Mathematical and Computational Applications, Vol. 12, No. 1, pp. 21-30, 2007.

A GENERALIZED KBM METHOD FOR STRONGLY NONLINEAR OSCILLATORS WITH SLOWLY VARYING PARAMETERS

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Abstract- A generalized Krylov-Bogoliubov-Mitropolsky (KBM) method is extended for the study of strongly nonlinear oscillators with slowly varying parameters. The asymptotic amplitude and phase are derived and then the asymptotic solutions of arbitrary order are obtained theoretically. Cubic nonlinear oscillators with polynomial damping are studied in detail. Three examples are considered: a generalized Van der Pol oscillator, a Rayleigh equation and a pendulum with variable length. Comparisons are also made with numerical solutions to show the efficiency and accuracy of the present method.
Keywords- strongly nonlinear oscillation, KBM method, slowly varying parameter

1. INTRODUCTION

This paper is to study the following strongly nonlinear oscillator

$$\frac{d^2 y}{dt^2} + g(y, \tilde{t}) = \varepsilon\, h(y, \frac{dy}{dt}, \tilde{t}) \tag{1}$$

where $\tilde{t} = \varepsilon t$ is the slow scale. Assume that g and h are arbitrary nonlinear functions of their arguments and Eq.(1) has periodic solutions when $\varepsilon = 0$. Many problems in engineering are modeled as Eq.(1), such as pendulum with varying length[1], machines with variable mass[2], and motion of electron in free-electron laser(FEL)[3]. For the case of linear spring in $g(y, \tilde{t})$, the classical KBM method [4] is effective to deal with Eq.(1). While $g(y, \tilde{t})$ is a cubic polynomial in y, the KB method is extended as the elliptic KB (EKB) method by using Jacobian elliptic functions [5,6], and adiabatic invariants coupled with elliptic functions are applied to find the asymptotic solutions [7]. However, the EKB method can give only the first order approximation and the classical KBM method is effective only for weakly nonlinear oscillations. For the case of linear damping in $h(y, \frac{dy}{dt}, \tilde{t})$, Kuzmak proposed a multiple scales method to obtain the conditions of periodicity and asymptotic solutions of first order [8], and then Luke extended it to higher order [9]. Kevorkian and

Li reviewed and compared the Kuzmak-Luke method and that of near-identity averaging transformations [10]. Bourland and Haberman used a two-variable procedure to give a careful analysis of Eq.(1) and derived the equation governing the slowly varying phase [11], which has been summarized by Kevorkian and Cole [12]. Recently, the author developed the Kuzmak-Luke method to obtain the asymptotic solutions of Eq. (1) and applied it to quadratic and cubic nonlinear oscillators [13]. For Eq. (1) without slowly varying parameters, that is

$$\frac{d^2 y}{dt^2} + g(y) = \varepsilon\ h(y, \frac{dy}{dt})$$

a generalized KBM method is proposed to obtain the asymptotic solutions of arbitrary order [14]. This method is effective for general nonlinear function $g(y)$. In this paper, this generalized KBM method will be developed to treat strongly nonlinear oscillators with slowly varying parameters. The asymptotic amplitude and phase are derived and then the asymptotic solutions of arbitrary order are obtained theoretically. As an application, cubic nonlinear oscillators with polynomial damping are studied in detail. Three typical examples are considered: a generalized Van der Pol oscillator, a Rayleigh equation and a pendulum with variable length. Comparisons are also made with numerical solutions to show the efficiency and accuracy of the present method.

2. THE GENERALIZED KBM METHOD

In this section we extend the generalized KBM method [14] to Eq. (1). Assume that the asymptotic solutions of Eq.(1) can be expanded as

$$y(a,\psi,\tilde{t}) = y_0(a,\psi) + \varepsilon\, y_1(a,\psi,\tilde{t}) + \varepsilon^2\, y_2(a,\psi,\tilde{t}) + \cdots \tag{2}$$

where a is the amplitude, and ψ is the phase factor. y_n $(n = 0,1,2,\cdots)$ are periodic functions of ψ with a constant period normalized to be T. a and ψ satisfy

$$\frac{da}{dt} = \varepsilon\, A_1(a,\tilde{t}) + \varepsilon^2 A_2(a,\tilde{t}) + \cdots \tag{3}$$

$$\frac{d\psi}{dt} = B_0(a,\tilde{t}) + \varepsilon\, B_1(a,\tilde{t}) + \varepsilon^2 B_2(a,\tilde{t}) + \cdots \tag{4}$$

For simplicity, the initial conditions are assumed to be

$$t = 0,\ \psi = 0,\ y(a,0,0) = a_0,\ \dot{y}(a,0,0) = 0 \tag{5}$$

From Eqs.(3) and (4), we have the derivative transformations

$$\frac{d}{dt} = \dot{a}\frac{\partial}{\partial a} + \dot{\psi}\frac{\partial}{\partial \psi} + \varepsilon\frac{\partial}{\partial \tilde{t}}$$

$$= B_0(a,\tilde{t})\frac{\partial}{\partial\psi} + \varepsilon[A_1(a,\tilde{t})\frac{\partial}{\partial a} + B_1(a,\tilde{t})\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\tilde{t}}] + \cdots \tag{6}$$

$$\frac{d^2}{dt^2} = B_0^2(a,\tilde{t})\frac{\partial^2}{\partial\psi^2} + \varepsilon[2B_0(a,\tilde{t})A_1(a,\tilde{t})\frac{\partial^2}{\partial a\partial\psi} + 2B_0(a,\tilde{t})B_1(a,\tilde{t})\frac{\partial^2}{\partial\psi^2}$$

$$+ 2B_0(a,\tilde{t})\frac{\partial^2}{\partial\psi\,\partial\tilde{t}} + A_1(a,\tilde{t})\frac{\partial B_0}{\partial a}\frac{\partial}{\partial\psi}] + \cdots \tag{7}$$

Substituting (2) into (1) and equating same powers of ε give the equations

$$O(1):\quad B_0^2(a,\tilde{t})\frac{\partial^2 y_0}{\partial\psi^2} + g(y_0,\tilde{t}) = 0 \tag{8}$$

$$O(\varepsilon^n):\ B_0^2(a,\tilde{t})\frac{\partial^2 y_n}{\partial\psi^2} + g_y(y_0,\tilde{t})\,y_n = F_n(y_0,y_1,\cdots,y_{n-1},\tilde{t}) \tag{9}$$

where F_n are known functions, $n = 1,2,\cdots$. In particular, F_1 is worked out as

$$F_1 = -(2B_0A_1\frac{\partial^2 y_0}{\partial a\partial\psi} + 2B_0B_1\frac{\partial^2 y_0}{\partial\psi^2} + B_0\frac{\partial^2 y_0}{\partial\psi\,\partial\tilde{t}} + A_1\frac{\partial B_0}{\partial a}\frac{\partial y_0}{\partial\psi}) + h(y_0, B_0\frac{\partial y_0}{\partial\psi},\tilde{t}) \tag{10}$$

According to the generalized KBM method [14], we assume that Eq.(8) has a periodic solution y_0 and B_0 can be found out. It is easy to verify that a solution of the homogeneous equation (9) has the form

$$y_{\mathrm{I}} = \frac{\partial y_0}{\partial\psi} \tag{11}$$

The other solution linearly independent of y_{I} can be found by the reduction of order

$$y_{\mathrm{II}} = y_{\mathrm{I}}\int_0^{\psi}\frac{1}{y_{\mathrm{I}}^2}d\varphi \tag{12}$$

Using variation of parameters, the general solutions of Eq.(9) are

$$y_n = D_{1n}(\tilde{t})y_{\mathrm{I}} + D_{2n}(\tilde{t})y_{\mathrm{II}} - \frac{y_{\mathrm{I}}}{g(a,\tilde{t})}\int_0^{\psi}F_n y_{\mathrm{II}}d\psi + \frac{y_{\mathrm{II}}}{g(a,\tilde{t})}\int_0^{\psi}F_n y_{\mathrm{I}}d\psi \tag{13}$$

where coefficients D_{1n} and D_{2n} can be determined by the initial conditions (5). For

$n=1$, we have $D_{11}=\dfrac{B_0(a,\tilde{t})A_1(a,\tilde{t})}{g(a,\tilde{t})}$, $D_{21}=0$. Multiplying Eq.(9) respectively by y_{I} and y_{II} and integrating from 0 to T with respect to ψ, we obtain the solvability conditions for Eq.(9)

$$\int_0^T F_n y_{\mathrm{I}} d\psi = 0 \tag{14}$$

$$\int_0^T F_n y_{\mathrm{II}} d\psi = 0 \tag{15}$$

Then A_n and B_n can be determined by Eqs.(14) and (15) respectively. Particularly, A_1 and B_1 can be worked out as

$$A_1 = \frac{B_0}{T\,g(a,\tilde{t})}\int_0^T y_{\mathrm{I}}\, h(y_0, B_0\frac{\partial y_0}{\partial\psi}, \tilde{t})d\psi \tag{16}$$

$$B_1 = \frac{1}{2g(a,\tilde{t})}\frac{\partial B_0}{\partial a}\int_0^T y_{\mathrm{I}}\, h(y_0, B_0\frac{\partial y_0}{\partial\psi}, \tilde{t})d\psi - \frac{B_0}{T\,g(a,\tilde{t})}\int_0^T y_{\mathrm{II}}\, h(y_0, B_0\frac{\partial y_0}{\partial\psi}, \tilde{t})d\psi \tag{17}$$

Hence, the asymptotic solution to $O(\varepsilon^2)$ is

$$y(t,\varepsilon) = y_0(a,\psi) + \varepsilon\, y_1(a,\psi,\tilde{t}) \tag{18}$$

and a and ψ can be solved from the following equations

$$\frac{da}{dt} = \varepsilon\, A_1(a,\tilde{t}) \tag{19}$$

$$\frac{d\psi}{dt} = B_0(a,\tilde{t}) + \varepsilon\, B_1(a,\tilde{t}) \tag{20}$$

The procedure can be carried out up to desired order, although the calculations are rather involved.

3. APPLICATION TO CUBIC NONLINEAR OSCILLATORS

As an application of the generalized KBM method, we consider the following cubic nonlinear oscillator

$$\frac{d^2y}{dt^2} + c_1(\tilde{t})y + c_2(\tilde{t})y^3 = \varepsilon\, k(y,\tilde{t})(\frac{dy}{dt})^n \tag{21}$$

$$y(0) = a_0\,, \quad \dot{y}(0) = 0$$

where n is a positive integer. For $n \geq 2$, the explicit approximations of Eq.(21) are difficult to obtain by the multiple scales method [11,12,13]. Suppose that the solution of Eq. (21) can be developed in the form of asymptotic expression (2). Note that $y_0(a,\psi)$ satisfies

$$B_0^2(a,\tilde{t})\frac{\partial^2 y_0}{\partial \psi^2} + c_1(\tilde{t})y_0 + c_3(\tilde{t})y_0^3 = 0 \qquad (22)$$

Eq. (22) has an exact analytical solution expressed by Jacobian elliptic functions in the case of $c_1(\tilde{t}) > 0$ and $c_3(\tilde{t}) > 0$

$$y_0 = a(\tilde{t})cn(K(v)\psi, v(\tilde{t})) = a(\tilde{t})cn(u,v) \qquad (23)$$

where $K(v)$ is the complete elliptic integral of the first kind with the modulus $\sqrt{v}$. Substituting Eq. (23) into Eq. (22), we can find that

$$B_0^2(a,\tilde{t}) = \frac{1}{K^2(v)}(c_1(\tilde{t}) + c_3(\tilde{t})a^2)$$

$$v^2(\tilde{t}) = \frac{c_3(\tilde{t})a^2}{2(c_1(\tilde{t}) + c_3(\tilde{t})a^2)}$$

From Eq. (16), $A_1(a,\tilde{t})$ can be solved as

$$A_1(a,\tilde{t}) = \frac{(-1)^{n+1}B_0^{n+1}K^n a^n}{4(c_1 + c_3 a^2)}\int_0^{4K} k(acn(u,v),\tilde{t})(sn(u,v)dn(u,v))^{n+1}du \qquad (24)$$

By the formulas of elliptic integrals [15], $A_1(a,\tilde{t})$ can be worked out and the amplitude $a(\tilde{t})$ is determined by Eq. (19). Similarly, we can carry out $B_1(a,\tilde{t})$, ψ, $y_1(a,\psi,\tilde{t})$ and so on.

When $c_1(\tilde{t}) > 0$ and $c_3(\tilde{t}) < 0$, the solution of Eq. (22) can be expressed by

$$y_0 = a(\tilde{t})sn(K(v)\psi, v(\tilde{t})) = a(\tilde{t})sn(u,v)$$

and then

$$B_0^2(a,\tilde{t}) = \frac{1}{K^2(v)}(c_1(\tilde{t}) + \frac{1}{2}c_3(\tilde{t})a^2)$$

$$v^2(\tilde{t}) = -\frac{c_3(\tilde{t})a^2}{2(c_1(\tilde{t}) + \frac{1}{2}c_3(\tilde{t})a^2)}$$

$$A_1(a,\tilde{t}) = \frac{B_0^{n+1}K^n a^n}{4(c_1 + c_3a^2)}\int_0^{4K} k(asn(u,v),\tilde{t})(cn(u,v)dn(u,v))^{n+1}du \tag{25}$$

Similarly, when $c_1(\tilde{t}) < 0$ and $c_3(\tilde{t}) > 0$, the solution of Eq. (22) can be expressed by

$$y_0 = a(\tilde{t})dn(K(v)\psi, v(\tilde{t})) = a(\tilde{t})dn(u,v)$$

and then

$$B_0^2(a,\tilde{t}) = \frac{1}{2K^2(v)}c_3(\tilde{t})a^2$$

$$v^2(\tilde{t}) = 2(1 + \frac{c_1(\tilde{t})}{c_3(\tilde{t})a^2})$$

$$A_1(a,\tilde{t}) = \frac{(-1)^{n+1}B_0^{n+1}K^n a^n v^{2(n+1)}}{4(c_1 + c_3a^2)}\int_0^{4K} k(adn(u,v),\tilde{t})(sn(u,v)cn(u,v))^{n+1}du \tag{26}$$

4. EXAMPLES

Example 1 Consider the following generalized Van der Pol oscillator

$$\frac{d^2y}{dt^2} + (1+\varepsilon t)y - (1+\varepsilon t)^3 y^3 + \varepsilon(\frac{1}{1+\varepsilon t} - y^2)\frac{dy}{dt} = 0 \tag{27}$$

$$y(0) = 0.6, \quad \dot{y}(0) = 0$$

From Eqs.(19) and (25) , we have

$$\frac{da}{dt} + \varepsilon\frac{a - 0.5(1+\varepsilon t)^2a^3}{[1-(1+\varepsilon t)^2a^2]K(v)}[\frac{1}{1+\varepsilon t}J_1(v) - a^2J_2(v)] = 0$$

with

$$J_1(v) = \int_0^K cn^2(u,v)dn^2(u,v)du = \frac{1}{3v}[(1+v)E(v) - (1-v)K(v)]$$

$$J_2(v) = \int_0^K sn^2(u,v)cn^2(u,v)dn^2(u,v)du$$

$$= \frac{1}{15v^2}[(1-v)(v-2)K(v) + 2(v^2 - v + 1)E(v)]$$

$$v = \frac{(1+\varepsilon t)^2 a^2}{2-(1+\varepsilon t)^2 a^2}$$

and $E(v)$ is the complete elliptic integral of the second kind associated with the modulus $\sqrt{v}$. Comparison of numerical solution and asymptotic amplitude obtained by Eqs.(19) and (25) with $\varepsilon = 0.01$ is shown in Fig. 1. In this paper the symbolic language *Mathematica* is used to implement the asymptotic and numerical solutions.

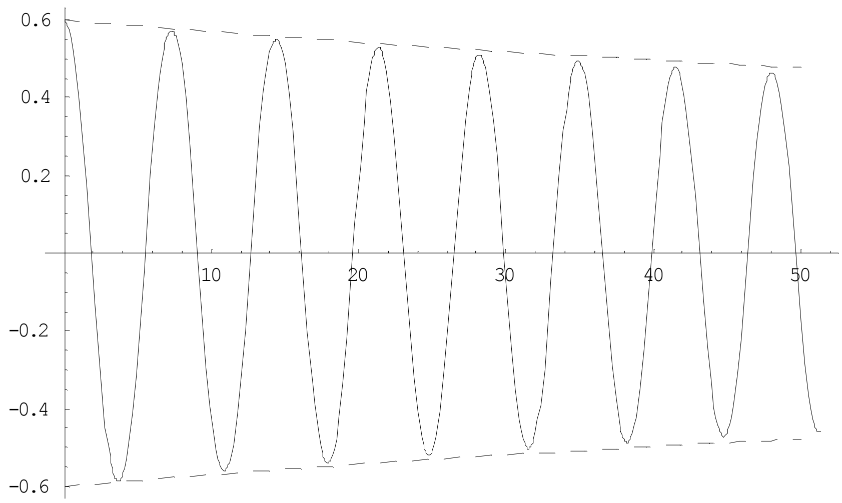

Fig. 1. Asymptotic amplitude and numerical solution of Eq. (27) with $\varepsilon = 0.01$
___, numerical solution and - - -, asymptotic amplitude

Example 2 Consider a cubic nonlinear oscillator with variable mass

$$\frac{d}{dt}(m(\tilde{t})\frac{dy}{dt}) + \varepsilon(\frac{dy}{dt})^3 + y + \frac{1}{6}y^3 = 0 \tag{28}$$

$$y(0) = 0.5, \quad \dot{y}(0) = 0$$

where $m(\tilde{t}) = 1 - \tilde{t} + \frac{1}{3}\tilde{t}^3$ and $\tilde{t} = \varepsilon t$. Eq. (28) is indeed a Rayleigh equation

$$(1-\varepsilon t + \frac{1}{3}(\varepsilon t)^3)\frac{d^2 y}{dt^2} + \varepsilon((-1+(\varepsilon t)^2)\frac{dy}{dt} + (\frac{dy}{dt})^3) + y + \frac{1}{6}y^3 = 0$$

From Eqs.(19) and (24), we have

$$\frac{da}{dt} + \varepsilon\{\frac{(1+\frac{1}{6}a^2)a^3 K(v)}{[1-\varepsilon t + \frac{1}{3}(\varepsilon t)^3]^2} J_3(v) + \frac{[-1+(\varepsilon t)^2]a}{[1-\varepsilon t + \frac{1}{3}(\varepsilon t)^3]K(v)} J_4(v)\} = 0$$

with

$$J_3(v)=\int_0^K sn^4(u,v)dn^4(u,v)du$$
$$=\frac{1}{35v^2}[(8v^3-15v^2+3v+2)K(v)-2(8v^3+6v^2-22v+1)E(v)]$$
$$J_4(v)=\int_0^K sn^2(u,v)dn^2(u,v)du=\frac{1}{3v}[(1-v)K(v)+(2v-1)E(v)]$$
$$v=\frac{a^2}{12+2a^2}$$

Comparison of numerical solution and asymptotic amplitude obtained by Eqs.(19) and (24) with $\varepsilon=0.01$ is shown in Fig. 2

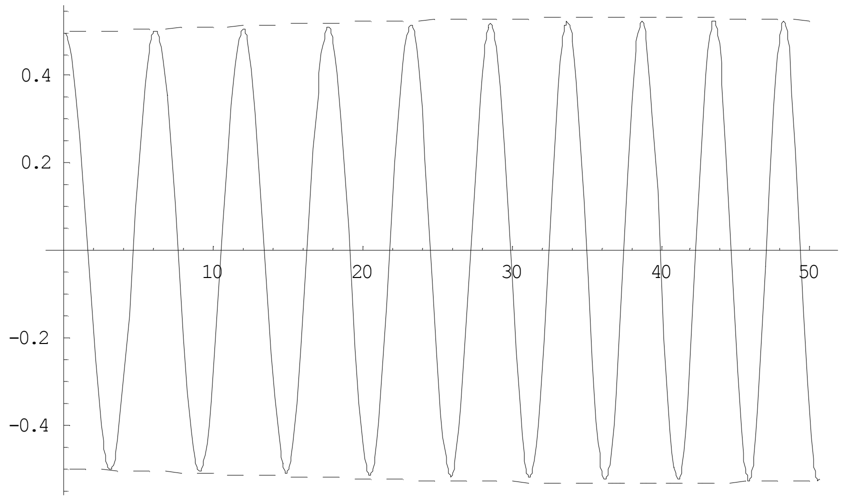

Fig. 2. Asymptotic amplitude and numerical solution of Eq. (28) with $\varepsilon=0.01$
___, numerical solution and - - -, asymptotic amplitude

Example 3 Consider a pendulum with slowly varying length

$$\frac{d}{dt}(l^2(\tilde{t})\frac{d\theta}{dt})+g\,l(\tilde{t})\sin\theta=0 \tag{29}$$

where θ is the angle of deviation of the pendulum from the vertical, g is the gravitational acceleration, $l(\tilde{t})$ is the slowly varying length. For not large oscillations, we can approximate $\sin\theta$ by the first two terms of the power series expansion, and then equation (29) becomes

$$\frac{d^2\theta}{dt^2}+\varepsilon\frac{2l'(\tilde{t})}{l(\tilde{t})}\frac{d\theta}{dt}+\frac{g}{l(\tilde{t})}\theta-\frac{g}{6l(\tilde{t})}\theta^3=0 \tag{30}$$

where $l' = \dfrac{dl}{d\tilde{t}}$. When $l(\tilde{t}) = (1+\tilde{t})^{\frac{1}{2}}$ and $g = 9.8$, Eq.(30) becomes

$$\frac{d^2\theta}{dt^2} + \varepsilon \frac{1}{1+\varepsilon t}\frac{d\theta}{dt} + \frac{9.8}{\sqrt{1+\varepsilon t}}\theta - \frac{9.8}{6\sqrt{1+\varepsilon t}}\theta^3 = 0 \tag{31}$$

From Eqs.(19) and (25), we have

$$\frac{da}{dt} + \varepsilon \frac{(12-a^2)a}{2(1+\varepsilon t)(6-a^2)K(v)} J_1(v) = 0$$

with $v = \dfrac{a^2}{12-a^2}$. Comparison of numerical solution and asymptotic amplitude obtained by Eqs.(19) and (25) with $\varepsilon = 0.01$ is shown in Fig. 3, where the initial conditions are $\theta(0) = \frac{1}{3}\pi$, $\dot{\theta}(0) = 0$.

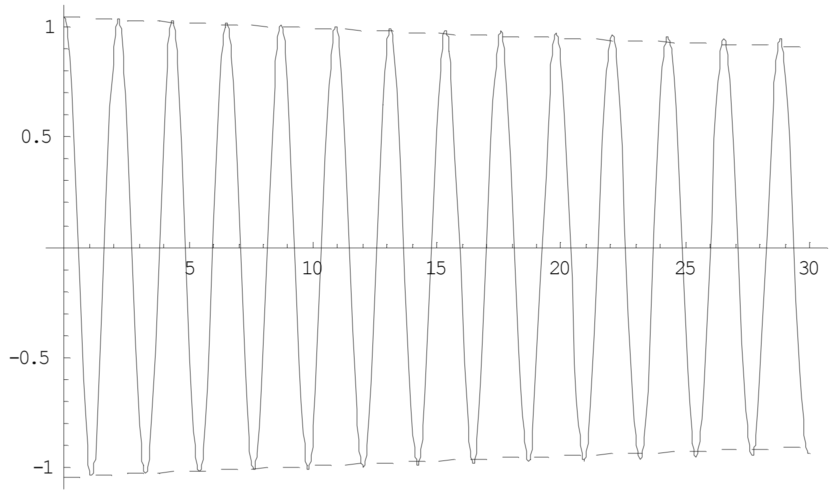

Fig. 3. Asymptotic amplitude and numerical solution of Eq. (31) with $\varepsilon = 0.01$
___, numerical solution and - - -, asymptotic amplitude

5. CONCLUSIONS

The generalized KBM method is effective for strongly nonlinear oscillators with slowly varying parameters and can obtain asymptotic solutions of arbitrary order theoretically, while the classical KBM method works only for weakly nonlinear oscillations and the EKB method can give only the leading order approximation.

Cubic nonlinear oscillators are studied in detail to illustrate the present method. Three examples are considered: a generalized Van der Pol oscillator, a Rayleigh equation and a pendulum with slowly varying length. The asymptotic results are in good agreement with the numerical solutions.

Acknowledgement-Research is supported by China Postdoctoral Science Foundation under grant No.2004036517, the Foundation of Advanced Research Center of Zhongshan University under grant No. 06M13, and Innovation Foundation for Young Technological Talents in Fujian Province, China, under grant No.2005J055.

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