Mathematical and Computational Applications, Vol. 14, No. 2, pp. 159-168, 2009.

ASYMPTOTIC SOLUTIONS AND COMPARISONS OF A GENERALIZED VAN DEL POL OSCILLATOR WITH SLOWLY VARYING PARAMETER

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Abstract- A generalized Van del Pol oscillator with slowly varying parameter is studied. The leading order approximate solutions are obtained respectively by three methods and comparisons are made with numerical results. Different amplitudes are also made to compare the accuracy of the three methods.
Keywords- Van del Pol equation, slowly varying parameter, approximate potential method, equivalent nonlinearization method, multiple scales method, numerical verification

1. INTRODUCTION

For general strongly nonlinear oscillator with slowly varying parameter, many perturbations are difficult to be applied strictly. The problem has caused many researcher's attention and been researched widely in recent years. This paper is to study the following strongly nonlinear oscillator of the form

$$\frac{d^2x}{dt^2} + \varepsilon k(x,\tilde{t})\frac{dx}{dt} + g(x,\tilde{t}) = 0 \tag{1}$$

where $\tilde{t} = \varepsilon t$ is the slow scale. For some special cases of k and g, we can obtain Van del Pol oscillator, Rayleigh equation and pendulum equation. We assume that functions k and g are arbitrary nonlinear functions and Eq.(1) has periodic solutions when $\varepsilon = 0$. For the case of quadratic and cubic nonlinear function $g(x,\tilde{t})$, Kuzmak-Luke multiple scales method [1-4] can be applied efficiently, and the asymptotic solutions expressed by Jacobian elliptic functions can also be obtained [4]. For general nonlinear functions $g(x,\tilde{t})$, Taylor series expansions are often used to approximate them but they are effective only for small amplitudes. Many efforts have been done to overcome the difficulty, such as Fourier series [3], equivalent linearization combined averaging method [5]. Approximate potential method was first proposed by Li in Ref.[6] to deal with a generalized pendulum equation resulted from the free electron laser (FEL). In Ref.[6] the potential for the nonlinear oscillator is expressed by a polynomial of degree three such that the leading approximation is expressible in terms of elliptic functions. In Ref.[7] Cai first proposed equivalent nonlinearization method to overcome the difficulty of some kinds of nonlinearity. This method use quadratic or cubic nonlinear polynomial to approximate nonlinear function $g(x,\tilde{t})$, and the least-squares fit method is used to decide the coefficients. Bosely presented a technique that uses numerical solutions to verify the order of the accuracy of an asymptotic expansion for several types of problems [8].

In this paper, we first obtain three approximate cubic nonlinear oscillators respectively by Taylor series expansions method, approximate potential method and equivalent nonlinearization method to approximate a generalized Van del Pol oscillator. Secondly, the leading order approximate solutions of these three approximate cubic nonlinear oscillators are obtained by the K-L multiple scales method. The numerical order verification is applied to verify that the asymptotic solutions are valid when the parameter ε is small for the three approximate cubic nonlinear oscillators but not uniformly valid for the original equation. The reason is that these three approximate cubic nonlinear oscillators have errors with the original equation. Finally, error analysis of the leading order approximate solutions shows that the errors are about one-tenth of the value of the small parameter ε . Error analysis also shows that Taylor series expansions method is better than approximate potential method and equivalent nonlinearization method when the amplitude is small, while equivalent nonlinearization method is better than Taylor series expansions method and approximate potential method when the amplitude is large. It also shows that Taylor series expansions method has large error for relatively large oscillations.

2. ASYMPTOTIC SOLUTION OF STRONGLY NONLINEAR OSCILLATOR

Van der Pol obtained the following equation

$$\ddot{x}-\varepsilon\dot{x}(1-\delta x^2)+\omega_0{}^2x=0 \tag{2}$$

which is negative damp for small oscillation, and positive damp for large oscillation. A modified Van del Pol oscillator has been recently proposed to describe a self-excited body sliding on a periodic potential [9], which is described by the following equation

$$M\frac{d^2x}{dt^2}+\Gamma(x^2-1)\frac{dx}{dt}+\frac{2\pi b}{\lambda}\sin(2\pi x/\lambda)+Kx=0 \tag{3}$$

Consider a generalized strongly nonlinear oscillator with slowly varying parameter in the form

$$\frac{d}{dt}(m(\tilde{t})\frac{dx}{dt})+\varepsilon(c(\tilde{t})x^2-d(\tilde{t}))\frac{dx}{dt}+a(\tilde{t})\sin x+b(\tilde{t})x=0 \tag{4}$$

where $\tilde{t}=\varepsilon t$ ($0<\varepsilon<<1$) is the slow scale.

Eq.(4) can transform to

$$\frac{d^2x}{dt^2}+\varepsilon\frac{c(\tilde{t})x^2-d_1(\tilde{t})}{m(\tilde{t})}\frac{dx}{dt}+\frac{a(\tilde{t})}{m(\tilde{t})}\sin x+\frac{b(\tilde{t})}{m(\tilde{t})}x=0 \tag{5}$$

where $d_1(\tilde{t})=d(\tilde{t})-m'(\tilde{t})$.

In order to obtain the asymptotic solution of Eq.(5) by using K-L multiple scales method, we obtain three approximate cubic nonlinear oscillators respectively by Taylor series expansions method, approximate potential method and equivalent nonlinearization method. Next, we give a brief introduction to the three methods.

2.1. Taylor series expansions method

For small oscillator, $\sin x \approx x - \frac{1}{6}x^3$, so Eq.(5) turns into

$$\frac{d^2 x}{dt^2} + \varepsilon \frac{c(\tilde{t})x^2 - d_1(\tilde{t})}{m(\tilde{t})}\frac{dx}{dt} + \frac{a(\tilde{t}) + b(\tilde{t})}{m(\tilde{t})}x - \frac{a(\tilde{t})}{6m(\tilde{t})}x^3 = 0 \qquad (6)$$

2.2. Approximate potential method

The fast scale t^+, following Kuzmak [1], is defined as $\frac{dt^+}{dt} = \omega(\tilde{t})$ with an unknown $\omega(\tilde{t})$ to be determined by the periodicity of the solution of Eq.(5). Suppose that the solution of Eq. (5) can be developed in the multiple scales form

$$x(t,\varepsilon) = x_0(t^+,\tilde{t}) + \varepsilon x_1(t^+,\tilde{t}) + \varepsilon^2 x_2(t^+,\tilde{t}) + \cdots \qquad (7)$$

Substituting (7) into (5) and equating powers of ε gives the leading order approximate equation

$$\omega^2(\tilde{t})\frac{\partial^2 x_0}{\partial t^{+2}} + \frac{a(\tilde{t})}{m(\tilde{t})}\sin x_0 + \frac{b(\tilde{t})}{m(\tilde{t})}x_0 = 0 \qquad (8)$$

We obtain the energy integral

$$\frac{\omega^2(\tilde{t})}{2}\left(\frac{\partial x_0}{\partial t^+}\right)^2 + V(x_0) = E_0(\tilde{t}) \qquad (9)$$

where

$$V(x_0) = \frac{a(\tilde{t})}{m(\tilde{t})}(1 - \cos x_0) + \frac{b(\tilde{t})}{2m(\tilde{t})}{x_0}^2 \qquad (10)$$

is the potential (For simplity, we set $V(0) = 0$) and $E_0(\tilde{t})$ is the slowly varying energy of the system. According to the character of the potential function, we may fit a polynomial of degree four to the potential (10) such that the periodic solution is expressible in terms of elliptic functions, which will be discussed in detail later.

2.3. Equivalent nonlinearization method

According to the character of $\frac{a(\tilde{t})}{m(\tilde{t})}\sin x + \frac{b(\tilde{t})}{m(\tilde{t})}x$, we may seek a polynomial of the form $c_1(\tilde{t})x + d_1(\tilde{t})x^3$ to fit

$$F(c_1,d_1) = \int_{x1}^{x2}\left(\frac{a(\tilde{t})}{m(\tilde{t})}\sin x + \frac{b(\tilde{t})}{m(\tilde{t})}x - c_1(\tilde{t})x - d_1(\tilde{t})x^3\right)^2 dx \to \min$$

The coefficients are chosen such that $\frac{\partial F}{\partial c_1} = 0$ and $\frac{\partial F}{\partial d_1} = 0$, and x_1 and x_2 can be chosen around the center x_r.

In the following, we choose $m(\tilde{t}) = 10 + \varepsilon t$, $c(\tilde{t}) \equiv 1$, $d(\tilde{t}) = 10 + \varepsilon t$, $a(\tilde{t}) = 5 + \varepsilon t$ and $b(\tilde{t}) = (5 + \varepsilon t)^2$ as an example to consider.

Following Taylor series expansions method, the approximate equation of Eq.(5) becomes

$$\frac{d^2x}{dt^2}+\varepsilon\frac{x^2-(9+\varepsilon t)}{10+\varepsilon t}\frac{dx}{dt}+\frac{(5+\varepsilon t)(6+\varepsilon t)}{10+\varepsilon t}x-\frac{5+\varepsilon t}{6(10+\varepsilon t)}x^3=0 \tag{11}$$

According to approximate potential method [6], the potential V is "U--Shaped", so Eq.(11) has periodic solutions around $x_r=0$. Denoting

$$\overline{V}(x)=\frac{1}{2}a_1(\widetilde{t})x^2+\frac{1}{4}b_1(\widetilde{t})x^4$$

where the coefficients are chosen such that

$\overline{V}=V$ at $x=0$, $x=\frac{\pi}{3}$ and $x=\frac{2\pi}{3}$

the fitting points $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ can be chosen according to concerned range of amplitude, and $\overline{V}'=0$ at $x=0$, then we have

$$\overline{V}(x)=\frac{1}{2}\frac{(5+\varepsilon t)(5.98788+\varepsilon t)}{10+\varepsilon t}x^2-\frac{1}{4}\frac{0.692954+0.138591\varepsilon t}{10+\varepsilon t}x^4 \tag{12}$$

Substituting $\overline{V}$ for V in Eq.(9), we can obtain the approximate equation of Eq.(5)

$$\frac{d^2x}{dt^2}+\varepsilon\frac{x^2-(9+\varepsilon t)}{10+\varepsilon t}\frac{dx}{dt}+\frac{(5+\varepsilon t)(5.98788+\varepsilon t)}{10+\varepsilon t}x-\frac{0.692954+0.138591\varepsilon t}{10+\varepsilon t}x^3=0 \tag{13}$$

According to equivalent nonlinearization method [7], we consider the range of amplitude of $x\in[-\frac{2\pi}{3},\frac{2\pi}{3}]$ because Eq.(13) has periodic solutions around $x_r=0$, then we get $c_1(\widetilde{t})=0.966589a(\widetilde{t})+b(\widetilde{t})$, $d_1(\widetilde{t})=-0.129881a(\widetilde{t})$.

So the equivalent nonlinearization equation of Eq.(5) is

$$\frac{d^2x}{dt^2}+\varepsilon\frac{x^2-(9+\varepsilon t)}{10+\varepsilon t}\frac{dx}{dt}+\frac{(5+\varepsilon t)(5.966589+\varepsilon t)}{10+\varepsilon t}x-\frac{0.129881(5+\varepsilon t)}{10+\varepsilon t}x^3=0 \tag{14}$$

We now apply K-L multiple scales method to obtain the leading order approximate solutions of these three approximate cubic nonlinear oscillators. Firstly, we introduce the application of K-L multiple scales method in cubic nonlinear oscillators. For system (1), suppose that the solution can be developed in the form of asymptotic expression (7), where $\omega(\widetilde{t})$ will be determined by the periodicity of the solution of Eq.(5). If the periodic is normalized to be 1, we have

$$\omega(\widetilde{t})=\frac{c}{\int_0^1 f_\varphi^2 \mathrm{d}\varphi}\exp(-\int_0^{\widetilde{t}}k(x_r,\tau)d\tau) \tag{15}$$

where x_r is the resonance center. We denote $x_0=f(\varphi,\widetilde{t})$ which is the leading order approximate solution, and $\varphi=t^++\varphi_0$. Constant c and φ_0 can be determined by the initial values of the system.

When $g(x,\widetilde{t})=a(\widetilde{t})x+b(\widetilde{t})x^3$, substituting expression (7) into Eq.(5) should give a leading order approximate equation

$$\omega^2(\widetilde{t})\frac{\partial^2 x_0}{\partial t^{+2}}+a(\widetilde{t})x_0+b(\widetilde{t})x_0^3=0 \tag{16}$$

Its energy integral is

$$\frac{\omega^2(\tilde{t})}{2}(\frac{\partial x_0}{\partial t^+})^2 + V(x_0, a, b) = E_0(\tilde{t}) \tag{17}$$

where

$$V(x_0, a, b) = \frac{1}{2}a(\tilde{t})x_0^2 + \frac{1}{4}b(\tilde{t})x_0^4 \tag{18}$$

is the potential, and $E_0(\tilde{t})$ is the slowly varying energy of the system. When $a(\tilde{t}) > 0$ and $b(\tilde{t}) < 0$ (resonance center is at $x_r = 0$), we can obtain x_0 in terms of elliptic function of t^+ by integrating (17)

$$x_0 = A_0(\tilde{t})\ sn[K(v)\varphi, v(\tilde{t})] \tag{19}$$

where $K(v)$ is the complete elliptic integral of the first kind associated with the modulus $\sqrt{v}$ and

$$A_0 = \sqrt{\frac{-2av}{b(1+v)}} \tag{20}$$

The modulus $\sqrt{v}$ is determined by equation

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{c^2b^2}{4a^3}\exp(-2\int_0^{\tilde{t}} k(0,\tau)d\tau) \tag{21}$$

where

$$L(v) = \int_0^K cn^2(u,v)dn^2(u,v)\mathrm{d}u = \frac{1}{3v}[(1+v)E(v) - (1-v)K(v)] \tag{22}$$

and $E(v)$ is the complete elliptic integral of the second kind associated with the modulus $\sqrt{v}$. More details of deduce, readers can refer to Ref.[10].

We assume the initial conditions are

$$x(0) = 1,\ \dot{x}(0) = 0 \tag{23}$$

For Eqs.(11) and (23)、(13) and (23)、(14) and (23), using K-L multiple scales method, we obtain the following asymptotic solutions expressed by Jacobian elliptic functions

$$x_0^{(1)} = \sqrt{\frac{12v}{1+v}(6+\varepsilon t)}sn[K(v)\varphi, v] \tag{24}$$

$$x_0^{(2)} = \sqrt{\frac{2v}{1+v}\frac{(5+\varepsilon t)(5.98788+\varepsilon t)}{0.692954+0.138591\varepsilon t}}sn[K(v)\varphi, v] \tag{25}$$

$$x_0^{(3)} = \sqrt{\frac{2v}{1+v}\frac{(5.966589+\varepsilon t)}{0.129881}}sn[K(v)\varphi, v] \tag{26}$$

Comparisons of the three leading order approximate solutions and numerical solutions of Eqs.(5) and (23) are shown respectively in Fig.1- Fig.3.

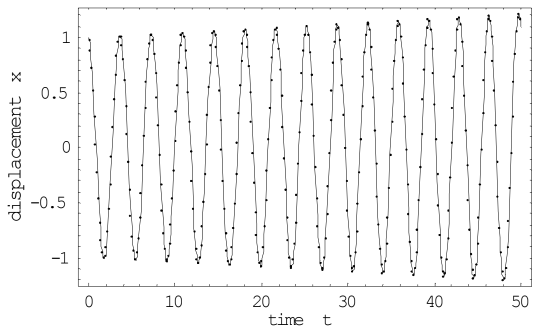


Fig. 1 Comparison of asymptotic solution (24) and numerical solution

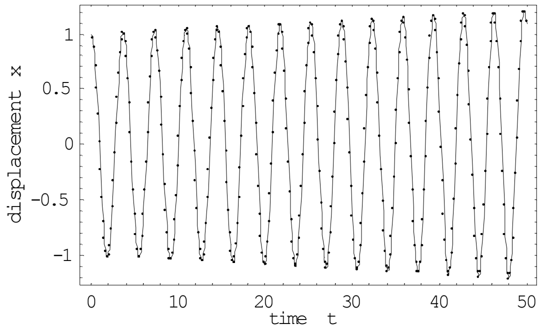


Fig. 2 Comparison of asymptotic solution (25) and numerical solution

___ numerical solution . . . . asymptotic solution

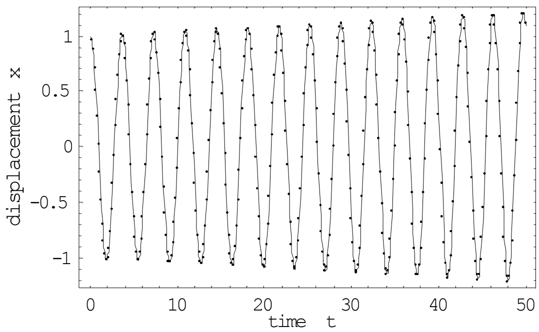


Fig. 3 Comparison of asymptotic solution (26) and numerical solution

____ numerical solution . . . . asymptotic solution

So the accuracies of the three leading order approximate solutions are quite satisfactory.

3. NUMERICAL ORDER VERIFICATION OF APPROXIMATE SOLUTION

Following Bosley's technique of numerical order verification [8], we assume the solution of Eq.(1) is

$$x_{asym}(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots + \varepsilon^N x_N(t) + O(\varepsilon^{N+1}) \tag{27}$$

The error of the asymptotic expansion is

$$\begin{aligned} Error = E_N(t,\varepsilon) &= \left| x_{exact}(t,\varepsilon) - x_{asym}(t,\varepsilon) \right| \\ &= \left| x_{exact}(t,\varepsilon) - \sum_{n=0}^{N} \varepsilon^n x_n(t) \right| = O(\varepsilon^{N+1}) = K\varepsilon^{N+1} \end{aligned} \tag{28}$$

where K is a constant. Taking the logarithm of both sides of equation (28) yields

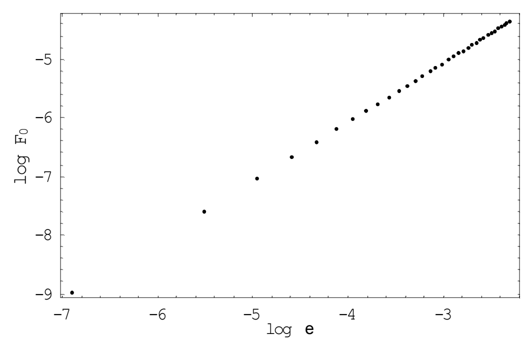
$$\log(Error) = \log(E_N) = \log K + (N+1)\log\varepsilon \tag{29}$$

which means the value of $\log(E_N)$ as a function of $\log\varepsilon$ should be linear with slope $N+1$. Therefore, when we graph $\log(E_N)$ versus $\log\varepsilon$ for different values of ε, these points should be nearly on a line and the linear equation that interpolates these points using a linear least-squares fit should have slope $N+1$.

In order to give a better overall estimation of difference between the exact (or numerical) and asymptotic solutions, instead of Ref.[8] with a fixed time $t = t_0$, an average error is introduced in Ref.[11]

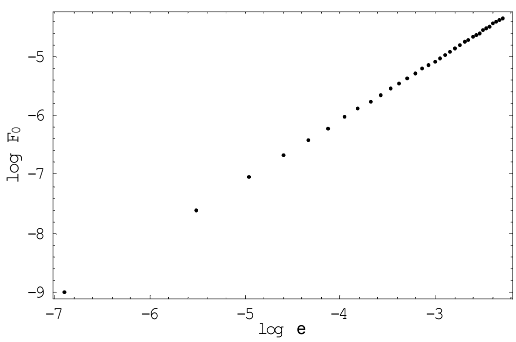
$$Average\ Error = F_N(t,\varepsilon) = \frac{1}{m}\sum_{i=1}^{m} E_N(t_i,\varepsilon) \tag{30}$$

where $t = t_i (i = 1, 2, \cdots m)$ are fixed points in the concerned domain of time t. To verify the order of asymptotic expansions(24)-(26), we first find the numerical solutions of Eq.(11), Eq.(13), Eq.(14) and (23) with $t_i = 0.02i\ (i = 1, 2, \cdots 10)$ when $\varepsilon \in [0.001, 0.1]$ by a step size 0.003. Next, we evaluate the asymptotic expansions (24)-(26) at the same values of ε and t_i. The exact solution $x_{exact}(t,\varepsilon)$ in Eq.(28) is replaced by the numerical solutions of Eq.(11), Eq.(13) and Eq.(14) respectively. Fig.4-Fig.6 plot the values of the errors at these 34 points, and these points are nearly on a line. The least-square fit of the data is used to determine the slopes 1.00537, 1.00777, and 1.00736, which are in good agreement with the theoretical slope $N + 1 = 1 (N = 0)$.



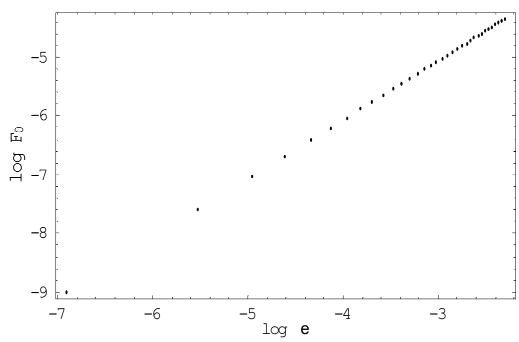
$\log F_0 = -2.03199 + 1.00537 \log \varepsilon$

Fig. 4 Numerical verification of the solution (24) with ε starting from 0.001 to 0.1



$\log F_0 = -2.02946 + 1.00777 \log \varepsilon$

Fig. 5 Numerical verification of the solution (25) with ε starting from 0.001 to 0.1



$\log F_0 = -2.03118 + 1.00736 \log \varepsilon$

Fig. 6 Numerical verification of the solution (26) with ε starting from 0.001 to 0.1

Therefore, we can conclude that K-L multiple scales method is valid when the parameter is small for the three approximate cubic nonlinear oscillators. If the exact solution $x_{exact}(t,\varepsilon)$ in Eq.(28) is replaced by the numerical solution of Eq.(5), the result is not asymptotic valid. For example, we consider $\varepsilon \in [0.001, 0.01]$. The lines obtained by Taylor series expansions method and equivalent nonlinearization method are

$\log F_0 = -4.09279 + 0.510403 \log \varepsilon$ and $\log F_0 = -4.63895 + 0.337526 \log \varepsilon$, respectively. And the result obtained by approximate potential method is not a line. The reason is that the three approximate cubic nonlinear oscillators have errors with Eq.(5), and the errors bring larger error between the asymptotic solutions with the numerical solution of Eq.(5).

4. ANALYSIS OF ERROR

Now we will show a numerical comparison of these three methods for parameter ε by using technique [8, 11]. Fig.7 plots F_0 versus ε of the asymptotic Eqs.(24), (25) and (26) obtained by the three methods, where $t_i = 0.02i\ (i = 1, 2, \cdots 10)$, and ε starts from 0.001 and ends at 0.1 by a step size 0.003. The exact solution $x_{exact}(t, \varepsilon)$ in Eq.(28) is replaced by the numerical solution of Eq.(5), and the amplitude is $x(0) = 1$.

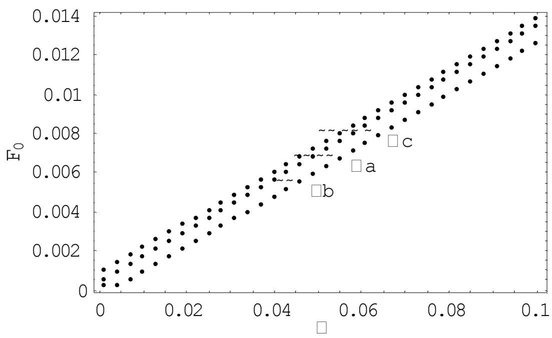


Fig. 7 The relation between the average error F_0 and parameter ε ($x(0) = 1$)
a: Taylor expansions method b: approximate potential method c: equivalent nonlinearization method

Note that the average error F_0 increases lineary as parameter ε increases, and the errors are about one-tenth of the small parameter ε. Obviously, approximate potential method has higher accuracy at this time. We can compare further the accuracy of the three methods for different amplitudes.

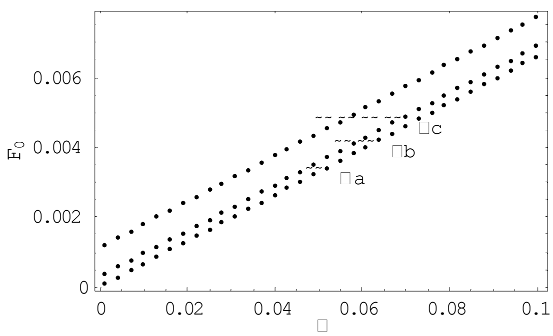


Fig. 8 The relation between the average error F_0 and parameter ε ($x(0) = 0.5$)

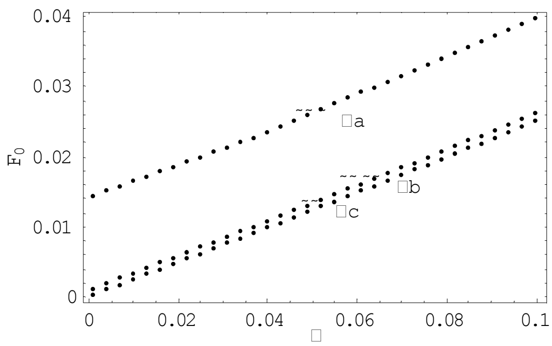


Fig.9 The relation between the average error F_0 and parameter ε ($x(0) = 2$)

a: Taylor expansions method b: approximate potential method c: equivalent nonlinearization method

5. CONCLUSIONS

K-L multiple scales method can be applied validly to strongly nonlinear oscillators with slowly varying parameter. For quadratic nonlinear oscillators, the asymptotic solutions expressed by Jacobian elliptic functions can be obtained. The error analysis also shows that the asymptotic solutions are still valid for $\varepsilon = 0.1$, which is not very small.

Taylor series expansions method is better when the amplitude is small, and equivalent nonlinearization method is better when the amplitude is large. It also shows that Taylor series expansions method has large error for large oscillations.

Acknowledgement- Research is supported by the Foundation for supporting Universities in Fujian Province under grant No 2007F5099.

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