq.(15) will not suffice to get physically acceptable solutions. It is for this reason that there exist above mentioned solutions for $\alpha_1 > 0$ in the case of suction, on the other hand, for $\alpha_1 < 0$ in the case of blowing. The variations of $f/\Omega\ell$ and $g/\Omega\ell$ for various values of parameters are plotted against ζ in Figure 1.

3. DISCUSSION

When a porous disk and a fluid at infinity rotate eccentrically with the same angular velocity Ω, there exists a single point in each plane *z*=constant where the velocity vector has only the axial component and about which the fluid layer rotates as a rigid body with the angular velocity Ω. The coordinates of this point are given by $x = -g(z)/\Omega$ and $y = f(z)/\Omega$ for $0 \le z < \infty$. Figure 2 shows these space curves for various values of the parameters β, *e* and *N*. The graphs $f/\Omega\ell$ and $-g/\Omega\ell$ plotted versus ζ in Figure 1 are the projections of the above mentioned space curves on the *yz*-plane and the *xz*-plane, respectively.

The following conclusions can be extracted from our analysis:

1. The positive sign of the material modulus α_1 brings out physically acceptable solutions for the suction case, whereas its negative sign is meaningful for the blowing case.
2. It is a well-known fact that suction and blowing have opposite characteristics on the boundary layer flows. It is clearly shown that the suction causes a thinning of the boundary layer, whereas the blowing leads to a reverse effect.
3. The presence of externally applied magnetic field brings about a thin boundary layer near the disk.
4. The effect of magnetic field on any flow is an important problem related to many practical applications as in the case of boundary layer flow control. Since the blowing causes an increment in the boundary layer thickness, it is shown that the boundary layer can be controlled by applying a magnetic field.
5. Increasing the fluid elasticity causes thickening of the boundary layer for the suction case, whereas the reverse is true for the blowing case.
6. There is no effect of the material modulus α_2 on the velocity field since both the disk and the fluid at infinity rotate with the same speed.

Acknowledgement- The author would like to thank the referee for his valuable comments.

4. REFERENCES

1. J. Coirier, Rotations non coaxiales d'un disque et d'un fluide à l'infini, *Journal de Mécanique* **11**, 317-340, 1972.
2. M. E. Erdoğan, Flow due to eccentric rotating a porous disk and a fluid at infinity, *Transactions of the ASME: Journal of Applied Mechanics* **43E**, 203-204, 1976.
3. S. N. Murthy and R. K. P. Ram, MHD flow and heat transfer due to eccentric rotations of a porous disc and a fluid at infinity, *International Journal of Engineering Science* **16**, 943-949, 1978.
4. H. V. Ersoy, MHD flow of an Oldroyd-B fluid due to non-coaxial rotations of a porous disk and the fluid at infinity, *International Journal of Engineering Science* **38**, 1837-1850, 2000.
5. H. V. Ersoy and S. Barış, Flow of a second order/grade fluid due to non-coaxial rotation of a porous disk and the fluid at infinity, *International Journal of Applied Mechanics and Engineering* **7**, 1189-1199, 2002.
6. T. Hayat, M. I. Hameed, S. Asghar and A. M. Siddiqui, Some steady MHD flows of the second order fluid, *Meccanica* **39**, 345-355, 2004.
7. A. Chakraborti, A. S. Gupta, B. K. Das and R. N. Jana, Hydromagnetic flow past a rotating porous plate in a conducting fluid rotating about a noncoincident parallel axis, *Acta Mechanica* **176**, 107-119, 2005.
8. I. Pop, Unsteady flow due to noncoaxially rotating disk and a fluid at infinity, *Bulletin of the Technical University of Istanbul* **32**, 14-18, 1979.
9. M. E. Erdoğan, Unsteady flow of a viscous fluid due to non-coaxial rotations of a disk and a fluid at infinity, *International Journal of Non-Linear Mechanics* **32**, 285-290, 1997.
10. T. Hayat, S. Asghar and A. M. Siddiqui, Unsteady flow of an oscillating porous disk and a fluid at infinity, *Meccanica* **34**, 259-265, 1999.
11. M. E. Erdoğan, Flow induced by non-coaxial rotation of a disk executing non-torsional oscillations and a fluid rotating at infinity, *International Journal of Engineering Science* **38**, 175-196, 2000.
12. T. Hayat, S. Asghar, A.M. Siddiqui and T. Haroon, Unsteady MHD flow due to non-coaxial rotations of a porous disk and a fluid at infinity, *Acta Mechanica* **151**, 127-134, 2001.
13. T. Hayat, M. Zamurad, S. Asghar and A. M. Siddiqui, Magnetohydrodynamic flow due to non-coaxial rotations of a porous oscillating disk and a fluid at infinity, *International Journal of Engineering Science* **41**, 1177-1196, 2003.
14. H. V. Ersoy, Unsteady viscous flow induced by eccentric-concentric rotation of a disk and the fluid at infinity, *Turkish Journal of Engineering and Environmental Sciences* **27**, 115-123, 2003.
15. T. Hayat, R. Ellahi and S. Asghar, Unsteady periodic flows of a magnetohydro-dynamic fluid due to noncoaxial rotations of a porous disk and a fluid at infinity, *Mathematical and Computer Modelling* **40**, 173-179, 2004.
16. M. Guria, S. Das and R. N. Jana, Hall effects on unsteady flow of a viscous fluid due to non-coaxial rotation of a porous disk and a fluid at infinity, *International Journal of Non-Linear Mechanics* **42**, 1204-1209, 2007.

17. S. Asghar, K. Hanif and T. Hayat, The effect of the slip condition on unsteady flow due to non-coaxial rotations of disk and a fluid at infinity, *Meccanica* **42**, 141-148, 2007.
18. M. Guria, B. K. Das and R. N. Jana, Oscillatory flow due to eccentrically rotating porous disk and a fluid at infinity, *Meccanica* **42**, 487-493, 2007.
19. A. M. Siddiqui, T. Haroon, T. Hayat and S. Asghar, Unsteady MHD flow of a non-Newtonian fluid due to eccentric rotations of a porous disk and a fluid at infinity, *Acta Mechanica* **147**, 99-109, 2001.
20. T. Hayat, T. Haroon, S. Asghar and A. M. Siddiqui, MHD flow of a third-grade fluid due to eccentric rotations of a porous disk and a fluid at infinity, *International Journal of Non-Linear Mechanics* **38**, 501-511, 2003.
21. T. Hayat and Y. Wang, Magnetohydrodynamic flow due to noncoaxial rotations of a porous disk and a fourth-grade fluid at infinity, *Mathematical Problems in Engineering* **2003**, 47-64, 2003.
22. T. Hayat, R. Ellahi and S. Asghar, Flow induced by non-coaxial rotation of a porous disk executing non-torsional oscillations and a second grade fluid rotating at infinity, *Applied Mathematical Modelling* **28**, 591-605, 2004.
23. H. V. Ersoy, Unsteady flow of a Maxwell fluid induced by non-coaxial rotation of a disk and the fluid at infinity, *International Journal of Applied Mechanics and Engineering* **10**, 159-165, 2005.
24. Y. Wang and W. Wu, Time-dependent magnetohydrodynamic flow induced by non-coaxial rotations of a non-torsionally oscillating porous plate and a third-order fluid at infinity, *Mathematical and Computer Modelling* **46**, 1277-1293, 2007.
25. T. Hayat, R. Ellahi and S. Asghar, Unsteady magnetohydrodynamic non-Newtonian flow due to non-coaxial rotations of disk and a fluid at infinity, *Chemical Engineering Communications* **194**, 37-49, 2007.
26. S. Asghar, K. Hanif, T. Hayat, C. M. Khalique, MHD non-Newtonian flow due to non-coaxial rotations of an accelerated disk and a fluid at infinity, *Communications in Nonlinear Science and Numerical Simulation* **12**, 465-485, 2007.
27. T. Hayat, R. Ellahi and S. Asghar, Hall effects on unsteady flow due to non-coaxially rotating disk and a fluid at infinity, *Chemical Engineering Communications* **195,** 958-976, 2008.
28. R. Rao and P. R. Rao, MHD flow of a second grade fluid in an orthogonal rheometer, *International Journal of Engineering Science* **23**, 1387-1395, 1985.
29. H. V. Ersoy, MHD flow of an Oldroyd-B fluid between eccentric rotating disks, *International Journal of Engineering Science* **37**, 1973-1984, 1999.
30. S. Asghar, M. R. Mohyuddin and T. Hayat, Effects of Hall current and heat transfer on flow due to a pull of eccentric rotating disks, *International Journal of Heat and Mass Transfer* **48**, 599–607, 2005.
31. A. M. Siddiqui, M. A. Rana and N. Ahmed, Effects of Hall current and heat transfer on MHD flow of a Burgers' fluid due to a pull of eccentric rotating disks, *Communications in Nonlinear Science and Numerical Simulation* **13**, 1554-1570, 2008.
32. T. Hayat, K. Maqbool and M. Khan, Hall and heat transfer effects on the steady flow of a generalized Burgers' fluid induced by a sudden pull of eccentric rotating disks, *Nonlinear Dynamics* **51**, 267-276, 2008.

33. R. S. Rivlin and J. L. Ericksen, Stress-deformation relations for isotropic materials, *Journal of Rational Mechanics and Analysis* **4**, 323-425, 1955.
34. J. E. Dunn and R. L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity two and fluids of second grade, *Archive for Rational Mechanics Analysis* **56**, 191-252, 1974.
35. R. L. Fosdisk and K. R. Rajagopal, Anomalous features in the model of second order fluids, *Archive for Rational Mechanics Analysis* **70**, 145-152, 1979.
36. K. R. Rajagopal, On boundary conditions for fluids of the differential type, In: Navier-Stokes Equations and Related Non-Linear Problems (A. Sequiera, Ed.), Plenum Press, New York, 1994.
37. K. R. Rajagopal and A. S. Gupta, An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate, *Meccanica* **19**, 158-160, 1984.