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NATURAL FREQUENCIES OF BEAM-MASS SYSTEMS IN TRANSVERSE MOTION FOR DIFFERENT END CONDITIONS

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Abstract-In this study, an Euler-Bernoulli type beam carrying masses at different locations is considered. Natural frequencies for transverse vibrations are investigated for different end conditions. Frequency equations are obtained for two and three mass cases. Analytical and numerical results are compared with each other.
Keywords- beam vibrations, concentrated masses.

1. INTRODUCTION

Transverse vibrations of beam-mass systems have been investigated by many researchers. Approximate and exact analysis were used to calculate the natural frequencies. The effects of mass, rotary inertia and springs were investigated [1-7]. Gürgöze and Batan [8] considered the numerical solution of the transcendental frequency equation. The characteristic equation was obtained using Rayleigh-Ritz method [9] and free vibrations were analyzed using Laplace transform and Rayleigh-Ritz method [10]. Maurizi and Belles [11] compared two fundamental theories of beam vibrations. Chai and Low [12] investigated the natural frequencies of a beam with a mass near the beam's ends. Low *et al.* [13] found that the results of experiments and the theory did not match well for beams of large slenderness ratio for centre loaded beams. Hamdan and Abdel Latif [14] compared Rayleigh-Ritz, Galerkin, Finite Elements and Exact solutions and showed finite elements method was preferable due to numerical stability and accuracy and these methods had a reasonably good accuracy and convergence rate for small attached inertia values. Özkaya *et al.* [15] analyzed non-linear free and forced vibrations of a beam-mass system by considering five different sets of boundary conditions by considering the effects of the location and the magnitude of the mass on the natural frequencies. Different assumed shape functions to obtain the kinetic and potential energies of the three classical beams carrying a concentrated mass were presented [16,17]. Low *et al.* [18] presented both experimental and theoretical results using Rayleigh-Ritz procedure and showed that the correlation between theory and experiments was much improved when stretching effects were included. Auciello and Nole [19] determined the free vibration frequencies of a beam composed of two tapered beam sections with different physical characteristics with a mass at its end. Özkaya and Pakdemirli [20] obtained the frequencies for the clamped-clamped beam with mass and searched approximate solutions for free and forced non-linear vibrations using a perturbation method. The solutions were compared with the results of both analytical and artificial neural network method [21]. Naguleswaran [22] presented the frequency equations for all the combinations of the classical boundary conditions and for various magnitudes and positions of a single particle mass. Öz [23] and Özkaya [24] calculated the frequencies of a beam carrying mass using FEM and analytical methods, and compared with other solutions. Turhan [25] considered the problem with a single mass for various classical end conditions using approximate methods and showed that resulting formulae can be put in reasonably simple forms in the special cases where the beam is symmetrically supported.

In this study, an Euler-Bernoulli type beam carrying masses on different locations is considered. Natural frequencies for transverse vibrations are investigated for different end conditions. Analytical and numerical results are compared with each other.

2. EQUATIONS OF MOTION

In this section, equations of motion for different cases will be derived. The Lagrangian of the system consisting of n masses can be written as follows

$$£=\frac{1}{2}\sum_{m=0}^{n}\int_{x_m}^{x_{m+1}}\rho A\dot{w}_{m+1}^{*\,2}dx^{*}+\frac{1}{2}\sum_{m=1}^{n}M_m\dot{w}_m^{*\,2}\left(x_m,t^{*}\right)-\frac{1}{2}\sum_{m=0}^{n}\int_{x_m}^{x_{m+1}}EIw_{m+1}^{*''\,2}dx^{*}\;,x_0=0,\;x_{n+1}=L \quad (1)$$

where n denotes number of concentrated masses, ρA is the mass per unit length of the beam, w_{m+1} is the displacement of the different portions of the beam which are separated by concentrated masses, M_m is the concentrated mass at location m, EI is the flexural rigidity of the beam, ($\dot{}$) and ( )′ are derivatives with respect to time and spatial variables. The first two terms in equation (1) are the kinetic energies of the beam and concentrated masses respectively, the last term is the elastic energy due to bending of the beam.

Invoking Hamilton's principle

$$\delta\int_{t_1}^{t_2} £dt^{*}=0 \quad (2)$$

and substituting the Lagrangian from equation (1), performing the necessary algebra, we finally obtain the following set of linear differential equations

$$\rho A\ddot{w}_{m+1}^{*}+EIw_{m+1}^{*\,iv}=0 \qquad m=0,1,2...n \quad (3)$$

In equation (3), the number of equations is n+1. The boundary conditions are as follows

Simple-simple ends: $w_1^{*}\left(0,t^{*}\right)=0,\; w_1^{*''}\left(0,t^{*}\right)=0,\; w_{n+1}^{*}\left(L,t^{*}\right)=0,\; w_{n+1}^{*''}\left(L,t^{*}\right)=0$ (4)

Fixed-fixed ends: $w_1^{*}\left(0,t^{*}\right)=0,\; w_1^{*'}\left(0,t^{*}\right)=0,\; w_{n+1}^{*}\left(L,t^{*}\right)=0,\; w_{n+1}^{*'}\left(L,t^{*}\right)=0$ (5)

Simple-fixed ends: $w_1^{*}\left(0,t^{*}\right)=0,\; w_1^{*''}\left(0,t^{*}\right)=0,\; w_{n+1}^{*}\left(L,t^{*}\right)=0,\; w_{n+1}^{*'}\left(L,t^{*}\right)=0$ (6)

Fixed-free ends: $w_1^{*}\left(0,t^{*}\right)=0,\; w_1^{*'}\left(0,t^{*}\right)=0,\; w_{n+1}^{*''}\left(L,t^{*}\right)=0,\; w_{n+1}^{*'''}\left(L,t^{*}\right)=0$ (7)

General end conditions for masses are as follows

$$w_p^{*}\left(x_p^{*},t^{*}\right)=w_{p+1}^{*}\left(x_p^{*},t^{*}\right),\; w_p^{*'}\left(x_p^{*},t^{*}\right)=w_{p+1}^{*'}\left(x_p^{*},t^{*}\right),\; w_p^{*''}\left(x_p^{*},t^{*}\right)=w_{p+1}^{*''}\left(x_p^{*},t^{*}\right), \quad (8)$$

$$EIw_p^{*'''}\left(x_p^{*},t^{*}\right)-EIw_{p+1}^{*'''}\left(x_p^{*},t^{*}\right)-M_p\ddot{w}_p^{*}\left(x_p^{*},t^{*}\right)=0,\quad p=1,2,\ldots n$$

Dimensionless parameters are introduced

$$x=\frac{x^{*}}{L},\; w_p=\frac{w_p^{*}}{L},\; \eta_{\mathrm{p}}=\frac{x_p^{*}}{L},\; t=\frac{t^{*}}{L^2}\sqrt{\frac{EI}{\rho A}},\; \alpha_{\mathrm{p}}=\frac{M_p}{\rho AL} \quad (9)$$

where α_p is the ratio of concentrated masses to the mass of the beam. R is the radius of inertia. Also $\eta_0 = 0$, $\eta_{n+1} = 1$. After inserting non-dimensional parameters, we obtain the equations of motion and boundary conditions for masses as follows

$$\ddot{w}_{m+1} + w_{m+1}^{iv} = 0 \qquad \text{m=0,1,2,...n} \tag{10}$$

$$w_p(\eta_p,t) = w_{p+1}(\eta_p,t), \quad w_p'(\eta_p,t) = w_{p+1}'(\eta_p,t), \quad w_p''(\eta_p,t) = w_{p+1}''(\eta_p,t) \tag{11}$$

$$w_p'''(\eta_p,t) - w_{p+1}'''(\eta_p,t) - \alpha_p \ddot{w}_p(\eta_p,t) = 0$$

These equations will be solved analytically for different end conditions and masses in the next section.

3. ANALYTICAL SOLUTION

One can assume

$$w_{m+1} = Y_{m+1}(x)e^{i\omega t} + cc \tag{12}$$

for the solution of equation (10), where cc stands for complex conjugate and ω is the natural frequency of the vibrations. Inserting equation (12) into equation (10),

$$Y_{m+1}^{iv} - \omega^2 Y_{m+1} = 0 \tag{13}$$

is obtained. The end conditions are as follows

Simple-simple ends: $Y_1(0) = Y_1''(0) = 0$, $\quad Y_{n+1}(1) = Y_{n+1}''(1) = 0$ (14)

Fixed-fixed ends: $Y_1(0) = Y_1'(0) = 0$, $\quad Y_{n+1}(1) = Y_{n+1}'(1) = 0$ (15)

Simple-fixed ends: $Y_1(0) = Y_1''(0) = 0$, $\quad Y_{n+1}(1) = Y_{n+1}'(1) = 0$ (16)

Fixed-free ends: $Y_1(0) = Y_1'(0) = 0$, $\quad Y_{n+1}''(1) = Y_{n+1}'''(1) = 0$ (17)

The boundary conditions for concentrated masses are as follows

$$Y_p(\eta_p) = Y_{p+1}(\eta_p), \quad Y_p'(\eta_p) = Y_{p+1}'(\eta_p), \quad Y_p''(\eta_p) = Y_{p+1}''(\eta_p),$$

$$Y_p'''(\eta_p) - Y_{p+1}'''(\eta_p) - \alpha_p \omega^2 Y_p(\eta_p) = 0 \tag{18}$$

The solution for equation (13) yields the mode shapes

$$Y_{m+1} = C_{m+1,1}\sin(kx) + C_{m+1,2}\cos(kx) + C_{m+1,3}\sinh(kx) + C_{m+1,4}\cosh(kx) \tag{19}$$

where $k = \sqrt{\omega}$. If the assumed solution (19) is arranged using the end conditions in equations (13)-(17), then frequency equations are obtained for each end conditions. Frequency equations were given for a single mass (n=1) and for some end conditions in references [15, 20, 21, 23]. For a simple- simple beam with two concentrated masses, the frequency equation is

$$
\begin{aligned}
&-2\alpha_1\alpha_2 k^2\{\cos(1+\eta_1-\eta_2)k-\cos(-1+\eta_1+\eta_2)k\}\cosh(1+\eta_1-\eta_2)k\\
&+\cosh(-1+\eta_1+\eta_2)k\{2\alpha_1\alpha_2k^2\cos(1+\eta_1-\eta_2)k-2\alpha_1\alpha_2k^2\cos(-1+\eta_1+\eta_2)k\}\\
&+\cosh(k-2\eta_1k)\{-\alpha_1\alpha_2k^2\cos k+\alpha_1\alpha_2k^2\cos(k-2\eta_2k)\}\\
&+\cosh(k-2\eta_2k)\{-\alpha_1\alpha_2k^2\cos k+\alpha_1\alpha_2k^2\cos(k-2\eta_1k)\}\\
&+k\cosh k\ \{2\alpha_1\alpha_2k\cos k-\alpha_1\alpha_2k\cos(k-2\eta_1k)-\alpha_1\alpha_2k\cos(k-2\eta_2k)+4\alpha_1\sin k\\
&+4\alpha_2\sin k\}+\sinh k\{4\alpha_1k\cos k+4\alpha_2k\cos k-4\alpha_1k\cos(k-2\eta_1k)-4\alpha_2k\cos(k-2\eta_2k)\\
&+16\sin k+\alpha_1\alpha_2k^2\sin(1+2\eta_1-2\eta_2)k+\alpha_1\alpha_2k^2\sin(k-2\eta_1k)\\
&-\alpha_1\alpha_2k^2\sin(k-2\eta_2k)\}+\sin k\{-\alpha_1\alpha_2k^2\ \sinh(1+2\eta_1-2\eta_2)k-\alpha_1\alpha_2k^2\sinh(k-2\eta_1k)\\
&+\alpha_1\alpha_2k^2\ \sinh(k-2\eta_2k)-4\alpha_1k\cosh(k-2\eta_1k)\ -4\alpha_2k\cosh(k-2\eta_2k)\}
\end{aligned} \tag{20}
$$

For a simple- fixed beam with two concentrated masses, the frequency equation is

$$
\begin{aligned}
&-4\alpha_2k\cos k\ \cosh(k-2\eta_2k)+\alpha_1\alpha_2k^2\cosh(1+2\eta_1-2\eta_2)k\ \sin k\\
&+\sin(1+\eta_1-\eta_2)k\{2\alpha_1\alpha_2k^2\cosh(1+\eta_1-\eta_2)k-2\alpha_1\alpha_2k^2\cosh(-1+\eta_1+\eta_2)k\}\\
&+\sin\eta_2k\{-4\alpha_1\alpha_2k^2\cosh(2\eta_1-\eta_2)k+4\alpha_1\alpha_2k^2\cosh\eta_2k\}\\
&+\sin(-1+\eta_1+\eta_2)k\{2\alpha_1\alpha_2k^2\cosh(1+\eta_1-\eta_2)k-2\alpha_1\alpha_2k^2\cosh(-1+\eta_1+\eta_2)k\}\\
&-\alpha_1\alpha_2k^2\cosh(k-2\eta_2k)\ \sin(k-2\eta_1k)+\cosh k\{4\alpha_1k\cos(k-2\eta_1k)+4\alpha_2k\cos(k-2\eta_2k)\\
&-16\sin k-2\alpha_1\alpha_2k^2\sin k-\alpha_1\alpha_2k^2\sin(1+2\eta_1-2\eta_2)k+2\alpha_1\alpha_2k^2\sin(k-2\eta_2k)\}\\
&-\alpha_1k\cosh(k-2\eta_1k)\{4\cos k+\alpha_2k[-2\sin k+\sin(k-2\eta_2k)]\}+\{16-2\alpha_1\alpha_2k^2\}\cos k\ \sinh k\\
&+\sinh k\{\alpha_1\alpha_2k^2\cos(1+2\eta_1-2\eta_2)k+2\alpha_1\alpha_2k^2\cos(k-2\eta_1k)+4\alpha_1k\sin(k-2\eta_1k)\\
&+4\alpha_2k\sin(k-2\eta_2k)\}-8k\sin k\sinh k(\eta_1+\eta_2)+16\alpha_1k\sin k\eta_1\ \sinh k\eta_1\\
&-\alpha_1\alpha_2k^2\cos k\ \sinh(1+2\eta_1-2\eta_2)k+\sinh(1+\eta_1-\eta_2)k\{2\alpha_1\alpha_2k^2\cos(1+\eta_1-\eta_2)k\\
&-2\alpha_1\alpha_2k^2\cos(-1+\eta_1+\eta_2)k\}+\sinh\eta_2k\{-4\alpha_1\alpha_2k^2\cos(2\eta_1-\eta_2)k+4\alpha_1\alpha_2k^2\cos\eta_2k\\
&+16\alpha_2k\sin\eta_2k\}+\sinh(-1+\eta_1+\eta_2)k\{2\alpha_1\alpha_2k^2\cos(1+\eta_1-\eta_2)k\\
&-2\alpha_1\alpha_2k^2\cos(-1+\eta_1+\alpha_2)k\}+\sinh(k-2\eta_1k)\{-\alpha_1\alpha_2k^2\cos(k-2\eta_2k)+4\alpha_1k\sin k\}\\
&+\sinh(k-2\eta_2k)\{-\alpha_1\alpha_2k^2\cos(k-2\eta_1k)+2\alpha_1\alpha_2k^2\cos k\ +4\alpha_2k\sin k\}
\end{aligned} \tag{21}
$$

The symbolic calculations for 3 and more masses are very difficult. That's why numerical methods will be better for calculating frequencies of beams having 3 or more concentrated masses. The frequencies calculated with equations (20) and (21) and Finite Element Methods (FEM) [23] will be given in the next section.

4. NUMERICAL SOLUTIONS

Numerical values for the natural frequencies for the first five modes will be given in this section. In Tables 1-8, the first five frequencies are presented for simple-simple, fixed-fixed, simple-fixed and fixed-free boundary conditions. The frequencies are calculated for beams having two and three masses from equations (20) and (21) for analytical and from equations in reference [23] for FEM solutions. Analytical and FEM results are close to each other as shown in the tables. It is difficult to find the frequency equations (determinants) for three and more concentrated mass systems, that's why the numerical solutions will be simpler and faster for these cases.

Table 1. Natural frequencies of a simple-simple beam with two masses

α_1,α_2	η_1,η_2	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1	0.1,0.3	6.1182	6.1182	26.5060	26.5457	55.4118	55.4118	99.0970	99.0971
1,1	0.5,0.7	4.7297	4.7314	25.1279	25.1281	60.8832	60.8833	141.289	141.2894
1,10	0.1,0.3	2.5095	2.5094	26.0754	26.0754	51.0693	51.0693	94.5054	94.5055
1,10	0.5,0.7	2.3875	2.3867	17.9251	17.9250	59.5695	59.5696	136.993	136.9939
10,1	0.1,0.3	4.5140	4.5139	18.5627	18.5627	38.5780	38.5779	96.6938	96.6938
10,1	0.5,0.7	2.0777	2.0771	22.0363	22.0366	54.6468	54.6468	140.866	140.8656
10,10	0.1,0.3	2.3567	2.3569	16.2569	16.2569	29.9752	29.9753	92.8631	92.8630
10,10	0.5,0.7	1.6769	1.6739	9.8120	9.8114	53.5165	53.5165	136.5350	136.5348

Table 2. Natural frequencies of a simple-simple beam with three masses

$\alpha_1,\alpha_2,\alpha_3$	η_1,η_2,η_3	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1,1	0.1,0.4,0.8	5.1305	5.1303	18.9150	18.9150	40.6683	40.6683	101.949	101.9495
1,1,10	0.1,0.4,0.8	3.0114	3.0114	11.7311	11.7311	39.4456	39.4456	98.7132	98.7132
1,10,1	0.1,0.4,0.8	2.1818	2.1819	17.1861	17.1866	37.3559	37.3559	99.3226	99.3224
10,1,1	0.1,0.4,0.8	4.1416	4.1416	13.0206	13.0203	25.9585	25.9585	99.4389	99.4389
10,10,10	0.1,0.4,0.8	1.8639	1.8640	6.67504	6.6752	14.1606	14.1606	93.7742	93.7744
1,1,1	0.2,0.5,0.7	4.4113	4.4106	18.2005	18.2003	39.1895	39.1894	137.980	137.9803
1,1,10	0.2,0.5,0.7	2.3503	2.3519	13.4689	13.4689	35.0008	35.0009	134.77	134.7699
1,10,1	0.2,0.5,0.7	2.0482	2.0485	18.1854	18.1855	29.3781	29.3781	137.958	137.9582
10,1,1	0.2,0.5,0.7	2.8578	2.8574	10.7711	10.7708	35.3795	35.3794	137.274	137.2738
10,10,10	0.2,0.5,0.7	1.5399	1.5449	6.3834	6.3845	13.5785	13.5790	134.252	134.2524

Table 3. Natural frequencies of a fixed-fixed beam with two masses

α_1,α_2	$\eta_1,\ \eta_2$	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1	0.1,0.4	12.3268	12.3268	51.0975	51.0974	74.7559	74.7559	148.3840	148.3835
1,1	0.5,0.8	11.2458	11.2456	35.0097	35.0094	78.6324	78.6324	170.7570	170.7589
1,10	0.1,0.4	4.5635	4.5636	49.9799	49.9800	71.6430	71.6430	147.5960	147.5945
1,10	0.5,0.8	7.4207	7.4212	18.7562	18.7561	76.8184	76.8184	166.6700	166.6706
10,1	0.1,0.4	11.5266	11.5265	25.8796	25.8800	57.3110	57.3095	143.2110	143.2112
10,1	0.5,0.8	4.2764	4.2760	33.2652	33.2653	75.4796	75.4797	169.0700	169.0685
10,10	0.1,0.4	4.5279	4.5280	24.4679	24.4680	53.6742	53.6742	143.0190	143.0191
10,10	0.5,0.8	4.0362	4.0361	12.4227	12.4216	74.1371	74.1370	164.5810	164.5808

Table 4. Natural frequencies of a fixed-fixed beam with three masses

$\alpha_1,\alpha_2,\alpha_3$	η_1,η_2,η_3	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1,1	0.1,0.4,0.8	11.7922	11.7922	30.7215	30.7211	67.7822	67.7822	110.967	110.9672
1,1,10	0.1,0.4,0.8	7.63826	7.6387	17.1981	17.1982	66.3658	66.3658	105.849	105.8486
1,10,1	0.1,0.4,0.8	4.5411	4.5410	28.6699	28.6697	66.8277	66.8276	107.863	107.8624
10,1,1	0.1,0.4,0.8	11.1358	11.1357	24.0862	24.0862	34.1821	34.1820	105.604	105.6039
10,10,10	0.1,0.4,0.8	4.29001	4.2895	10.8779	10.8765	25.0046	25.0046	97.4892	97.4891
1,1,1	0.2,0.5,0.7	9.8651	9.8652	28.388	28.3881	47.2248	47.2249	170.693	170.6932
1,1,10	0.2,0.5,0.7	5.2111	5.2099	20.5397	20.5397	42.5816	42.5815	170.676	170.6761
1,10,1	0.2,0.5,0.7	4.1922	4.1927	27.9181	27.9180	39.0948	39.0947	168.932	168.9323
10,1,1	0.2,0.5,0.7	7.1547	7.1546	15.5748	15.5747	41.0639	41.0638	166.417	166.4168
10,10,10	0.2,0.5,0.7	3.4315	3.4318	9.9660	9.9659	15.9884	15.9884	164.119	164.1189

Table 5. Natural frequencies of a simple-fixed beam with two masses

α_1,α_2	η_1,η_2	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1	0.1,0.4	8.0935	8.0936	35.2747	35.2747	60.4191	60.4190	145.022	145.0226
1,1	0.5,0.8	8.4048	8.4048	32.2827	32.2825	58.0458	58.0457	153.725	153.7242
1,10	0.1,0.4	3.1113	3.1115	34.4019	34.4019	56.0053	56.0054	144.673	144.6738
1,10	0.5,0.8	6.2389	6.2387	15.8460	15.8459	55.2028	55.2027	150.501	150.5008
10,1	0.1,0.4	5.7707	5.7707	18.4104	18.4104	56.7723	56.7874	141.896	141.8950
10,1	0.5,0.8	3.2258	3.2256	31.7872	31.7869	53.3825	53.3825	153.661	153.6618
10,10	0.1,0.4	2.9428	2.9427	13.3775	13.3775	53.3560	53.3560	141.793	141.7945
10,10	0.5,0.8	3.0904	3.0906	11.9999	12.0000	51.2347	51.2347	150.3000	150.2992

Table 6. Natural frequencies of a simple-fixed beam with three masses

$\alpha_1,\alpha_2,\alpha_3$	η_1,η_2,η_3	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1,1	0.1,0.4,0.8	7.8991	7.8991	27.3065	27.3064	42.5047	42.5046	107.4010	107.4006
1,1,10	0.1,0.4,0.8	6.1687	6.1683	13.3744	13.3742	39.7342	39.7343	101.7990	101.7989
1,10,1	0.1,0.4,0.8	3.1008	3.1011	27.0739	27.07410	38.5989	38.5990	105.0960	105.0958
10,1,1	0.1,0.4,0.8	5.7193	5.7191	17.2349	17.2349	32.9085	32.9085	104.8830	104.8825
10,10,10	0.1,0.4,0.8	2.8545	2.8544	9.9384	9.9385	14.5616	14.5616	96.9660	96.9660
1,1,1	0.2,0.5,0.7	6.9686	6.9689	21.2964	21.2960	43.3097	43.3098	161.7880	161.7883
1,1,10	0.2,0.5,0.7	4.2861	4.2876	14.3862	14.3864	35.9369	35.9370	161.6720	161.6722
1,10,1	0.2,0.5,0.7	3.1284	3.1280	20.6621	20.6621	34.4942	34.4943	159.3900	159.3905
10,1,1	0.2,0.5,0.7	3.8659	3.8660	14.1970	14.1974	40.5099	40.5101	159.7870	159.7866
10,10,10	0.2,0.5,0.7	2.4381	2.4377	7.3628	7.3629	14.8312	14.8312	156.9830	156.9858

Table 7. Natural frequencies of a fixed-free beam with two masses

α_1,α_2	η_1,η_2	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1	0.1,0.4	3.1802	3.1810	13.5261	13.5262	50.8105	50.8105	74.6163	74.6163
1,1	0.5,0.8	1.8602	1.8589	12.7765	12.7763	54.0891	54.0890	82.0620	82.0620
1,10	0.1,0.4	1.8816	1.8824	8.4921	8.4928	49.5416	49.5417	71.4882	71.4882
1,10	0.5,0.8	0.7404	0.7223	12.1158	12.1157	50.9673	50.9674	79.7553	79.7552
10,1	0.1,0.4	3.1645	3.1640	12.6406	12.6406	26.0392	26.0391	56.8499	56.8499
10,1	0.5,0.8	1.1085	1.2397	7.0279	7.0276	53.6151	53.6151	78.1078	78.1078
10,10	0.1,0.4	1.8773	1.8777	8.4071	8.4076	24.6129	24.6129	53.0191	53.0191
10,10	0.5,0.8	0.6799	0.6778	4.8469	4.8464	50.3070	50.3070	76.2814	76.2815

Table 8. Natural frequencies of a fixed-free beam with three masses

$\alpha_1,\alpha_2,\alpha_3$	η_1,η_2,η_3	ω_1 Exact	ω_1 FEM	ω_2 Exact	ω_2 FEM	ω_3 Exact	ω_3 FEM	ω_4 Exact	ω_4 FEM
1,1,1	0.1,0.4,0.8	1.9294	1.9219	12.6448	12.6447	46.7473	46.7478	70.4596	70.4597
1,1,10	0.1,0.4,0.8	0.7445	0.7540	12.2807	12.2807	45.0895	45.0845	69.2074	69.2089
1,10,1	0.1,0.4,0.8	1.5090	1.5079	6.0015	6.0008	45.3993	45.3983	68.6999	68.6900
10,1,1	0.1,0.4,0.8	1.9263	1.9192	11.8006	11.8006	26.0284	26.0284	50.1420	50.1420
10,10,10	0.1,0.4,0.8	0.7144	0.6986	4.7084	4.7081	24.5768	24.5767	45.1396	45.1396
1,1,1	0.2,0.5,0.7	2.0627	2.0612	13.3815	13.3814	32.9549	32.9550	50.7696	50.7695
1,1,10	0.2,0.5,0.7	0.8896	0.9167	13.2647	13.2646	30.0958	30.0958	44.8998	44.8999
1,10,1	0.2,0.5,0.7	1.2885	1.2882	8.7245	8.7241	32.7169	32.7169	43.4452	43.4450
10,1,1	0.2,0.5,0.7	2.0042	2.0035	8.4292	8.4293	20.1125	20.1126	47.0179	47.0177
10,10,10	0.2,0.5,0.7	0.7825	0.7632	5.7935	5.7928	13.4773	13.4772	31.3119	31.3122

5. CONCLUDING REMARKS

The transverse vibrations of an Euler-Bernoulli type beam carrying concentrated masses are investigated using analytical and numerical methods. The natural frequencies are calculated for several boundary conditions and the comparison of frequencies is presented. FEM and analytical solution are close to each other. Since it is tedious and difficult to obtain the frequency equations, FEM will be appropriate to calculate the natural frequencies of beams having three and more concentrated masses.

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