Mathematical and Computational Applications, Vol. 13, No. 3, pp. 147-151, 2008.

AN ITERATIVE METHOD FOR SOLVING LINEAR FRACTION PROGRAMMING (LFP) PROBLEM WITH SENSITIVITY ANALYSIS

Said Tantawy
Department of Mathematics
Helwan University (11795)
Cairo, Egypt.
saidtantawy@hotmail.com

Abstract- In this paper an iterative method for solving linear fraction programming (LFP) problem is proposed, it will be shown that this method can be used for sensitivity analysis when a scalar parameter is introduced in the objective function coefficients and our task of this sensitivity investigation is to maintain the optimality of the problem under consideration. A simple example is given to illustrate this proposed method.

Key Words - Linear fraction program, sensitivity analysis.

1-INTRODUCTION

Linear fraction maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning. Several methods for solving this problem are proposed in [5], their method depends on transforming this (LFP) to an equivalent linear program. Another method is called updated objective function method derived from [3] is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function .Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by [6] and [6] in his paper made a useful study about the optimality condition in fractional programming. More recent works on fractional programming theory and methods can be found in [1,2].The suggested method in this paper depends mainly on the updated method in iterative manner then the optimality condition for a given basic feasible solution of (LFP) is defined. Also we may be interested in the effects of small changes in the objective function coefficient on the optimality condition for a given optimal solution and the task of this sensitivity investigation is to maintain the optimality of the given problem. In section 2 some notations and definitions of the (LFP) problem is given while in section 3 the problem of sensitivity analysis when scalar parameter is introduced into the objective function coefficient is discussed followed by a simple example in section 4 to illustrate this sensitivity study and finally a conclusion about this sensitivity investigation is made in section 5.

2-NOTAIONS AND DEFINITIONS

Linear fraction program (LFP) problem arises when a linear ratio function has to be maximized (minimized) on the set X ={x; Ax = b; x ≥ 0}, hence the (LFP) problem can be written as

$$\text{Maximize } Z = \frac{c^T x + \alpha}{d^T x + \beta} \tag{1}$$

Subject to

$$x \in X$$

Here c and d are vectors in R^n represent the objective function coefficients, α and β are scalars, A is an m x n matrix and b ∈ R^m, also it is assumed that $d^T x + \beta > 0$ for every x ∈ X. For the (LFP) problem we have $(c^T - \overline{Z}\, d^T)\, x = \beta \overline{Z} - \alpha$ represents the $\overline{Z}$ - level curve of the linear fraction objective function and when $\overline{Z}$ is arbitrary it is seen that each level curve of (LFP) is linear over X, and so one extreme of X has to be the optimal solution for (LFP). Consider the linear programming problem

$$\text{Maximize } Z^* = (c^T - Z^0 d^T)\, x$$

Subject to (2)

$$x \in X$$

Here Z^0 is a given constant; hence we have the following proposition

Proposition 2-1 if x^0 solves the linear fractional programming (LFP) defined by (1) with optimal value Z^0 then x^0 solves the linear programming (2) with optimal value $Z^* = \beta Z^0 - \alpha$.
Proof: Since the level curve of the objective function of the (LFP) defined by (2) is in the form $(c^T - \overline{Z}\, d^T)\, x = \beta \overline{Z} - \alpha$, the proof is Straight forward.

Suppose x is a basic feasible solution of Ax = b, x ≥ 0 with corresponding basic decomposition then Ax = b, x ≥ 0 can be written $Bx_B + Nx_N = b$, $x_B \geq 0$ and $x_N \geq 0$. If we contract a tableau for x in the form

$$T(x) = \begin{bmatrix} I & B^{-1}N & B^{-1}b \\ 0 & r & Z_B \end{bmatrix} \tag{3}$$

Here $x = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}^T$, $Z_B = \dfrac{c_B^T x_B + \alpha}{d_B^T x_B + \beta}$, and r is a raw vector in R^{n-m},

$$r = (c_B^T B^{-1} N - c_N^T) - Z_B \quad (d_B^T B^{-1} N - d_N^T) \tag{4}$$

Suppose x_B is a basic feasible solution for the linear program problem defined by (2) with corresponding tableau T(x), if any $r_i \in r$; i ∈ {1,2,…n-m} is negative then pivoting in r_i yields a new basic feasible solution and the optimality condition occurs when $r \geq 0$ (5)

Remark 2-1.Condition (5) reduces to the well known optimality condition in linear programming problem when d is the zero vectors.

Remark 2-2. In this method pivoting in $r_i \in r$; $i \in \{1,2,\ldots n\text{-}m\}$ yields a new basic feasible solution x_B^1 with an increment in the objective function value given by $\dfrac{\theta r_i}{d_B^T x_B^1 + \beta}$, but in the case of the ordinary linear program when d = 0 the increment of the objective function value equals θr_i.

3- SENSITIVITY ANAYSIS IN LINEAR FRACTION PROGRAMMING

Introducing a scalar parameter in the objective function coefficient of the (LFP) problem defined by (1) may affect the optimality of the given problem due to some changes in the data of the objective function. In this case the (LFP) problem defined by (1) becomes

$$\text{Maximize } Z(\mu) = \frac{(c^T + \mu \bar{c}^T)x + \alpha}{(d^T + \mu \bar{d}^T)x + \beta} \tag{6}$$

Subject to

$$x \in X,$$

where $\bar{c}$, $\bar{d}$ are given vectors in R^n and μ is scalar parameter introduced to the objective function coefficients, hence to maintain the optimality of a given basic feasible solution determined when μ = 0 the condition

$$((c_B^T(\mu)B^{-1}N - c_N^T(\mu)) - Z_B(\mu)(d_B^T(\mu)B^{-1}N - d_N^T(\mu)) \geq 0 \tag{7}$$

Must be satisfied, solving (7) for μ will give a range on μ such that the optimal basic feasible solution doesn't change.

3.1. An application on production example

Let us consider a company that manufactures two kinds of products A_1, A_2 with a profit of 4, 2 dollar per unit respectively. .However the cost for each one unit of the above products is 1, 2 dollar. It is assumed that a fixed cost of 5 dollars is added to the cost function due to expected duration through the process of production and also a fixed amount of 10 dollar is added to the profit function. If the objectives of this company is to maximize the profit in return to the total cost, provided that the company has a raw materials for manufacturing and suppose the material needed per pounds are 1, 3 and the supply for this raw material is restricted to 30 pounds, it is also assumed that twice of production of A_2 is more than the production of A_1 at most by 5 units . In this case if we consider x_1 and x_2 to be the amount of units of A_1 , A_2 to be produced then the above problem can be formulation as

$$\text{Maximize } Z = \frac{4x_1 + 2x_2 + 10}{x_1 + 2x_2 + 5}$$

Subject to;

$$x_1 + 3x_2 \leq 30$$
$$-x_1 + 2x_2 \leq 5$$
$$x_1 \geq 0, \; x_2 \geq 0.$$

Now consider the linear fraction programming problem when a scalar parameter μ is introduced in the objective function in the form

$$\text{Maximize } Z(\mu) = \frac{(4+\mu)x_1 + (2-\mu)x_2 + 10}{(1+2\mu)x_1 + (2-\mu)x_2 + 5}$$

Subject to

$$x_1 + 3x_2 \leq 30$$
$$-x_1 + 2x_2 \leq 5$$
$$x_1 \geq 0, \; x_2 \geq 0.$$

For μ = 0 the optimal basic feasible solution is given by

Basic	x_1	x_2	x_3	x_4	solution
x_1	1	3	1	0	30
x_4	0	5	1	1	35
Z_B	0	$\frac{8}{35}$	$\frac{2}{35}$	0	$\frac{130}{35}$

Now if the scalar parameter is introduced to the objective function coefficient, then the corresponding tableau becomes

Basic	x_1	x_2	x_3	x_4	solution
x_1	1	3	1	0	30
x_4	0	5	1	1	35
Z_B	0	$(\frac{8-10\mu}{35+60\mu})$	$(\frac{2-3\mu}{35+60\mu})$	0	$\frac{130+30\mu}{35+60\mu}$

From the above tableau we have the set of inequalities in the form
$8 - 10\mu \geq 0$, $2 - 3\mu \geq 0$ $35 + 60\mu > 0$ which gives
$\mu \in [\frac{-7}{12}, \frac{2}{3}]$ to be the set of all parameters for which the optimal basic feasible solution remains optimal.

4. CONCLUDING REMARKS

Sensitivity analysis in linear fraction program due to some changes in the objective function coefficient is investigated to maintain the optimality of the given problem also we hope to extend this method to the case of multiple objective linear fraction programming problems because in many real problems the data of the objective functions are dynamically changes.

5. REFERENCES

1- E.B. Bajalinove, Linear Fractional Programming: Theory, Methods, Applications and Software, Kluwer Academic Publishers, 2003.

2- E.B. Bajalinove, A. Tangian, Adjusting an objective function to a given optimal solution in linear and linear fractional programming, in: A. Tangian, J. Gruber (Eds.), Constructing and Applying Objective Functions, in: Lecture Notes in Economics and Mathematical Systems, 510, Springer, 2001.

3-Bitran .G.R. and A.J. Novaes. Linear programming with a fractional objective function. Operations Research, 21, 22-29, 1973.

4-Bitran .G.R. and T.L Magnanti Duality and sensitivity analysis with fractional objective function. Operation Research, 24, 675-699, 1976.

5-Chaners .A. and W.W.Cooper. Programming with linear fractional functional. Naval Research logistics Quarterly, 9, 181-186, 1962.

6-Singh .C. Optimality condition in fractional programming. Journal of optimization Theory and Applications, 33, 287-294, 1981.