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AN ANALYTICAL MODEL OF MOUNTAIN WAVE FOR WIND WITH SHEAR

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Abstract: A 2-D analytical mesoscale hydrostatic model of a stably stratified orographic barrier has been considered. Expressions for surface pressure perturbation, mountain drag and energy flux across Pirpanjal mountains of Kashmir valley for variable wind have been derived. These are further evaluated for realistic vertical profile of wind and temperature. Also study has been extended to obtain all these parameters for different vertical wind profile.
Keywords- Energy flux, Mountain drag, Surface pressure perturbation and Pirpanjal Mountains.

1. INTRODUCTION

Lyra [2] first addressed the study of 2-D mountain wave problem. He considered a 2-D model and obtained solutions using Green's function. Later Queney [3 and 4] purposed a theory in a stratified and rotating atmosphere and applied this theory to the flow over a 2-D bell shaped mountain. Afterward many studies have been done related to mountain wave.

Recently Teixeira et al. [6] developed an analytical model to predict the surface drag exerted by internal gravity waves on an isolated axisymmetric mountain for a velocity profile that varies relatively slowly with height based on Wentzel- Kramers-Brillouin (WKB) approximation. They showed that drag is proportional to inverse Richardson number R_i^{-1} and it decreases as R_i decreases for wind varies linearly with height. Afterward Teixeira and Miranda [7] modified the model of Teixeira et al. [6] to calculate the mountain drag exerted by a stratified flow over a 2-D mountain ridge. They showed that drag is strongly affected by the vertical variation of the background velocity than an axisymmetric mountain and calculated mountain drag and pressure perturbation at surface analytically.

In this paper, the aim is to evaluate the surface pressure perturbation, mountain drag and energy flux for variable wind across double ridge profile of Pirpanjal mountains of Kashmir valley using the model purposed by Teixeira and Miranda [7] for single ridge. Further study has been extended to obtain all these parameters for different vertical wind profile.

2. THEORETICAL MODEL

The analytical expression of Pirpanjal hills of Kashmir valley (Kumar et al. [1]), whose profile is shown in figure 1 is given by

$$h(x) = \frac{H}{1+x^2/a^2} + \frac{d}{dx}\left(\frac{b}{1+x^2/c^2}\right) + \frac{d}{1+(x-e)^2/f^2} \tag{1}$$

where, $H = 1.9km$; $a = 8.6019796km$; $b = 39.353073km$; $c = 49.038303km$; $d = 1.8835262km$, $e = 36.0km$ and $f = 178.43532km$

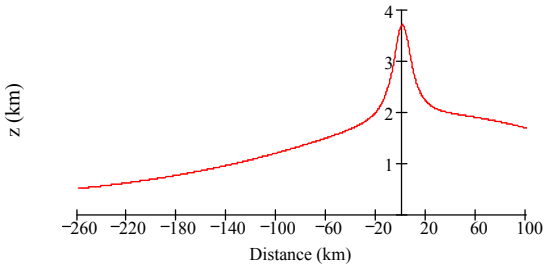

Fig. 1, Profile of Pirpanjal hills of Kashmir Valley

By Fourier transform of equation (1), we have

$$\hat{h}(k) = aHe^{-ak} + ikbce^{-ck} + dfe^{-(f-ei)k} \tag{2}$$

and

$$\begin{aligned}\hat{h}(k)\hat{h}^*(k) &= a^2H^2e^{-2ak} + k^2b^2c^2e^{-ck} + d^2f^2e^{-2fk} \\ &+ 2adfHe^{-(a+f)k}\cos(ek) + 2bcdfe^{-(c+f)k}k\sin(ek)\end{aligned} \tag{3}$$

To evaluate surface pressure perturbation across Pirpanjal mountains of Kashmir valley, we take the following expression of surface pressure perturbation

$$\hat{p}(z=0) = i\rho_0 NU_0\left[1 + \frac{i}{2}\frac{U_0'}{N} - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0U_0''}{N^2}\right)\right]\hat{h}(k) \tag{4}$$

(Teixeira and Miranda [7])

where $N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}$ is Brunt-Vaisala frequency, ρ_0 is mean density , U_0 is the unperturbed surface wind velocity, U_o' and U_0'' are the first order and second order derivative of U_0, $\hat{h}(k)$ is the Fourier transform of the profile of orographic barrier $h(k)$.

By inverse Fourier transform of equation (4), we have

$$p'(z=0) = i\rho_0 NU_0\left[1 + \frac{i}{2}\frac{U_0'}{N} - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0U_0''}{N^2}\right)\right]\int_{-\infty}^{\infty}\hat{h}(k)e^{ikx}dk \tag{5}$$

Now substituting equation (2) into equation (5) for real solution, we have

$$\begin{aligned}p'(z=0) &= i\rho_0 NU_0\left[1 + \frac{i}{2}\frac{U_0'}{N} - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0U_0''}{N^2}\right)\right] \\ &\times\int_0^{\infty}\left(aHe^{-ak} + ikbce^{-ck} + dfe^{-(f-ei)k}\right)e^{ikx}dk\end{aligned}$$

$$= -\rho_0 N U_0 \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]\left[\frac{aHx}{a^2+x^2} + \frac{bc\left(c^2-x^2\right)}{\left(c^2+x^2\right)^2} + \frac{df(e+x)}{f^2+(e+x)^2}\right]$$
$$-\frac{1}{2}\rho_0 U_0 U_0'\left[\frac{a^2 H}{a^2+x^2} - \frac{2bc^2 x}{\left(c^2+x^2\right)^2} + \frac{df^2}{f^2+(e+x)^2}\right] \quad (6)$$

Equation (6) is the analytical expression of surface pressure perturbation for 2- D profile of Pirpanjal mountains of Kashmir valley. Which contains two parts, first part is antisymmetric with respect to mountains and second part is symmetric with respect to mountains.

Now, the expression of mountain drag is

$$D = 2\pi\rho_0 N U_0 \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]\int_{-\infty}^{\infty} k\hat{h}(k)\hat{h}^*(k)dk \quad (7)$$

(Teixeira and Miranda [7])

Substitute equation (3) into equation (7), we get

$$D = 2\pi\rho_0 N U_0 \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]$$
$$\times \int_{-\infty}^{\infty} k\begin{pmatrix} a^2H^2e^{-2ak} + k^2b^2c^2e^{-2ck} + d^2f^2e^{-2fk} \\ + 2adfHe^{-(a+f)k}\cos(ek) + 2bcdfe^{-(c+f)k}k\sin(ek)\end{pmatrix}dk \quad (8)$$

So, mountain drag for real solution becomes

$$D = 2\pi\rho_0 N U_0 \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]$$
$$\times\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2+e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2+e^2\right)^3}\right] \quad (9)$$

Now the expression of energy flux at surface is

$$E = \int_{-\infty}^{\infty} p'(z=0)w'(z=0)dx = 2\pi\int_{-\infty}^{\infty}\hat{p}(z=0)\hat{w}^*(z=0)dk \quad (10)$$

As , $w'(z=0) = U_0\dfrac{\partial h}{\partial x}$

By its Fourier transform, we have

$$\hat{w}(z=0) = ikU_0\hat{h}(k) \quad (11)$$

Now substitute equation (10) into equation (11), we get

$$E = -2\pi i U_0\int_{-\infty}^{\infty} k\hat{p}(z=0)\hat{h}^*(k)dk \quad (12)$$

Substituting $\hat{p}(z=0)$ from equation (4) into equation (12), we have

$$E = 2\pi\rho_0 N U_0^2\left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]\int_{-\infty}^{\infty} k\hat{h}(k)\hat{h}^*(k)dk \quad (13)$$

Finally using equation (3) into equation (13) for real solution, we get

$$E = 2\pi\rho_0 N U_0^2 \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2}\right)\right]$$
$$\times \left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3}\right] \tag{14}$$

3. RESULTS AND CONCLUSIONS

The analytical expressions for surface pressure perturbation (equation 6), mountain drag (equation 9) and energy flux (equation 14) for 2-D profile of Pirpanjal mountains of Kashmir valley for variable wind have been derived. Further these expressions are evaluated for the real vertical profile of $U(z)$ and $T(z)$ for dated 02-08-2005 and 11-08-2005, which are shown in figures 2 and 3.

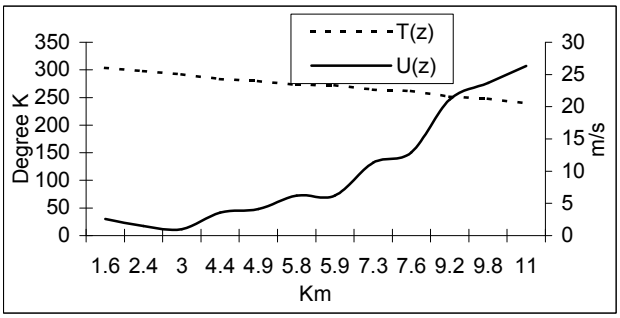

Fig. 2 Vertical Profile of U(z) and T(z) on 12Z of 02-08-05 at Srinagar

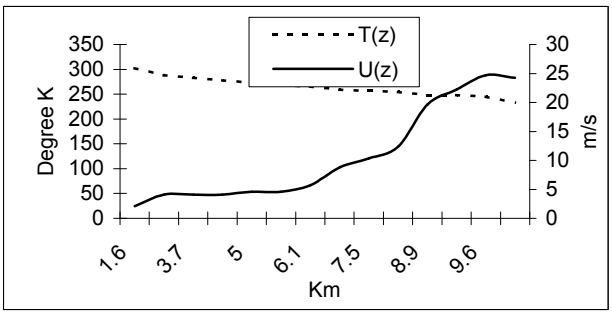

Fig. 3 Vertical Profile of U(z) and T(z) on 12Z of 11-08-05 at Srinagar

For dated 02-08-2005

$$P'(z=0) = -.999\left[\frac{ax}{a^2 + x^2} + \frac{bc(c^2 - x^2)}{H(c^2 + x^2)^2} + \frac{df(e+x)}{H\left(f^2 + (e+x)^2\right)}\right]$$
$$-.03615\left[\frac{a^2}{a^2 + x^2} - \frac{2bc^2 x}{H(c^2 + x^2)^2} + \frac{df^2}{H\left(f^2 + (e+x)^2\right)}\right]$$

$$D = .484\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3}\right]$$

$$E = 1.26\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3}\right]$$

For dated 11-08-2005

$$P'(z=0) = -\left[\frac{ax}{a^2 + x^2} + \frac{bc\left(c^2 - x^2\right)}{H\left(c^2 + x^2\right)^2} + \frac{df(e+x)}{H\left(f^2 + (e+x)^2\right)}\right]$$

$$-.0028\left[\frac{a^2}{a^2 + x^2} - \frac{2bc^2x}{H\left(c^2 + x^2\right)^2} + \frac{df^2}{H\left(f^2 + (e+x)^2\right)}\right]$$

$$D = .412\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3}\right]$$

$$E = .866\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3}\right]$$

The above expressions for surface pressure perturbation contain two parts, first part is antisymmetric with respect to mountains, second part is symmetric with respect to mountains and its contribution is negligible. Also as surface wind decreases in result magnitude of antisymmetric part increases and magnitude of symmetric part decreases. Also if we assume that wind is constant with height, in that case symmetric part becomes zero.

It can be noticed from the expressions of mountain drag and energy flux that as $f \to 0$, the magnitudes of last two factors increases, which are due to valley between the ridges, thus valley role becomes important for mountain drag and energy flux in case of $f \to 0$ and its profile is shown in figure 4.

When the distance between the ridges increases, the valley's contribution started decreases and for $f \to \infty$, last two terms of mountain drag and energy flux approaches zero. Thus for $f \to \infty$, there is no contribution of valley between the ridges for mountain drag and energy flux. The profiles of Pirpanjal mountains for $f = 10km$ and $f \to \infty$ are shown in figure 5 and 6 respectively.

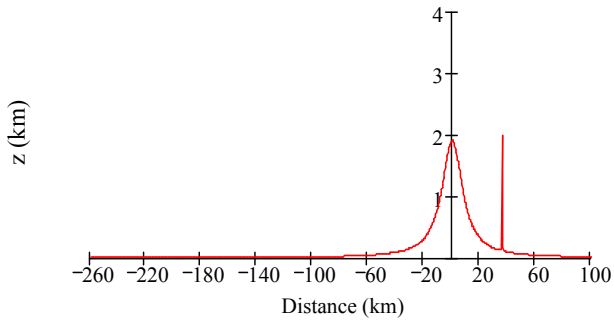

Fig. 4 Profile of Pirpanjal hills for $f \to 0$

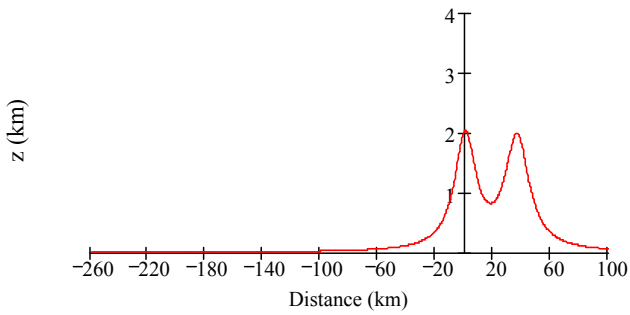

Fig. 5 Profile of Pirpanjal hills for $f = 10\,km$

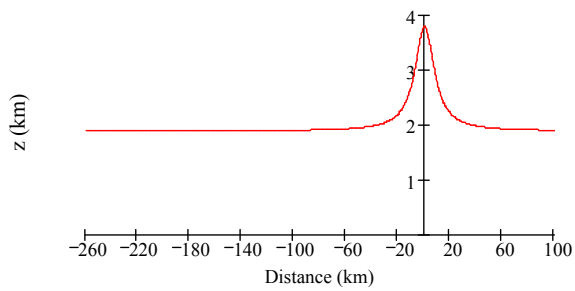

Fig. 6 Profile of Pirpanjal hills for $f = \infty$

For constant wind velocity, mountain drag (equation 9) and energy flux (equation 14) reduce to

$$D_0 = 2\pi\rho_0 N U_0 \left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2} \right) + 2adfH \frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf \frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3} \right]$$

$$E_0 = 2\pi\rho_0 N U_0^2 \left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2} \right) + 2adfH \frac{(a+f)^2 - e^2}{\left((a+f)^2 + e^2\right)^2} + 4bcdf \frac{3e(c+f)^2 - e^3}{\left((c+f)^2 + e^2\right)^3} \right]$$

Using above expressions into equation (9) and (14) respectively, we get

$$\frac{D}{D_0} = \frac{E}{E_0} = \left[1 - \frac{1}{8}\left(\frac{U_0'^2}{N^2} + 2\frac{U_0 U_0''}{N^2} \right) \right]$$

This implies that normalized mountain drag is equal to normalized energy flux and independent on the orographic barrier.

If in case wind rotates with height at constant rate such that

$$U = U_0 \cos(\beta z) \quad \text{(Shutts and Gadian [5])}$$

where, β is constant, so equations (6), (9) and (14) become

$$P'(z=0) = -\left[1 - \frac{1}{8R_i}\right]\left[\frac{ax}{a^2+x^2} + \frac{bc(c^2-x^2)}{H(c^2+x^2)^2} + \frac{df(e+x)}{H(f^2+(e+x)^2)}\right]$$

$$D = 2\pi\rho_0 N U_0\left[1 - \frac{1}{8R_i}\right]\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2+e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2+e^2\right)^3}\right]$$

and

$$E = 2\pi\rho_0 N U_0^2\left[1 - \frac{1}{8R_i}\right]\left[\frac{1}{4}\left(H^2 + d^2 + \frac{3}{2}\frac{b^2}{c^2}\right) + 2adfH\frac{(a+f)^2 - e^2}{\left((a+f)^2+e^2\right)^2} + 4bcdf\frac{3e(c+f)^2 - e^3}{\left((c+f)^2+e^2\right)^3}\right]$$

where $R_i = \dfrac{N^2}{U_0^2\beta^2}$ and its contour for surface pressure perturbation is shown in figure 7.

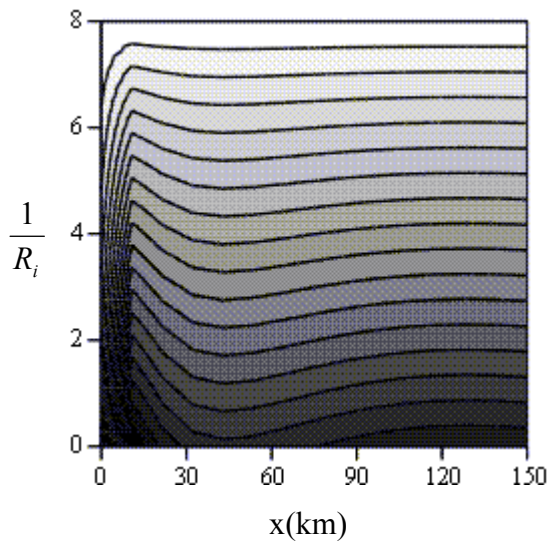

Fig. 7, Contour of surface pressure perturbation for $U = U_0 \cos(\beta z)$ between inverse Richardson number $\frac{1}{R_i}$ and distance x (km).

Further if wind decreases linearly with height such that $U = U_0\left(1 - \frac{z}{z_c}\right)$, where z_c is the constant. So surface pressure perturbation becomes

$$P'(z=0) = -\left\{ \begin{array}{l} \left[1 - \frac{1}{8R_i}\right]\left[\frac{ax}{a^2+x^2} + \frac{bc(c^2-x^2)}{H(c^2+x^2)^2} + \frac{df(e+x)}{H(f^2+(e+x)^2)}\right] \\ + \frac{1}{2\sqrt{R_i}}\left[\frac{a^2}{a^2+x^2} - \frac{bc^2x}{H(c^2+x^2)^2} + \frac{df^2}{H(f^2+(e+x)^2)}\right] \end{array} \right\}$$

where $R_i = \frac{z_c^2 N^2}{U_0^2}$ and its contour is shown in figure 8.

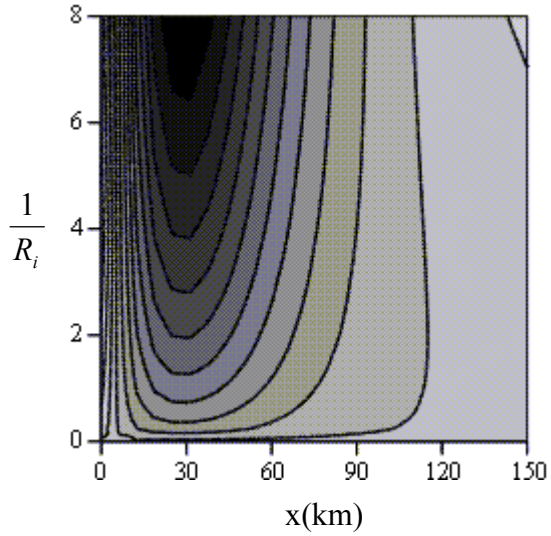

Fig. 8, Contour of surface pressure perturbation for $U = U_0\left(1 - \frac{z}{z_c}\right)$ between inverse Richardson number $\frac{1}{R_i}$ and distance x (km).

Now in case wind flows with parabolic profile such that

$$U = U_0\left(1 - \left(\frac{z}{z_c}\right)^2\right) \text{ (Teixeira and Miranda [7])}$$

where z_c is the constant. So in this case surface pressure perturbation from equation (6) reduces to

$$P'(z=0) = -\left[1 + \frac{1}{2R_i}\right]\left[\frac{ax}{a^2+x^2} + \frac{bc(c^2-x^2)}{H(c^2+x^2)^2} + \frac{df(e+x)}{H(f^2+(e+x)^2)}\right]$$

where $R_i = \frac{z_c^2 N^2}{U_0^2}$ and its contour is shown in figure 9.

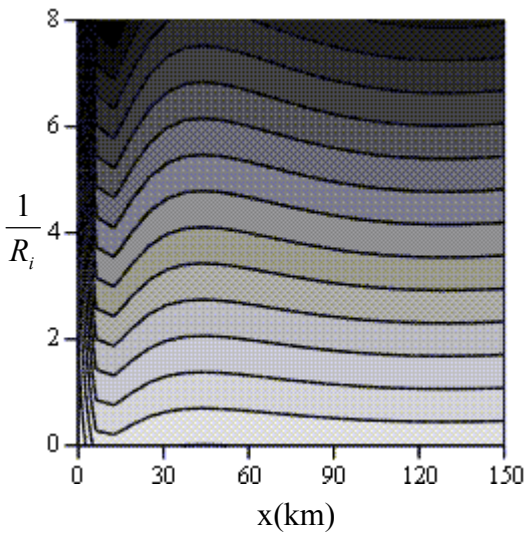

Figure 9, Contour of surface pressure perturbation for $U = U_0\left(1-\left(\frac{z}{z_c}\right)^2\right)$ between inverse Richardson number $\frac{1}{R_i}$ and distance x (km).

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