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THE OVERALL ASSURANCE INTERVAL FOR THE NON-ARCHIMEDEAN EPSILON IN DEA MODELS

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Abstract- Mehrabian, et. al. (1998) presented a procedure for determining overall assurance interval of ε. Solving n linear programs are needed for this propose, where n is the number of Decision Making Units involved in the evaluation. This paper proposes an efficient algorithm that can determine the overall assurance interval of ε by solving a few number of linear programs.
Keywords- Data Envelopment Analysis, Non-Archimedean Infinitesimal.

1. INTRODUCTION

Charnes, Cooper and Rhodes (1978) published a breakthrough paper which pioneered the Data Envelopment Analysis (DEA) approach, a methodology for determining the relative efficiencies of Decision Making Units (DMU`s). The data are input-output observations for a number of DMU`s using varying amounts of the same inputs to produce varying amounts of the same outputs.

In recent years DEA has enjoyed both rapid growth and widespread acceptance. A bibliography by Emrouznejad and Thanassolis (1997) contains almost 1500 studies employing the methodology of DEA. In these studies the two most frequently used models are the Charnes, Cooper and Rhodes (CCR) (1978) model and the Banker, Charnes and Cooper (BCC) (1984), model, which both involve the NON-Archimedean infinitesimal, ε.

As a theoretical construct, ε provides a lower bound for multipliers, to keep them away from zero. Some difficulties arise in representing an infinitesimal, because of finit tolerances in computer calculations.

Ali and Seiford (1993) have proposed an upper bound on ε for feasibility of the multiplier side and boundedness of the envelopment side of the CCR and BCC models (see also Chapter 4 of Charnes, et. al. (1994)). Mehrabian, et. al. (1998) showed that Ali and Seiford`s bound for ε cannot be valid. They also provided a procedure for determining an assurance interval of the non-Archimedean ε. Solving n linear programs are needed to determine the assurance interval where n is the number of decision making units under evaluation. This paper presents an efficient algorithm that can determine the assurance interval of the non-Archimedean ε by solving a few number of linear programs.

The paper is organized as follows. Section 2 presents definitions that are neede through the paper. This section also contains the procedure of determining the assurance interval of ε presented by Mehrabian, et. al. (1998). Section 3 presents the new efficient algorithm of determining the assurance interval of ε. Section 4 continues with an empirical example. Concluding remarks appear in section 5.

2. DEFINITIONS

Consider n Decision Making Units (DMU`s), each consuming varying amounts of m inputs in the production of s outputs. The $m \times n$ matrix of inputs is denoted by X and the $s \times n$ matrix of outputs by Y. Further, x_{ij} denotes the amount consumed of the ith input by the jth decision making unit and y_{rj} denotes the amount produced of its rth output. Finally, X_j and Y_j denotes, respectively, the vectors of inputs and outputs for the jth DMU.

The input-oriented linear programming problem formulation for the CCR and BCC models for evaluation of DMU$_o$ (both the envelopment and the multiplier side) are as follows:

CCR_p : *Envelopment Side* $\qquad$ CCR_d : *Multiplier Side*

$$\begin{array}{ll}
\min \theta - \varepsilon(1s^o + 1s^i) & \max UY_o \\
s.t: & s.t: \\
\theta X_o - X\lambda - s^i = 0, & VX_o = 1. \\
Y\lambda - s^o = Y_o, & UY_j - VX_j \le 0, \qquad j = 1,\dots,n \\
\lambda \ge 0, s^i \ge 0, s^o \ge 0 & U \ge \varepsilon 1, V \ge \varepsilon 1
\end{array}$$

BCC_p : *Envelopment Side* $\qquad$ BCC_d : *Multiplier Side*

$$\begin{array}{ll}
\min \theta - \varepsilon(1s^o + 1s^i) & \max UY_o + u_0 \\
s.t & s.t. \\
\theta X_o - X\lambda - s^i = 0, & VX_o = 1. \\
Y\lambda - s^o = Y_O & UY_j - VX_j + 1u_0 \le 0, \qquad j = 1,\dots,n \\
1\lambda = 1 & U \ge \varepsilon 1, \quad V \ge \varepsilon 1 \\
\lambda \ge 0, \quad s^i \ge 0, s^o \ge 0 &
\end{array}$$

Where ε is an non-Archimedean infinitesimal.

Mehrabian, et. al. (1998) defined P_o and $\overline{P}_o$ as the following LP problems corresponding to DMU$_o$:

$$\begin{array}{ll}
P_o : \max \varepsilon & \overline{P}_o : \max \varepsilon \\
s.t. & s.t. \\
VX_o = 1, & VX_o \le 1, \\
UY_j - VX_j \le 0, \quad j = 1,\dots,n & UY_j - VX_j \le 0, \quad j = 1,\dots,n \\
U \ge \varepsilon 1,, \quad V \ge \varepsilon 1 & U \ge \varepsilon 1, \quad V \ge \varepsilon 1
\end{array}$$

where here ε is a scalar variable. Note that the optimal solution of P_o is equal to the optimal solution of $\overline{P}_o$.

We review some definitions from Mehrabian, et. al. (1998).

Definition 1. The *assurance interval* for feasibility / boundedness of the CCR-model for the evaluation of DMU $_o$ is defined as the interval $[0, \varepsilon_o^*]$, where ε_o^* is the optimal value for P_o.

Definition 2. The intersection of all assurance interval for feasibility / boundedness of the CCR-model for the evaluation of all DMU`s defines the *overall assurance interval* $[0, \varepsilon^*]$ with $\varepsilon^* = \min\{\varepsilon_1^*, ..., \varepsilon_n^*\}$.

Definition 3. Each element of the overall assurance interval $[0, \varepsilon^*]$ defines an *assurance value* of the non-Archimedean ε for feasibility / boundedness of the CCR-model for evaluation of all DMU`s.

Based on Definition 2 the overall assurance interval $[0, \varepsilon^*]$ can be achieved by solving n linear programs: $P_1, P_2, ..., P_n$.

Similar statements can be given for the BCC-model.

3. A NEW ALGORITHM

This section presents an algorithm for obtaining the overall assurance interval of the non-Archimedean ε in the DEA models using a few linear programs, which is of computational important.

Consider the following LP problem:

$$
\begin{aligned}
P' : \max \quad & \varepsilon \\
s.t. \quad & \\
& VX_j \leq 1, \quad j = 1,2,...,n \\
& UY_j - VX_j \leq 0, \quad j = 1,2,...,n \\
& U \geq \varepsilon 1, \quad V \geq \varepsilon 1.
\end{aligned}
$$

Let $(\varepsilon^o, U^o, V^o)$ be the optimal solution of P'. Mehrabian et. al. (1998) showed that ε^o is a nonzero assurance value and also they showed that if $\{j : V^o X_j = 1\}$ is singleton, then $\varepsilon^o = \varepsilon^*$, i.e., solving only one LP problem leads to the overall assurance interval of the non-Archimedean ε. Here, this idea will be extended.

Let $J_p \subseteq \{1, ..., n\}$, consider P'_{J_p} as the following LP problem:

$$
\begin{aligned}
P'_{J_p} : \max \, & \varepsilon \\
s.t. \quad & \\
& VX_j \leq 1, \quad j \in J_p \\
& UY_j - VX_j \leq 0, \quad j = 1,...,n \\
& U \geq \varepsilon 1, \quad V \geq \varepsilon 1
\end{aligned}
$$

Lemma. If $(\varepsilon^{J_p}, U^{J_p}, V^{J_p})$ is the optimal solution of P'_{J_p} and $\{j : V^{J_p} X_j = 1\} = \{t\}$ is the singleton, then $\varepsilon^{J_p} = \min\{\varepsilon_j^* : j \in J_p\}$, where ε_j^* is the optimal solution of P_j.

Proof: Consider the dual of P'_{J_p} as follows:

$$D'_{J_p} : \min \sum_{j\in J_p} \theta_j$$
$$s.t.$$
$$1W + 1Z = 1$$
$$\sum_{j\in J_p} \theta_j X_j - X\lambda - Z = 0$$
$$Y\lambda - W = 0$$
$$W \geq 0, Z \geq 0, \lambda \geq 0, \theta_j \geq 0, j \in J_p.$$

By hypothesis, $V^{J_p} X_j < 1, \quad j \in J_p \setminus \{t\}$. Then the complementary slackness condition implies that in the optimal solution of $D'_{J_p}, \theta_j^{J_p} = 0, \quad j \in J_p \setminus \{t\}$. Hence, $\theta_t^{J_p}$ is the optimal value of the following problem:

$$\min \theta_t$$
$$s.t.$$
$$1W + 1Z = 1$$
$$\theta_t X_t - X\lambda - Z = 0$$
$$Y\lambda - W = 0$$
$$W \geq 0, Z \geq 0, \lambda \geq 0, \theta_t \geq 0$$

and this is a dual problem for $\overline{P}_t$, which has the optimal value of ε_t^*, then $\varepsilon^{J_p} = \varepsilon_t^*$. Since $\varepsilon^{J_p} \leq \min\{\varepsilon_j^* : j \in J_p\}$, then $\varepsilon^{J_p} = \min\{\varepsilon_j^* : j \in J_p\}$.

Extension of the foregoing lemma is trivially leaded to the following theorem:

Theorem. Let $J_1, J_2, ..., J_k$ be k subset of $\{1,2,...,n\}$ such that $J_1 \cup J_2 \cup ... \cup J_k = \{1,2,...,n\}$ and $(\varepsilon^{J_p}, U^{J_p}, V^{J_p})$ be the optimal solution of P'_{J_p} for which $\{j : V^{J_p} X_j = 1\}$ is singleton, for all $p, p = 1,2,,...,k$, then

$$\varepsilon^* = \min\{\varepsilon^{J_1}, \varepsilon^{J_2}, ..., \varepsilon^{J_k}\}$$

As a result of the theorem is to construct the following efficient algorithm that is used for finding the overall assurance interval of the non-Archimedean ε:

Step 0: Set $T = \{\}, \quad J = \{1,2,...,n\}$.

Step 1: Solve P'_J and find its its optimal solution: $(\varepsilon^J, U^J, V^J)$.

Step 2: Find $K = \{j : V^J X_j = 1\}$

If $|K| = 1$

then $T \leftarrow T \cup K$ and continue from Step 3,

else $J \leftarrow J \setminus K, \quad T \leftarrow T \cup K$ and come back to Step 1.

Step 3: Solve P_j and save its optimal solution ε_j^*, for all $j, \quad j \in T$.

Step 4: $\varepsilon^* = \min\{\varepsilon_j^* : j \in T\}$.

4. EMPIRICAL EXAMPLE

The algorithm of determining the overall assurance interval of ε is illustrated by means of the 44-unit data set of table 1 from Alirezai and Alamdar (1998). The data are concerned with 44 Power Plant Systems of Iran for the year 1997 and consist of three outputs (units performance factor, capability factor, and availability factor) and two inputs (production conditions and number of personnel).

The algorithm implication leads the following results:

$$T = \{33,\ 42,\ 39,\ 41,\ 13,\ 38,\ 12,\ 40,\ 23,\ 25,\ 44\}$$

The following table presents ε_j^* for $j \in T$.

j	ε_j^*
33	0.0025465641
42	0.0028170620
39	0.0027096794
41	0.0025388063
13	0.0037240236
38	0.0025148081
12	0.0026216156
40	0.0027549076
23	0.0026506155
25	0.0025719935
44	0.0026071612

Then

$$\varepsilon^* = \min\{\varepsilon_j^* : j \in T\} = 0.0025148081$$

Hence, the overall assurance interval is $[0,\ \varepsilon^*] = [0,\ 0.0025148081]$.

Note that the foregoing algorithm gives the overall assurance interval of non-Archimedean ε by solving a few number of LP problems and it is computationally efficient in comparison with procedure presented by Mehrabian, et. al. (1998) in which solving n LP problems were needed.

5. CONCLUSION

This paper presented an efficient algorithm for determining the overall assurance interval of the non-Archimedean ε in DEA models. Solving a few number of LP problems are needed for the propose of the algorithm while for the procedure presented by Mehrabian, et. al. (1998), n LP problems have to solve- where n is the number of units under evaluation.

An empirical test using 44 units data set confirmed the capability of proposed algorithm

Table 1: The data for 44 Power Plants Systems.

DMU`s	I_1	I_2	O_1	O_2	O_3
1	1.06	0.333	88.54	64.35	86.27
2	1.05	0.72	89.76	71.1	89.72
3	1.07	0.4	89.96	55.62	85.68
4	1.02	0.7	89.13	82.95	90.43
5	1.07	0.498	78.32	75.99	75.57
6	1.14	0.55	73.36	73.81	61.01
7	1.05	0.75	90.22	79.65	89.97
8	1.21	0.516	79.87	76.38	65.97
9	1.19	0.525	77.21	39.981	82.94
10	1.23	0.363	73.34	45.048	88.88
11	1.24	0.612	78.34	84.61	76.11
12	1.48	1.563	74.54	37.5	69.03
13	1.03	4.15	89.39	83.34	77.9
14	1.15	3.32	83.36	66.38	81.33
15	1.23	1.007	67.05	92.8	76.81
16	1.19	0.959	80.4	81.21	87.87
17	1.28	0.937	68.84	74.42	77.00
18	1.29	0.694	70.38	73.13	57.07
19	1.36	0.32	81.29	92.45	85.66
20	1.33	0.314	88.92	74.99	89.98
21	1.31	0.479	86.88	57.78	89.33
22	1.41	0.313	73.42	73.28	69.37
23	1.55	0.81	33.05	46.25	64.76
24	1.32	0.635	61.1	49.68	82.4
25	1.51	1.222	57.95	77.63	62.6
26	1.49	0.347	34.09	96.94	71.35
27	1.38	0.347	90.25	80.57	90.49
28	1.31	0.78	50.66	66.28	83.54
29	1.39	0.304	83	56.57	89.73
30	1.31	0.625	65.63	88.93	59.12
31	1.46	05	43.77	77.78	74.06
32	1.49	0.6	73.92	70.93	77.35
33	1.67	0.711	12.77	78.07	96.43
34	1.34	0.732	82.74	66.38	88.03
35	1.41	0.25	90.44	50.55	55.05
36	1.42	0.346	92.35	57.16	62.23
37	1.37	0.444	82.04	74.68	41.65
38	1.61	0.909	94.49	70.72	51.39
39	1.43	2	99.36	94.08	54.69
40	1.6	0.526	87.6	65.14	26.88
41	1.64	0.795	76.11	68.6	24.5
42	1.37	3.333	100	57.39	48.82
43	1.43	0.588	91.29	91.18	42.1
44	1.49	1.111	89.43	100	47.12

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