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THE COSINE RULE II FOR A SPHERICAL TRIANGLE ON THE DUAL UNIT SPHERE $\tilde{S}^2$

M. Kazaz[1], H. H. Uğurlu[2] and A. Özdemir[1]

[1]Celal Bayar Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümü, Manisa, Türkiye
[2] Gazi Üniversitesi, Gazi Eğitim Fakültesi, Orta Öğretim Fen ve Matematik Alanlar Eğitimi Bölümü, Ankara, Türkiye
mustafa.kazaz@bayar.edu.tr, hugurlu@gazi.edu.tr, aozdemir@hotmail.com

Abstract- In this work, we proved the Cosine Rule II for a spherical triangle on the dual unit sphere $\tilde{\boldsymbol{S}}^2$.

Keywords- Cosine Rule II, dual unit sphere, dual spherical triangle.

1. INTRODUCTION

Dual numbers had been introduced by W. K. Clifford (1845-1879) as a tool for his geometrical investigations. After him, E. Study used dual numbers and dual vectors in his research on line geometry and kinematics (see [3]). He devoted special attention to the representation of directed line by dual vectors and defined the mapping that is said with his name. There exists one to one correspondence between the vectors of dual unit sphere $\tilde{\boldsymbol{S}}^2$ and the directed lines of the space $\boldsymbol{R}^3$ of lines (E.Study's mapping).

In plane geometry it is studied points, lines, triangles, etc. On the sphere, there are points, but there are no straight lines, at least not in the usual sense. However, straight lines in the plane are characterized by the fact that they are the shortest paths between points. The curves on the sphere with same property are the great circles. Thus it is natural to use great circles as replacements for lines.

The Sine Rule I and Cosine Rule I for the dual and real spherical trigonometries have been well known for a long time (see, [1], [4], [5]). In this study, we prove the Cosine Rule II for a spherical triangle on the dual unit sphere $\tilde{\boldsymbol{S}}^2$.

2. DUAL NUMBERS AND DUAL VECTORS

Definition 2.1. *A dual number has the form* $\hat{\lambda} := \lambda + \varepsilon\lambda^*$, *where* λ *and* λ^* *are real numbers and* ε *stands for the dual unit which is subject to the rules:*

$$\varepsilon \neq 0,\ \ \varepsilon^2 = 0,\ \ 0\varepsilon = \varepsilon 0 = 0,\ \ 1\varepsilon = \varepsilon 1 = \varepsilon.$$

We denote the set of dual numbers by $\boldsymbol{D}$:

$$\boldsymbol{D} = \left\{ \hat{\lambda} = \lambda + \varepsilon\lambda^* :\ \lambda,\ \lambda^* \in \boldsymbol{R},\ \ \varepsilon^2 = 0 \right\}.$$

Equality, addition and multiplication are defined in $\boldsymbol{D}$ by

$$\lambda+\varepsilon\lambda^* = \beta+\varepsilon\beta^* \quad \text{if and only if } \lambda=\beta \text{ and } \lambda^*=\beta^*,$$

$$(\lambda+\varepsilon\lambda^*)+(\beta+\varepsilon\beta^*)=(\lambda+\beta)+\varepsilon(\lambda^*+\beta^*),$$

and

$$(\lambda+\varepsilon\lambda^*)(\beta+\varepsilon\beta^*)=\lambda\beta+\varepsilon(\lambda\beta^*+\lambda^*\beta),$$

respectively. Then it is easy to show that $(\boldsymbol{D}, +, .)$ is a commutative ring with unity. The numbers $\varepsilon\lambda^*$ $(\lambda^* \in \boldsymbol{R})$ are divisors of 0. We note that if λ and β are two nonzero elements of a ring R such that $\lambda\beta = 0$, then λ and β are *divisors of 0* (or *0 divisors*).

Moreover, if $\hat{\lambda} = \lambda+\varepsilon\lambda^*$, $\hat{\beta}=\beta+\varepsilon\beta^* \in \boldsymbol{D}$ with $\beta \neq 0$ then the division is given by

$$\frac{\hat{\lambda}}{\hat{\beta}}=\frac{\lambda+\varepsilon\lambda^*}{\beta+\varepsilon\beta^*}=\frac{\lambda}{\beta}+\varepsilon\left(\frac{\lambda^*}{\beta}-\frac{\lambda\beta^*}{\beta^2}\right).$$

Now let f be a differentiable function. Then the Maclaurin series generated by f is $f(\hat{x})=f(x+\varepsilon x^*)=f(x)+\varepsilon x^* f'(x)$, where $f'(x)$ is the derivative of f. Then we have

$$Cos\,(x+\varepsilon x^*) = Cos\,x - \varepsilon x^*\ Sin\,x, \tag{1}$$

$$Sin\,(x+\varepsilon x^*) = Sin\,x + \varepsilon x^*\ Cos\,x, \tag{2}$$

$$\sqrt{x+\varepsilon x^*} = \sqrt{x}+\varepsilon\frac{x^*}{2\sqrt{x}}, \quad (x>0). \tag{3}$$

The norm of $|\ \hat{x}\ |$ of a dual number $\hat{x} = x+\varepsilon x^*$ defined by

$$|\hat{x}| = |x+\varepsilon x^*| = \sqrt{\hat{x}^2} = \sqrt{x^2+2\varepsilon x x^*}.$$

Then the formula (3) allows us to write

$$|\hat{x}| = |x+\varepsilon x^*| = |x| + \varepsilon x^* \frac{x}{|x|} \quad (x \neq 0).$$

Thus we have

$$|\hat{x}| = \begin{cases} \hat{x}, & if \quad \hat{x}>0 \\ 0, & if \quad \hat{x}=0 \\ -\hat{x}, & if \quad \hat{x}<0 \end{cases}.$$

Let $\boldsymbol{D}^3$ be the set of all triples of dual numbers:

$$\boldsymbol{D}^3 = \{\tilde{a} = (a_1,\ a_2,\ a_3) \mid a_i \in \boldsymbol{D},\quad i = 1, 2, 3\}.$$

The elements of $\boldsymbol{D}^3$ are called *dual vectors*. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = a + \varepsilon a^*$, where a and a^* are the vectors of $\boldsymbol{R}^3$.

Now let $\tilde{a} = a + \varepsilon a^*$, $\tilde{b} = b + \varepsilon b^* \in \boldsymbol{D}^3$ and $\hat{\lambda} = \lambda_1 + \varepsilon\lambda_1^* \in \boldsymbol{D}$ then we define

$$\tilde{a} + \tilde{b} = a + b + \varepsilon(a^* + b^*),$$
$$\hat{\lambda}\tilde{a} = \lambda_1 a + \varepsilon(\lambda_1 a^* + \lambda_1^* a).$$

Then $\boldsymbol{D}^3$ becomes a unitary $\boldsymbol{D}$-module with these operations. It is called $\boldsymbol{D}$*-module* or *dual space*.

The *inner product* of two dual vectors $\tilde{a} = a + \varepsilon a^*$, $\tilde{b} = b + \varepsilon b^* \in \boldsymbol{D}^3$ is defined by

$$< \tilde{a},\ \tilde{b} > = < a,\ b > + \varepsilon(< a,\ b^* > + < a^*,\ b >),$$

where $< a,\ b >$ is the known inner product of the vectors a and b in the 3-dimentional vector space $\boldsymbol{R}^3$.

The cross product of two dual vectors $\tilde{a} = a + \varepsilon a^*$ and $\tilde{b} = b + \varepsilon b^* \in \boldsymbol{D}^3$ is defined by

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon(a^* \times b + a \times b^*),$$

where $a \times b$ is the known cross product in $\boldsymbol{R}^3$.

Lemma 2.2. L*et* $\tilde{a},\ \tilde{b},\ \tilde{c},\ \tilde{d} \in \boldsymbol{D}^3$. *Then we have*

$$< \tilde{a} \times \tilde{b},\ \tilde{c} > = -det(\tilde{a}, \tilde{b}, \tilde{c}),$$
$$\tilde{a} \times \tilde{b} = -\tilde{b} \times \tilde{a},$$
$$(\tilde{a} \times \tilde{b}) \times \tilde{c} = -< \tilde{a}\ ,\ \tilde{c} > \tilde{b} + < \tilde{b}, \tilde{c} > \tilde{a},$$
$$< \tilde{a} \times \tilde{b}, \tilde{c} \times \tilde{d} > = -< \tilde{a},\ \tilde{c} >< \tilde{b}, \tilde{d} > + < \tilde{a},\ \tilde{d} >< \tilde{b},\ \tilde{c} >,$$
$$< \tilde{a} \times \tilde{b},\ \tilde{a} > = 0,\ and\ < \tilde{a} \times \tilde{b},\ \tilde{b} > = 0\ .$$

■

Definition 2.3. Let $\tilde{a} = a + \varepsilon a^* \in \boldsymbol{D^3}$. Then $\tilde{a}$ is said to be dual unit vector if the vectors a and a^* satisfy the following equations

$$< a \ ,a > = \ 1, \ < a, \ a^* > = 0.$$

The set of all dual unit vectors is called the dual unit sphere, and is denoted by $\tilde{\boldsymbol{S}}^2$ (for more details, see [3], [5]).

Theorem 2.4 ***(E. Study's Mapping)*** *The dual unit vectors of the dual unit sphere* $\tilde{\boldsymbol{S}}^2$ *are in one to one correspondence with the directed lines of the 3-space* $\boldsymbol{R^3}$ *lines* [3]. ∎

3. THE COSINE RULE II FOR A DUAL SPHERICAL TRIANGLE

In this section we prove the Cosine Rule II for dual spherical triangles.

Let $\tilde{A}, \tilde{B}$ and $\tilde{C}$ be three points on dual unit sphere $\tilde{\boldsymbol{S}}^2$ given by the linearly independent dual unit vectors $\tilde{a} = a + \varepsilon a^*$, $\tilde{b} = b + \varepsilon b^*$ and $\tilde{c} = c + \varepsilon c^*$, respectively. These points together with the great-circle-arcs $\tilde{A}\tilde{B}, \tilde{B}\tilde{C}, \tilde{C}\tilde{A}$ form a dual spherical triangle $\Delta\tilde{A}\tilde{B}\tilde{C}$ see Figure 1.

We will suppose that $det(a, \ b, \ c) > 0$. We denote the dual unit vectors with the same sense as $\tilde{b} \times \tilde{c}, \ \tilde{c} \times \tilde{a}$ and $\tilde{a} \times \tilde{b}$ by $\tilde{n}_a, \ \tilde{n}_b$ and $\tilde{n}_c$, respectively. The side $\tilde{a}$ of $\Delta\tilde{A}\tilde{B}\tilde{C}$ is defined as the dual angle for which

$$< \tilde{b}, \ \tilde{c} > = Cos\,\tilde{a}, \ \ \tilde{b} \times \tilde{c} = \tilde{n}_a Sin\,\tilde{a}\,.$$

It can be given similar definitions for the other sides $\tilde{b}$ and $\tilde{c}$ of $\Delta\tilde{A}\tilde{B}\tilde{C}$. Thus we have

$$\begin{aligned} &< \tilde{a}, \ \tilde{b} > = Cos\,\tilde{c}, \ \ \tilde{a} \times \tilde{b} = \tilde{n}_c \, Sin\,\tilde{c}, \\ &< \tilde{c}, \ \tilde{a} > = Cos\,\tilde{b}, \ \ \tilde{c} \times \tilde{a} = \tilde{n}_b \, Sin\,\tilde{b} \ = -\tilde{a} \times \tilde{c}. \end{aligned}$$

Since we can write $\tilde{n}_a = n_a + \varepsilon n_a^*$, it is obvious that n_a is the real unit vector having the same sense as $b \times c$. If $\tilde{a} = a + \varepsilon a^*$, we have $Sin\,a > 0$. This means that $\left|Sin\,\tilde{a}\right| = Sin\,\tilde{a}$, and similarly $\left|Sin\,\tilde{b}\right| = Sin\,\tilde{b}$, $\left|Sin\,\tilde{c}\right| = Sin\,\tilde{c}$. It is obvious that $\tilde{a}, \ \tilde{b}$ and $\tilde{c}$ are the dual unit vectors having the same sense as $\tilde{n}_b \times \tilde{n}_c, \ \tilde{n}_c \times \tilde{n}_a$ and $\tilde{n}_a \times \tilde{n}_b$, respectively.

Definition 3.1. The angle $\tilde{\alpha}$ of $\Delta\tilde{A}\tilde{B}\tilde{C}$ is defined as the dual angle given by

$$< \tilde{n}_b,\ \tilde{n}_c > = -Cos\,\tilde{\alpha},\quad \tilde{n}_b \times \tilde{n}_c = \tilde{a}\,Sin\,\tilde{\alpha}\,.$$

The angles $\tilde{\beta}$ and $\tilde{\gamma}$ of $\Delta\tilde{A}\tilde{B}\tilde{C}$ can be defined similarly (for details, see [5]).

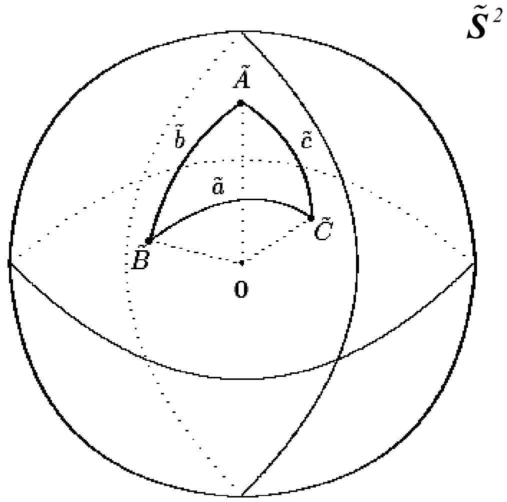

Figure 1: Dual Spherical Triangle

Lemma 3.2. *Let $\Delta\tilde{A}\tilde{B}\tilde{C}$ be a spherical triangle on the dual unit sphere $\tilde{S}^2$. Then the Sine Rule and Cosine Rule I are given by*

$$\frac{Sin\,\tilde{\alpha}}{Sin\,\tilde{a}} = \frac{Sin\,\tilde{\beta}}{Sin\,\tilde{b}} = \frac{Sin\,\tilde{\gamma}}{Sin\,\tilde{c}}\,, \tag{4}$$

$$Cos\,\tilde{\alpha} = \frac{Cos\,\tilde{a} - Cos\,\tilde{b}\,Cos\,\tilde{c}}{Sin\,\tilde{b}\,Sin\,\tilde{c}}\,, \tag{5}$$

respectively [4].

■

In the same way, we obtain Cosine Rule I for $\tilde{\beta}$ and $\tilde{\gamma}$ as follows:

$$Cos\,\tilde{\beta} = \frac{Cos\tilde{b} - Cos\tilde{a}\,Cos\tilde{c}}{Sin\,\tilde{a}\,Sin\,\tilde{c}}, \tag{6}$$

$$Cos\,\tilde{\gamma} = \frac{Cos\tilde{c} - Cos\tilde{a}\,Cos\tilde{b}}{Sin\,\tilde{a}\,Sin\,\tilde{b}}. \tag{7}$$

Using the equations (1)-(2) we obtain the following corollaries:

Corollary 3.3. *The real and dual part of the formula (4) are given by*

$$\frac{Sin\,\alpha}{Sin\,a}=\frac{Sin\,\beta}{Sin\,b}=\frac{Sin\,\gamma}{Sin\,c}, \tag{8}$$

$$\alpha^*\frac{Cos\,\alpha}{Sin\,a}-a^*Cot\,a\frac{Sin\,\alpha}{Sin\,a}=\beta^*\frac{Cos\,\beta}{Sin\,b}-b^*Cot\,b\frac{Sin\,\beta}{Sin\,b}=\gamma^*\frac{Cos\,\gamma}{Sin\,c}-c^*Cot\,c\frac{Sin\,\gamma}{Sin\,c},$$

respectively. ■

In corollary 3.3, the real part is known as the Sine Rule for a spherical triangle.

Corollary 3.4. *The real and dual parts of the formulas (5), (6) and (7) are given by*

$$Cos\,\alpha=\frac{Cos\,a-Cos\,bCos\,c}{Sin\,b\ \ Sin\,c},\ \ Sin\,\alpha=\frac{-Sin\,a}{\alpha^*Sin\,b\ \ Sin\,c}\left(b^*Cos\,\gamma+c^*Cos\beta-a^*\right);$$

$$Cos\,\beta=\frac{Cos\,b-Cos\,aCos\,c}{Sin\,aSin\,c},\ \ Sin\,\beta=\frac{-Sin\,b}{\beta^*Sin\,aSin\,c}\left(a^*Cos\,\gamma+c^*Cos\alpha-b^*\right);$$

$$Cos\,\gamma=\frac{Cos\,c-Cos\,aCos\,b}{Sin\,aSin\,b},\ \ Sin\,\gamma=\frac{-Sin\,c}{\gamma^*Sin\,aSin\,b}\left(a^*Cos\,\gamma+b^*Cos\beta-c^*\right),$$

respectively.

■

In corollary 3.4, the real parts give the Cosine Rule I for a spherical triangle.

Now we state and prove the correspondence of Cosine Rule II for hyperbolic spherical trigonometry given in [1]

Lemma 3.5 (The Dual Cosine Rule II). *Let $\Delta\tilde{A}\tilde{B}\tilde{C}$ be a spherical triangle on the dual unit sphere $\tilde{\boldsymbol{S}}^2$. Then the Cosine Rule II is given by*

$$Cos\,\tilde{c}=\frac{Cos\,\tilde{\alpha}\,Cos\,\tilde{\beta}+Cos\,\tilde{\gamma}}{Sin\,\tilde{\alpha}\,Sin\,\tilde{\beta}}. \tag{9}$$

Proof: For brevity, let $\tilde{A},\tilde{B}$ and $\tilde{C}$ be $Cos\,\tilde{a}$, $Cos\,\tilde{b}$ and $Cos\,\tilde{c}$, respectively. Then the Cosine Rule I yields

$$Cos\,\tilde{\alpha}=\frac{\tilde{A}-\tilde{B}\tilde{C}}{\left(1-\tilde{B}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)^{\frac{1}{2}}}, \tag{10}$$

$$Cos\,\tilde{\beta} = \frac{\tilde{B}-\tilde{A}\tilde{C}}{\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)^{\frac{1}{2}}}, \tag{11}$$

$$Cos\,\tilde{\gamma} = \frac{\tilde{C}-\tilde{A}\tilde{B}}{\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{B}^2\right)^{\frac{1}{2}}}. \tag{12}$$

Since $Cos^2\tilde{\alpha}+Sin^2\tilde{\alpha} = 1$, it follows that

$$Sin^2\tilde{\alpha} = \frac{\tilde{D}}{\left(1-\tilde{B}^2\right)\left(1-\tilde{C}^2\right)},$$

where $\tilde{D} = 1+2\tilde{A}\tilde{B}\tilde{C}-\tilde{A}^2-\tilde{B}^2-\tilde{C}^2$. We note that $\tilde{D}$ is positive and symmetric in $\tilde{A},\tilde{B}$ and $\tilde{C}$. Then we obtain

$$Sin\,\tilde{\alpha} = \frac{\sqrt{\tilde{D}}}{\left(1-\tilde{B}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)^{\frac{1}{2}}}, \tag{13}$$

$$Sin\tilde{\beta} = \frac{\sqrt{\tilde{D}}}{\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)^{\frac{1}{2}}}, \tag{14}$$

$$Sin\,\tilde{\gamma} = \frac{\sqrt{\tilde{D}}}{\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{B}^2\right)^{\frac{1}{2}}}. \tag{15}$$

If we write the formulas (10)-(15) in the right side of the formula (9), then the equality is satisfied:

$$\begin{aligned}
\frac{Cos\,\tilde{\alpha}Cos\,\tilde{\beta}+Cos\,\tilde{\gamma}}{Sin\,\tilde{\alpha}Sin\,\tilde{\beta}} &= \frac{\left(\tilde{A}-\tilde{B}\tilde{C}\right)\left(\tilde{B}-\tilde{A}\tilde{C}\right)+\left(\tilde{C}-\tilde{A}\tilde{B}\right)\left(1-\tilde{C}^2\right)}{\left(1-\tilde{B}^2\right)^{\frac{1}{2}}\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)}\frac{\left(1-\tilde{A}^2\right)^{\frac{1}{2}}\left(1-\tilde{B}^2\right)^{\frac{1}{2}}\left(1-\tilde{C}^2\right)}{\tilde{D}} \\
&= \frac{\tilde{A}\tilde{B}-\tilde{A}^2\tilde{C}-\tilde{B}^2\tilde{C}+\tilde{A}\tilde{B}\tilde{C}^2+\tilde{C}-\tilde{A}\tilde{B}-\tilde{C}^3+\tilde{A}\tilde{B}\tilde{C}^2}{\tilde{D}} \\
&= \frac{\tilde{C}\left(1+2\tilde{A}\tilde{B}\tilde{C}-\tilde{A}^2-\tilde{B}^2-\tilde{C}^2\right)}{\tilde{D}} \\
&= \tilde{C} \\
&= Cos\,\tilde{c}\ .
\end{aligned}$$

■

In the same way, we can give the similar formulas for $Cos\,\tilde{b}$ and $Cos\,\tilde{a}$ as follows:

$$Cos\,\tilde{b} = \frac{Cos\,\tilde{\gamma}Cos\,\tilde{\alpha} + Cos\,\tilde{\beta}}{Sin\,\tilde{\alpha}Sin\,\tilde{\gamma}}; \tag{16}$$

$$Cos\,\tilde{a} = \frac{Cos\,\tilde{\beta}Cos\,\tilde{\gamma} + Cos\,\tilde{\alpha}}{Sin\,\tilde{\beta}\,Sin\,\tilde{\gamma}} \tag{17}$$

Using the equations (1) and (2), we obtain the following corollary:

Corollary 3.6. *The real and dual parts of the formulas (9), (16) and (17) are given by*

$$Cos\,c = \frac{Cos\,\alpha Cos\,\beta + Cos\,\gamma}{Sin\,\alpha Sin\,\beta}, \quad Sin\,c = \frac{Sin\,\gamma}{c^{*}Sin\,\alpha Sin\,\beta}\left(\beta^{*}Cos\,a + \alpha^{*}Cos\,b + \gamma^{*}\right);$$

$$Cos\,b = \frac{Cos\,\gamma Cos\,\alpha + Cos\,\beta}{Sin\,\gamma Sin\,\alpha}, \quad Sin\,b = \frac{Sin\,\beta}{b^{*}Sin\,\alpha Sin\,\gamma}\left(\alpha^{*}Cos\,c + \gamma^{*}Cos\,a + \beta^{*}\right);$$

$$Cos\,a = \frac{Cos\,\beta Cos\,\gamma + Cos\,\alpha}{Sin\,\beta Sin\,\gamma}, \quad Sin\,a = \frac{Sin\,\alpha}{a^{*}Sin\,\beta Sin\,\gamma}\left(\gamma^{*}Cos\,b + \beta^{*}Cos\,c + \alpha^{*}\right),$$

respectively. ■

In Corollary 3.6, the real parts are known as the Cosine Rule II for a spherical triangle.

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