PADE EMBEDDED PIECEWISE DIFFERENTIAL TRANSFORM METHOD FOR THE SOLUTION OF ODE’S

Y. Çenesiz, A.B. Koç, B. Çitil, A. Kurnaz
Science Faculty, Department of Mathematics,
Selçuk University, Konya 42003, Turkey
ycenesiz@selcuk.edu.tr

Abstract- In this paper Pade Embedded Differential Transformation is proposed for the solution of higher order nonlinear or linear Ordinary Differential Equations (ODE’s). The proposed approach provides a better iterative procedure to find the spectrum of the analytic solutions compared to the classical differential transformation. Illustrative examples are presented to show the preciseness and effectiveness of the proposed method.

Keywords- Differential Transform Method, Pade Approximants

1. INTRODUCTION

In the 1980’s, Zhou [1] proposed the Differential Transformation (DT) Method for the solution of electrical circuits and, then, many researchers have applied this method for solving different types of differential equations [1].

It is important to note that a large amount of research works has been devoted to the application of DT method to a wide class of stochastic and deterministic problems involving differential, integro-differential and systems of such equations [6-10,13]. The method is well suited to physical problems since it makes unnecessary restrictive methods and assumptions [8-11] which may change the problem being solved, sometimes seriously.

The DT method was used by many researchers to investigate scientific applications. Chen applied DT method to handle nonlinear heat conduction problems [6] and used this method to solve the transient advective dispersive transport equation[11].

The differential transformation technique is based upon Taylor series expansion and provides iterative procedures to obtain higher-order power series. The basic definitions of the differential transformation are introduced as follows [5-11]

If \( y(t) \) is analytic in the domain \( T \) and continuously differentiable with respect to \( t \), then let

\[
\phi(t,k) = \frac{d^k y(t)}{dt^k} \quad \forall t \in T
\]

where \( k \) belongs to the set of nonnegative integers denoted as K-domain.

For \( t = t_i \), then \( \phi(t,k) = \phi(t_i,k) \)

Definition : The differential transform of the function \( y(t) \) is defined by
\[ Y(k) = \phi(t_i, k) = \frac{1}{k!} \left[ \frac{d^k y(t)}{dt^k} \right]_{t=t_i}, \ \forall k \in K \]  

(1)

where \( Y(k) \) is called the spectrum of \( y(k) \) at \( t = t_i \) in the \( K \) domain.

**Definition**: The inverse transformation of \( Y(k) \) is defined by

\[ y(t) = \sum_{k=0}^\infty (t-t_i)^k Y(k). \]  

(2)

If \( Y(k) \) is defined as

\[ Y(k) = M(k) \left[ \frac{d^k q(t)y(t)}{dt^k} \right]_{t=t_i}, \ \forall k \in K \]  

(3)

then the function \( y(t) \) can be represented by

\[ y(t) = \frac{1}{q(t)} \sum_{k=0}^\infty \frac{(t-t_i)^k Y(k)}{M(k)} \]  

(4)

where \( M(k) \neq 0, q(t) \neq 0 \). \( M(k) \) is called the weighting factor and \( q(t) \) is regarding as a kernel corresponding to \( y(t) \). If \( M(k) = 1 \) and \( q(t) = 1 \) then eq.(1) and eq.(3) are equivalent. In this paper we use the transformation with \( M(k) = q(t) = 1 \).

The fundamental mathematical operations performed by the differential transformation can be found in \[6,8-10\].

After transforming the differential equation in to the \( K \) domain, the solution can be obtained by finite-term Taylor series plus a remainder as

\[ y(t) = \sum_{k=0}^N (t-t_i)^k Y(k) + R_{N+1}(t) \]

where

\[ R_{N+1}(t) = \sum_{k=N+1}^\infty \frac{(t-t_i)^k Y(k)}{M(k)}. \]

**2. PADE APPROXIMANTS**

If the given power series converges to the same function for \( |z| < \Re \) with \( 0 < \Re < \infty \), then a sequence of Padé approximants can converge for \( z \in D \), where \( D \) is a domain larger than \( |z| < \Re \). We will then have extended our domain of convergence. This is frequently a practical approach to what amounts to analytic continuation [2]. On the other hand, if often happens that the Padé approximant converges to the true solution of a differential equation, even when the Taylor series solution diverges.

A Padé approximant [2-4,14] is a ratio of polynomials that contains the same information that a truncated power series does. The \( \left[ \frac{L}{M} \right] \) Padé approximant to the \( k \) th order Taylor series \( y(t) = \sum_{i=0}^k c_i t^i \) is given by
with the polynomial \( R_L(t) \) in the numerator having degree \( L \) and the polynomial \( Q_M(t) \) in the denominator having degree \( M \). Without loss of generality, we can take \( A_0 = 1 \).

The remaining \( L + M + 1 = k \) coefficients are chosen so that first \( L + M + 1 \) terms of the Taylor series expansion of \( P[L/M] \) match the first \( k \) terms of the Taylor series. For the case \( L \geq M - 1 \), the polynomials \( Q_M \) and \( R_L \) can be written respectively.

\[
Q_M = \begin{bmatrix}
c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\
c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\
t^M & t^{M-1} & \cdots & t & 1
\end{bmatrix}
\]

and

\[
R_L = \begin{bmatrix}
c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\
c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\
\sum_{i=0}^{L-M} c_i t^{M+i} & \sum_{i=0}^{L-M} c_i t^{M+i-1} & \cdots & \sum_{i=0}^{L} c_i t^i
\end{bmatrix}
\]

If the lower index on a sum exceeds the upper, the sum is replaced by zero.

**3. PADE EMBEDDED PIECE-WISE DT TRANSFORMATION.**

Since the Differential Transform Method provides series solution, it is, in general, difficult to establish its convergence or the number of terms to achieve a specified accuracy. Therefore, the time interval of interest, say \( t \in [0, T] \) which is not necessarily small, is divided into many subdomains for good approximations \([6, 8-12]\).

In this section, we describe the DT method with Pade approximants (P-DT) to obtain better approximations for higher order nonlinear problems.

A general initial value problem can be given as

\[
y^{(n)}(t) = f(t, y, y', \ldots, y^{(n-1)}), \quad a \leq t \leq b
\]

with the initial conditions

\[
y(a) = \alpha_1, \ y'(a) = \alpha_2, y''(a) = \alpha_3, \ldots, y^{(n-1)}(a) = \alpha_n
\]

Assume that eq.(5) is to be solved in the interval \((0, T]\). Define the following set of disjoint intervals \( h_n = [t_n, t_{n+1}] \) with \( t_0 = 0 \), so that \( \sum_{n=0}^{N} h_n = (0, T] \) where \( N \) denotes the number of disjoint subdomains of \((0, T]\). For simplicity applying the differential transform to eq.(5), we obtain
\[
\frac{(k + n)!}{k!} Y(k + n) = F(Y(k), Y(k + 1), ..., Y(k + n - 1))
\]  

(7)

where \( F \) denotes the transformed function of \( f(t, y, y', ..., y^{(n-1)}) \). From the initial conditions (6), we can write

\[
Y(0) = \alpha_i
\]

\[
Y(1) = \frac{\alpha_i}{1!}
\]

\[...
\]

\[
Y(n-1) = \frac{\alpha_i}{(n-1)!}
\]  

(8)

In the first subinterval \((t_0, t_1]\), \(y(t)\) can be described by \(y_0(t)\). From the recurrence relation (7) with the initial conditions (8), the \(2m\)th order differential approximation \(y(t)\) can be written as

\[
y_0(t) = \sum_{k=0}^{2m} Y_0(k)(t-a)^k = Y_0(0) + Y_0(1)(t-a) + ... + Y_0(2m)(t-a)^{2m}
\]  

(9)

Once the Taylor polynomial is obtained, the approximation of \(y(t)\) at the grid point \(t_1\) can be evaluated via the pade diagonal approximant \(Y_0^0 \left[ \frac{m}{m} \right] \) as

\[
y(t_1) = Y(0) = y_0^0 \left[ \frac{m}{m} \right] = \sum_{i=0}^{0} Y_0(i)(t_1-a)^{mi} \sum_{i=1}^{1} Y_0(i)(t_1-a)^{mi-1} ... \sum_{i=m}^{m} Y_0(i)(t_1-a)^{m}
\]  

(10)

Then, for the second subinterval, the artificial initial conditions at \(t = t_1\) can be borrowed from the Taylor series expansion (9) as follows:

\[
Y_i(1) = \sum_{k=1}^{2m} \frac{k!}{(k-1)!} Y_0(k)(t_1-a)^{k-1}
\]

\[
Y_i(2) = \sum_{k=2}^{2m} \frac{k!}{(k-2)!} Y_0(k)(t_1-a)^{k-2}
\]

\[
Y_i(n-1) = \sum_{k=n-1}^{2m} \frac{k!}{(k-n+1)!} Y_0(k)(t_1-a)^{k-n+1}
\]  

(11)
Hence, equations (10) and (11) determine the initial conditions for the recurrence relation (7) in the subinterval \((t_1, t_2]\). In a similar manner, the differential transform approximation in the second interval gives

\[
y(t) = \sum_{k=0}^{2m} Y_k(t_2 - t_1)^k = Y_0(t_1) + Y_1(t_2 - t_1) + \ldots + Y(2m)(t_2 - t_1)^{2m}
\]

Then, the approximation of \(y(t)\) at the grid point \(t_2\) can be evaluated by the Pade approximant \(Y^p_r[m/m]\) as follows:

\[
y(t_2) = Y_2(0) = \sum_{i=0}^{m} Y(i)(t_2 - t_1)^{i} - \sum_{i=1}^{m} \sum_{j=0}^{m-i} Y(i)(t_2 - t_1)^{m+i-j} \ldots \ldots \ldots \ldots \sum_{i=1}^{m} \sum_{j=0}^{m-i} Y(i)(t_2 - t_1)^{1}
\]

The same procedure is applied for each subinterval up to the final approximation

\[y(t_N) = Y^{N-1}_p[m/m]
\]

obtained.

4. EXAMPLES

In this section, we give some illustrative examples to understand the effectiveness of the method.

Example 1. Consider the linear initial-value problem \([12]\)

\[
\frac{dy}{dt} = y(t) - t^2 + 1, \quad y(0) = 0.5
\]

with \(h = 0.2\). Applying the DT method, gives us the recurrence relation

\[
Y_{i}(k + 1) = (Y_{i}(k) - \delta(k - 2) + \delta(k))/(k + 1)
\]

with \(Y_0(0) = 0.5\) where \(Y(.)\) is the transformed function and \(\delta(.)\) is the Dirac function.

Using the procedure mentioned (7-9) and taking the first six terms we find the approximation for the first subinterval \(0 \leq t < 0.2\),

\[
y_0(t) = 0.5 + 1.5t + 0.75t^2 - 0.08333t^3 - 0.020833t^4 - 0.004166t^5 - 0.000694t^6
\]

and the Pade approximant,

\[
Y^{0}[3/3] = \frac{0.5 + 1.40013t + 0.45268t^2 - 0.22603t^3}{0.99999 - 0.19973t + 0.00455t^2 + 0.00051t^3}
\]
By substituting \( t_1 = 0.2 \) in equation (15), the initial value for the second interval can be obtained

\[
y(t_1) = Y^0_p[\frac{3}{3}](t_1) = Y_1(0) = 0.82929
\]  

(16)

Then, for the second interval, substituting the initial value (16) in eq.(13) for \( i = 1 \), we get \( Y^1_p[\frac{3}{3}] \) approximant. By applying the same procedure in each subinterval, we can find the approximate solution of the problem. The exact solution of the problem is known \( y(t) = (t+1)^2 - \frac{1}{2}e^t \). Fig.1. shows the errors from the classical DT Method and the proposed P-DT Method.

![Figure 1. Comparison of classical DT Method and P-DT method in example 1](image)

**Example 2.** Now, we consider nonlinear initial value problem [12]

\[
\frac{dy}{dt} = -(y + 1)(y + 3), \quad y(0) = -2
\]

(17)

with \( h = 0.1 \). The recurrence relation for this problem will be

\[
Y_i(k+1) = -\left( \sum_{r=0}^{k} Y_r(0)Y_{k-r} + 4Y(k) + 3\delta(k) \right) / (k+1)
\]

(18)

with \( Y_0(0) = -2 \). Using the procedure mentioned above, we find the approximation for the first subinterval \( 0 \leq t < 0.1 \),

\[
y_0(t) = -2 + t - \frac{1}{3}t^3 + \frac{2}{15}t^5
\]

(19)
and, then, the Pade approximant,

\[
Y_{p}^{[3/3]} = \frac{-2 + t - \frac{4}{5} t^2 + \frac{1}{15} t^3}{1 + \frac{2}{5} t^2}
\]  

(20)

Applying the same procedure in each subinterval, we can find the approximate solution of the problem. This problem has the exact solution \( y(t) = -3 + 2(1 + e^{-2t})^{-1} \). Fig.2 shows the comparison of the classical DT Method and proposed P-DT Method.

Example 3. Finally we consider the stiff initial value problem [12]

\[
\frac{dy}{dt} = 5e^{5t} (y(t) - t)^2 + 1, \quad y(0) = -1
\]

(21)

with \( h = 0.05 \). Applying the DT method, the recurrence relation will be

\[
Y_{i}(k + 1) = \frac{5^{S_i}}{k!} \otimes \left[ (Y_{i}(k) - \delta(k-1)) \otimes (Y_{i}(k) - \delta(k-1)) \right] + \delta(k)
\]

(22)

with \( Y_{0}(0) = -1 \). Using the procedure mentioned above for the first 6 terms, we find the approximation for the first subinterval \( 0 \leq t < 0.05 \),

\[
y_{0}(t) = -1 + 6t - 12.5t^2 + 20.83333t^3 - 26.04166t^4 + 26.04166t^5 - 21.70138t^6
\]

(23)

and the Pade approximant,
Applying the same procedure in each subinterval we can find the approximate solution of the problem which is known to have the exact solution \( y(t) = t - e^{-3t} \). Fig.3 shows the comparison of the classical DT Method and P-DT Method.

![Figure 3. Comparison of classical DT Method and P-DT method in example3.](image)

5. CONCLUSION

The algorithmic computations and graphical representations associated with the examples discussed above were performed by using Maple 9. The proposed algorithm produced relatively better solutions compared with the classical DT method. Therefore, we can conclude that the convergency of the solutions by differential transformation can be improved by embedding the Pade approximants at the grid points.

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6. REFERENCES