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ON THE MODIFIED NEWTON'S METHOD FOR THE SOLUTION OF A CLASS OF NONLINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract- In this paper we investigate the solvability of a certain class of nonlinear singular integro-differential equations with Cauchy kernel in the usual Holder space and its generalization by producing the sufficient conditions for the convergence of the modified Newton's method.
Keywords- Nonlinear singular integro-differential equations , Modified Newton's method .

1. INTRODUCTION

Nonlinear singular integro-differential equations (NSIDE) has been studied in works of Guseinov and Mukhtarov [7] , Wolfersdorf [16] , Mal'sagov [11] , and many others. The method of Newton-Kantorovich , [1,3,4,5,7,11,13] , (the modified Newton's method) is frequently used to investigate the solvability for many classes of nonlinear singular integro-differential equations (NSIDE) .

In this paper we study the sufficient conditions for the applicability and convergence of the modified Newton's method for the following class of NSIDE with Cauchy kernel in the usual Holder space

$$P\big(u\big)(x) = u(x) - T[\, G\big(\,.,u(.),u'(.)\,\big)](x) = 0 \tag{1.1}$$

where $u(x)$ is the unknown function and

$$T[\, G\big(\, s,u(s),u'(s)\,\big)\,](x) = \frac{1}{2\pi}\int_a^b \frac{G\big(\, s,u(s),u'(s)\,\big)}{s-x}\, ds,$$

which is taken as a Cauchy principle value .

Also , we consider a generalized form of (1.1) which has the form :

$$P\big(u\big)(x) = u(x) - T[\, G\Big(\,.,u(.),u'(.),...,u^{(n)}(.)\Big)\,]\,(x) = 0\,, \tag{1.2}$$

where

$$T[\, G\Big(s,u(s),u'(s),...,u^{(n)}(s)\Big)\,](x) = \frac{1}{2\pi}\int_a^b \frac{G\Big(s,u(s),u'(s),...,u^{(n)}(s)\Big)}{s-x}\, ds.$$

2. BASIC NOTIONS AND AUXILIARY RESULTS

We shall introduce some notations , definitions and assumptions , which will be used in the sequel .

Definition 2.1 [5]

a) We denote by $c[\,a\,,b\,]$ to the space of all continuous functions defined on $[\,a,b\,]$.
For $u\in c[\,a\,,b\,]$, the norm of u is given by $\|\,u\,\|_c = \max\limits_{a\le x\le b}|\,u\,(\,x\,)\,|$.

b) By $H_\delta^{(k)}[a,b],(k=0,1,2,\ldots,n)$, we denote the space of all k-times differentiable functions whose k-derivatives satisfy Holder's condition with exponent δ, $0<\delta<1$, on the closed interval $[a,b]$, i.e:

For every u belongs to $H_{\delta}^{(k)}[a,b]$, we define the following norms:

$$\| u(x) \|_{\delta}^{(k)} = \max_{0<\ell<k} \left| u^{\ell}(x) \right|_{\delta} \quad , \quad \| \exp(u) \|_{\delta} = \left(1+\| u \|_{\delta}\right) \exp\left(\| u \|_{\delta}\right)$$

and
$$\| u(x) \|_{\delta} = \| u(x) \|_{c} + H_{\delta}\left(u(x) \right) , \tag{2.1}$$

where

$$H_{\delta}\left(u(x) \right) = \sup_{x_1, x_2 \in [a,b]} \frac{| u(x_1) - u(x_2) |}{| x_1 - x_2 |^{\delta}} . \tag{2.2}$$

Definition 2.2 (Frechet's derivative) [8]

Let X and Y be Banach spaces and P be a nonlinear operator defined from X into Y. Let h be an arbitrary element of X and x be a fixed element of X.

If $P'(x)$ is a linear continuous operator defined from X into Y such that :

$$P(x+h) - P(x) = P'(x)h + \Theta(x;h) \tag{2.3}$$

where

$$\lim_{\| h \|_X \to 0} \frac{\| \Theta(x;h) \|_Y}{\| h \|_X} = 0 . \tag{2.4}$$

Then, the linear operator $P'(x)$ is called Frechet's derivative of the nonlinear operator $P(x)$ at the point x with step h.

Assumptions

(i) Let the function G in equation (1.1) be defined and continuous on the region : $D = \{ (s,u,u') : s \in [a,b] , u \text{ and } u' \in (-\infty, \infty) \}$ and possess partial derivatives up to the second order and satisfy the following condition:

$$\left| \frac{\partial^m G(s_1,u_1(s_1),u_1'(s_1))}{\partial u_1^{\alpha_1} \partial u_1^{\alpha_2}} - \frac{\partial^m G(s_2,u_2(s_2),u_2'(s_2))}{\partial u_2^{\alpha_1} \partial u_2^{\alpha_2}} \right| \le$$

$$\le a_m \left\{ |s_1 - s_2|^{\delta} + \| u_1 - u_2 \| + \| u_1' - u_2' \| \right\}$$

where, $m = \alpha_1 + \alpha_2$, $m = 0,1,2$ and a_m are constants. (2.5)

(ii) Let the function G in equation (1.2) be defined and continuous on the region: $D = \{ (s,u^{(0)},...,u^{(n)}) : s \in [a,b] , u^{(j)}(s) \in (-\infty, \infty) , j = 0,1,...,n \}$ and possess partial derivatives up to the second order and satisfy the following condition:

$$\left| \frac{\partial^m G\left(s_1,u_1(s_1),u_1'(s_1), ...,u_1^{(n)}(s_1)\right)}{\partial u_1^{\alpha_1} \partial u_1^{\alpha_2} ... \partial u_1^{\alpha_\ell}} - \frac{\partial^m G\left(s_2,u_2(s_2),u_2'(s_2), ...,u_2^{(n)}(s_2)\right)}{\partial u_2^{\alpha_1} \partial u_2^{\alpha_2} ... \partial u_2^{(n)\alpha_\ell}} \right| \le$$

$$\le a_m \left\{ |s_1 - s_2|^{\delta} + \| u_1 - u_2 \| + \| u_1' - u_2' \| + ... + \left\| u_1^{(n)} - u_2^{(n)} \right\| \right\} \tag{2.6}$$

where $m = \alpha_1 + \alpha_2 + ... + \alpha_\ell$, $m = 0,1,2$ and a_m are constants .

Lemma 2.1

Let the function G in the equation (1.1) satisfies the assumption (i). Then , the following inequality is valid

$$\left\| \frac{\partial^m G(s,u(s),u'(s))}{\partial u^{\alpha_1} \, \partial u^{\alpha_2}} \right\|_\delta \le a_m \left\{ 1 + \| u(s) \|_\delta + \| u'(s) \|_\delta \right\} + \left\| \frac{\partial^m G(s,0,0)}{\partial u^{\alpha_1} \, \partial u^{\alpha_2}} \right\|_c . \quad (2.7)$$

Proof

It is easy to get the proof from definition 2.1 and condition (2.5).

Lemma 2.2

Let the function G in equation (1.2) satisfies the assumption (ii). Then , the following inequality is valid

$$\left\| \frac{\partial^m G\left(s,u(s),u'(s),...,u^{(n)}(s)\right)}{\partial u^{\alpha_1} \, \partial u^{\alpha_2} ... \, \partial u^{(n)^{\alpha_\ell}}} \right\| \le a_m \left\{ 1 + \| u(s) \|_\delta + \| u'(s) \|_\delta + ... + \left\| u^{(n)}(s) \right\|_\delta \right\} +$$

$$+ \left\| \frac{\partial^m G(s,0,0,...,0_n)}{\partial u^{\alpha_1} \, \partial u^{\alpha_2} ... \, \partial u^{(n)^{\alpha_\ell}}} \right\|_c . \quad (2.8)$$

Proof

It is easy to get the proof from definition 2.1 and condition (2.6).

Lemma 2.3 [14]

If the real or complex - valued coefficients $A_q(t)$ and $b(t)$, $(q = \overline{1,n}\ )$, are continuous in an interval J and if s belongs to J , then the initial value problem

$$Lu \equiv \sum_{q=0}^{n} A_q(t) \, u^{(q)}(t) = b(t) \ , \ u^{(\nu)}(s) = \kappa_\nu \quad , \ (\nu = \overline{0, n-1})$$

has exactly one solution . The solution exists in all of J and depends continuously on κ_ν and on $A_q(t)$ and $b(t)$ in each compact subinterval of J .

3. APPLICABILITY OF THE MODIFIED NEWTON'S METHOD TO A CERTAIN CLASS OF NSIDE

In this section we shall consider the applicability of the modified Newton's method to the class given by equation (1.1). The following two lemmas are
fundamental in our study.

Lemma 3.1

Let the function G in the equation (1.1) satisfies the assumption (i) . Then, the operator P defined in the equation (1.1) is Frechet differentiable at every fixed point of the space $H_\delta[a,b]$ with derivative given by :

$$P'(u) \, h(x) = h(x) - \frac{1}{2\pi} \int_a^b \frac{h(s) \;\, G_u\left(s,u(s),u'(s) \right) + h'(s) \;\, G_{u'}\left(s,u(s),u'(s) \right)}{s - x} \, ds . \quad (3.1)$$

Moreover , P' satisfies Lipschitz's condition:

$$\| P'(u_1) - P'(u_2) \|_\delta \le \xi \| u_1 - u_2 \|_\delta \tag{3.2}$$

in the sphere

$$S(u_0, r) = \{ u \in H_\delta[a,b], \| u - u_0 \|_\delta \le r \} \tag{3.3}$$

where ξ is a constant .

Proof

Let $u_0(x)$ be a fixed point in the space $H_\delta[a,b]$ and $h(x)$ be an arbitrary element in $H_\delta[a,b]$.

Then ,

$$P(u_0 + h) - P(u_0) = h(x) - \frac{1}{2\pi} \int_a^b \frac{G(s, u_0 + h, u_0' + h') - G(s, u_0, u_0')}{s - x} ds. \tag{3.4}$$

By applying Lagrange's formula ,[10], we get :

$$P(u_0 + h) - P(u_0) = P'(u_0) h(x) + \Theta_1(x) + \Theta_2(x) + \Theta_3(x) \tag{3.5}$$

where

$$\lim_{\| h \| \to 0} \frac{\| \Theta_j(x; h) \|}{\| h \|} = 0 \quad , \quad j = 1,2,3.$$

Hence , $P(u)$ is differentiable in the sense of Frechet and its derivative is given by:

$$P'(u) h(x) = h(x) - \frac{1}{2\pi} \int_a^b \frac{h(s)\ G_u(s, u(s), u'(s)) + h'(s)\ G_{u'}(s, u(s), u'(s))}{s - x} ds.$$

From equation (3.1) and Lagrange's formula ,[10],we get :

$$\begin{aligned} P'(u_1)h - P'(u_2)h = & \int_0^1 Z_1(s)(u_1 - u_2)\ h(s) \left(\frac{1}{s-x} \right) ds\ + \\ & + \int_0^1 Z_2(s)\ (u_1 - u_2) h'(s) \left(\frac{1}{s-x} \right) ds + \\ & + \int_0^1 Z_3(s)\ (u_1' - u_2') h(s) \left(\frac{1}{s-x} \right) ds\ + \\ & + \int_0^1 Z_4(s)\ (u_1' - u_2') h'(s) \left(\frac{1}{s-x} \right) ds\ , \end{aligned} \tag{3.6}$$

where:

$$\left.\begin{aligned}
Z_1(s) &= \frac{-1}{2\pi}\int_0^1 G_{uu}\left(s, u_2 + t(u_1 - u_2), u_2' + th'\right)dt\,, \\
Z_2(s) &= \frac{-1}{2\pi}\int_0^1 G_{uu'}\left(s, u_2 + t(u_1 - u_2), u_2' + th'\right)dt\,, \\
Z_3(s) &= \frac{-1}{2\pi}\int_0^1 G_{u'u}\left(s, u_2 + th, u_2' + t(u_1' - u_2')\right)dt \\
&\text{and} \\
Z_4(s) &= \frac{-1}{2\pi}\int_0^1 G_{u'u'}\left(s, u_2 + th, u_2' + t(u_1' - u_2')\right)dt.
\end{aligned}\right\} \tag{3.7}$$

Hence:

$$\begin{aligned}\| P'(u_1) - P'(u_2)\|_\delta \le \left(\rho_1\| Z_1\|_\delta + \rho_2\| Z_2\|_\delta\right)\| u_1 - u_2\|_\delta + \\ + \left(\rho_3\| Z_3\|_\delta + \rho_4\| Z_4\|_\delta\right)\| u_1' - u_2'\|_\delta\,. \end{aligned} \tag{3.8}$$

where

$$\| Z_1(s)\|_\delta \le \mu + \| G_{uu}(s,0,0)\|_c\,,$$

$$\| Z_2(s)\|_\delta \le \mu + \| G_{uu'}(s,0,0)\|_c\,,$$

$$\| Z_3(s)\|_\delta \le \mu + \| G_{u'u}(s,0,0)\|_c$$

and

$$\| Z_4(s)\|_\delta \le \mu + \| G_{u'u'}(s,0,0)\|_c \quad \text{with} \quad \mu = \eta + \alpha\,.$$

For all $u_1, u_2 \in H_\delta[a,b]$, there exists a constant υ such that:

$$\upsilon = \max_{a\le s\le b}\left| u_1'(s) - u_2'(s)\right|\,. \tag{3.9}$$

Therefore, from the assumption (i) , Lipschitz's condition (3.8) has the form :

$$\| P'(u_1) - P'(u_2)\|_\delta \le \xi\, \| u_1 - u_2\|_\delta$$

where:

$$\xi = \max\{d_1, d_2\} \tag{3.10}$$

and :

$$d_1 = \mu(\rho_1 + \rho_2) + \rho_1 \| G_{uu}(s,0,0)\|_c + \rho_2 \| G_{uu'}(s,0,0)\|_c\,,$$

$$d_2 = \upsilon\left(\mu(\rho_3 + \rho_4) + \rho_3 \| G_{u'u}(s,0,0)\|_c + \rho_4 \| G_{u'u'}(s,0,0)\|_c\right).$$

Thus the Lemma is true .

Lemma 3.2

Let the conditions of lemma (3.1) be satisfied . Then, the linear operator

$$P'(u)\, h(x) = h(x) - \frac{1}{2\pi}\int_a^b \frac{h(s)\;G_u\left(s, u(s), u'(s)\right) + h'(s)\;G_{u'}\left(s, u(s), u'(s)\right)}{s - x}\, ds,$$

has a bounded inverse, $\Gamma_0 = [P'(u_0)]^{-1}$,on the space $H_\delta[a,b]$ for any fixed point $u_0(x_0)$ and arbitrary element $h(x)$ belong to $H_\delta[a,b]$, such that:

$$h(a)=h(b) \tag{3.11}$$

Proof

To find the operator $\Gamma_0 = [P'(u_0)]^{-1}$,we investigate the solvability of the equation ,

$$P'(u_0)h(x)=f(x) \ , \tag{3.12}$$

that can be rewritten , by using (3.1) , as follows :

$$h(x)-\frac{A}{2\pi}\int_a^b \frac{h(s)}{s-x}ds-\frac{B}{2\pi}\int_a^b \frac{h'(s)}{s-x}ds=f(x) \tag{3.13}$$

where $f(x)$ is an arbitrary continuous function in $H_\delta[a,b]$, $A=G_u(s_0,u_0,u'_0)$ and $\mathrm{B}=G_{u'}(s_0,u_0,u'_0)$ are non - zero constants. Now , consider the holomorphic function

$$\Phi(z)=\frac{1}{2\pi i}\int_a^b \frac{ih(s)}{s-z}ds \qquad , \ i=\sqrt{-1} \tag{3.14}$$

and by using the condition (3.11) we can show that :

$$\Phi'(z)=\frac{1}{2\pi i}\int_a^b \frac{ih'(s)}{s-z}ds \qquad , \ i=\sqrt{-1} \tag{3.15}$$

hence , the Sokhotski formulae , [6] , are :

$$\Phi^{\pm}(s)=\pm\frac{1}{2}ih(s)+\Phi(s) \tag{3.16}$$

and

$$\Phi'^{\pm}(s)=\pm\frac{1}{2}ih'(s)+\Phi'(s). \tag{3.17}$$

Substituting from equations (3.14) , (3.15) into (3.13) we obtain :

$$\left[\Phi'^{+}(s)+\frac{A+i}{B}\Phi^{+}(s)\right]+\left[\Phi'^{-}(s)+\frac{A-i}{B}\Phi^{-}(s)\right]=\frac{-f(s)}{B} \quad , \tag{3.18}$$

which can be rewritten in the following Boundary value problem (B.V.P) :

$$F^{+}(s)=c(s)F^{-}(s)+g(s) \ , \tag{3.19}$$

where:

$$g(s)=\frac{-f(s)}{B} \quad , \ c(s)=-1 \tag{3.20}$$

and

$$F^{\pm}(s)=\Phi^{\pm'}(s)+\frac{A\pm i}{B}\Phi^{\pm}(s), \tag{3.21}$$

which are first order linear ordinary differential equations.

From the theory of linear singular integral equations [6] , the index of equation (3.19) equals zero and the coefficient function $c(s)$ equals -1 .

Putting $c(s)=\dfrac{X^{+}(s)}{X^{-}(s)}$ in the equation (3.19) , we obtain :

$$\frac{F^{+}(s)}{X^{+}(s)}=\frac{F^{-}(s)}{X^{-}(s)}+\frac{g(s)}{X^{+}(s)} \ , \tag{3.22}$$

where:

$$X^{+}(z)=\sqrt{c}\,\exp(\mathrm{E}(z)),\ X^{-}(z)=\frac{1}{\sqrt{c}}\exp(\mathrm{E}(z)) \text{ and } \mathrm{E}(z)=\frac{1}{2\pi i}\int_{a}^{b}\frac{\ln[c(\tau)]}{\tau-z}\,d\tau\,. \tag{3.23}$$

By using relations (3.23) , we obtain :

$$X^{+}(z)=i\sqrt{\frac{b-z}{z-a}} \qquad \text{and} \qquad X^{-}(z)=-i\sqrt{\frac{b-z}{z-a}}\ . \tag{3.24}$$

Therefore ,

$$F^{+}(z)-F^{-}(z)=\frac{1}{\pi}\int_{a}^{b}\left(g(\tau)\ \sqrt{\frac{(b-z)(\tau-a)}{(z-a)(b-\tau)}}\right)\left(\frac{1}{\tau-z}\right)d\tau=T\Lambda\ , \tag{3.25}$$

where

$$\Lambda=2g(\tau)\ \sqrt{\frac{(b-z)(\tau-a)}{(z-a)(b-\tau)}}\ .$$

The equation (3.21) is a first order linear ordinary differential equation has the following solution:

$$\Phi^{\pm}(x)=\Phi^{\pm}(a)\exp\left(\frac{A}{B}(a-x)\right)\cos\left(\frac{a-x}{B}\right)\pm i\Phi^{\pm}(a)\exp\left(\frac{A}{B}(a-x)\right)\sin\left(\frac{a-x}{B}\right)+$$

$$+\int_{a}^{x}F^{\pm}(s)\ \exp\left(\frac{A}{B}(s-x)\right)\cos\left(\frac{s-x}{B}\right)ds\pm i\int_{a}^{x}F^{\pm}(s)\ \exp\left(\frac{A}{B}(s-x)\right)\sin\left(\frac{s-x}{B}\right)ds. \tag{3.26}$$

Therefore ,

$$\begin{aligned}h(x)=-i\,(\Phi^{+}(x)-\Phi^{-}(x)) &= -i\left(\Phi^{+}(a)-\Phi^{-}(a)\right)\ \exp\left(\frac{A}{B}(a-x)\right)\cos\left(\frac{a-x}{B}\right)+\\ &+\left(\Phi^{+}(a)+\Phi^{-}(a)\right)\ \exp\left(\frac{A}{B}(a-x)\right)\ \sin\left(\frac{a-x}{B}\right)-\\ &-i\int_{a}^{x}\left(F^{+}(s)-F^{-}(s)\right)\ \exp\left(\frac{A}{B}(s-x)\right)\ \cos\left(\frac{s-x}{B}\right)ds+\\ &+\int_{a}^{x}\left(F^{+}(s)+F^{-}(s)\right)\ \exp\left(\frac{A}{B}(s-x)\right)\ \sin\left(\frac{s-x}{B}\right)ds\,.\end{aligned} \tag{3.27}$$

By using the condition (3.11), we get:

$$\Phi^{+}(a)-\Phi^{-}(a)=0 \qquad \text{and} \qquad \Phi^{+}(b)-\Phi^{-}(b)=0\ . \tag{3.28}$$

From (3.26) and (3.28), equation (3.27) can be written as:

$$h(x)=\exp\left(\frac{A}{B}(a-x)\right)\sin\left(\frac{a-x}{B}\right)\operatorname{cosec}\left(\frac{a-b}{\mathrm{B}}\right)\left[-\int_a^b g(s)\exp\left(\frac{A}{B}(s-a)\right)\sin\left(\frac{s-b}{B}\right)ds\right.$$

$$\left.+\,i\int_a^b \exp\left(\frac{A}{B}(s-a)\right)\cos\left(\frac{s-b}{\mathrm{B}}\right)T\Lambda\,ds\right]+\int_a^{\mathrm{x}} g(s)\exp\left(\frac{A}{B}(s-x)\right)\sin\left(\frac{s-x}{B}\right)ds$$

$$-\,i\int_a^x \exp\left(\frac{A}{B}(s-x)\right)\cos\left(\frac{s-x}{B}\right)T\Lambda\,ds=\Gamma_0\big(f(x)\big)\,. \tag{3.29}$$

Finally, To prove the boundedness of the operator $\Gamma_0=\left[P'(u_0)\right]^{-1}$, we have to evaluate the norm of each term in (3.29).

Hence,

$$\left\|\exp\left(\frac{A}{B}(a-x)\right)\right\|_\delta \le (1+\beta)\ \exp(\beta) \tag{3.30}$$

where

$$\beta=\frac{A|b-a|}{B}+\left|\frac{A}{B}\right|\big(2|b|\big)^{1-\delta}\equiv \text{constant.}$$

and

$$\left\|\sin\left(\frac{a-x}{B}\right)\right\|_\delta \le 1+2\lambda \qquad , \left\|\cos\left(\frac{a-x}{B}\right)\right\|_\delta \le 1+2\lambda$$

where

$$\lambda=\sup_{x,y\in[a,b]}\ |\,x-y\,|^{-1}\equiv\text{constant}. \tag{3.31}$$

Since , $g(s)=\dfrac{-f(s)}{B}$ and $f(s)$ is an arbitrary continuous function defined on a closed interval.

hence ,

$$\|\,g(s)\,\|_\delta \le R\equiv\text{constant}. \tag{3.32}$$

From (3.32) and (3.25) , we obtain :

$$\|\,T\Lambda\,\|_\delta \le \rho_5\widetilde{R}\,, \tag{3.33}$$

where $\widetilde{R}$ is a constant depends on R .

Moreover , let

$$\left\|\operatorname{cosec}\left(\frac{a-b}{B}\right)\right\|_\delta = Q\equiv\text{constant}. \tag{3.34}$$

Now , from (3.30) – (3.34) and (3.29) , we have :

$$\|\,h\,\|_\delta \le \big(Q\,W+1\,\big)\big(\,(b-a)\,W\,\big)\big(\rho_5\,\widetilde{R}+R\big)=N\equiv\text{constant}\ , \tag{3.35}$$

where

$$W=\big(1+2\lambda\big)\big(1+\beta\big)\exp(\beta).$$

Thus, the linear operator (3.1) has a bounded inverse and the lemma is true .

Now, we consider $\|\Gamma_0 \ P(u_0)\|_\delta$ where $\Gamma_0 = [P'(u_0)]^{-1}$, we get:

$$\| \Gamma_0 P(u_0) \|_\delta \le N\Big((1+\rho_6 a_0) \| u_0 \|_\delta - \rho_6 \left(a_0 (1 + \| u_0' \|_\delta) + \| G(s,0,0) \|_c \right) \Big)$$

$$= M \equiv \text{constant}. \qquad (3.36)$$

Thus, all the conditions of applicability and convergence of the modified Newton's method are satisfied .Hence , the following theorem is valid.

Theorem 3.1

Let the conditions of lemma (3 .2) be satisfied and let $u_0(s) \in H_\delta[a,b]$ be the initial approximation for the solution of the NSIDE (1.1) so, if :

$$\left\| [P'(u_0)]^{-1} P(u_0) \right\|_\delta \le M \quad , \quad \left\| [P'(u_0)]^{-1} \right\|_\delta \le N \quad \text{and} \quad \wp = MN\xi < \frac{1}{2} .$$

Then , the equation (1.1) has a unique solution v in the sphere

$$\| u - \mathrm{v} \|_\delta \le r_0 \quad , \; r > r_o = M(1 - \sqrt{1 - 2\wp})/\wp$$

to which the successive approximations

$$u_{n+1} = u_n - [P(u_0)]^{-1} P(u_n)$$

of the modified Newton's method converges and the rate of convergence is given by the inequality

$$\| u_n - \mathrm{v} \|_\delta \le M \frac{\left(1 - \sqrt{1 - 2\wp}\right)^n}{\sqrt{1 - 2\wp}} .$$

4. ON A GENERALIZATION FORM OF NSIDE

In this section , we can generalize the class of NSIDE represented by (1.1) to a more general class of NSIDE written in the form (1.2). By the same technique have used in section § 2 , we shall study the NSIDE (1.2).

Lemma 4.1

Let the function G in the equation (1.2) satisfies the assumption (ii) . Then, the operator P defined in the equation (1.2) is Frechet differentiable at every fixed point of the space $H_\delta[a,b]$ with derivative given by :

$$P'(u) \ h(x) = h(x) - \frac{1}{2\pi} \int_a^b \frac{1}{s-x} \sum_{\mathrm{i}=0}^{n} G_{u^{(\mathrm{i})}}(s, u^{(0)}, \dots, u^{(n)}) \, \mathrm{h}^{(\mathrm{i})}(s) \, ds . \qquad (4.1)$$

Moreover , P' satisfies Lipschitz's condition:

$$\| P'(u_1) - P'(u_2) \|_\delta \le \xi^* \| u_1 - u_2 \|_\delta \qquad (4.2)$$

in the sphere :

$$S(u_0^*, r^*) = \left\{ u \in \mathrm{H}_\delta[a,b], \left\| u - u_0^* \right\|_\delta \le r^* \right\} \qquad (4.3)$$

where ξ^* is a constant .

Proof

Let $u_0^*(x)$ be a fixed point in the space $H_\delta[a,b]$ and $h(x)$ be an arbitrary element in $H_{\delta}[a,b]$. Then

$$P\left(u_0^{*}+h\right)-P\left(u_0^{*}\right)=P'\left(u_0^{*}\right)h(x)+\Theta_{\mathrm{i},j}\left(s;h^{*}\right),\ h^{*}=(h^{(0)},\ldots,h^{(n)}) \qquad (4.4)$$

where

$$\lim_{\left\|h^{*}\right\|\to 0}\frac{\left\|\Theta_{\mathrm{i},j}(s;h^{*})\right\|}{\left\|h^{*}\right\|}=0 \quad , \quad (\mathrm{i},j=\overline{1,n}).$$

Hence, $P(u)$ is differentiable in the sense of Frechet and its derivative is given by:

$$P'(u)h(x)=h(x)-\frac{1}{2\pi}\int_a^b\frac{1}{s-x}\sum_{\mathrm{i}=0}^{n}G_{u^{(\mathrm{i})}}(s,u^{(0)},\ldots,u^{(n)})\,h^{(\mathrm{i})}(s)\,ds.$$

The Frechet derivative $P'(u)$ satisfies the following Lipschitz's condition:

$$\|P'(u_1)-P'(u_2)\|_{\delta}\le\xi^{*}\|u_1-u_2\|_{\delta},$$

where $\xi^{*}=\max(D_1,D_2)\equiv$ constant, such that :

$$D_1=\sum_{j=0}^{n}\rho_{0,j}\left(\mu_{0,j}+\left\|G_{u^{(0)}u^{(j)}}\left(s,0,0,\ldots,0_{(n)}\right)\right\|_c\right)$$

and

$$D_2=\sum_{\mathrm{i}=1}^{n}\sum_{j=0}^{n}\rho_{\mathrm{i},j}\left(\mu_{\mathrm{i},j}+\left\|G_{u^{(\mathrm{i})}u^{(j)}}\left(s,0,0,\ldots,0_{(n)}\right)\right\|_c\right)\beta_{\mathrm{i}}.$$

Thus, the lemma is true.

Lemma 4.2

Let the conditions of lemma (4.1) be satisfied, then the linear operator

$$P'(u)(x)=h(x)-\frac{1}{2\pi}\sum_{\omega=0}^{n}A_{\omega}\int_a^b\frac{h^{(\omega)}(s)}{s-x}ds=f(x), \qquad (4.5)$$

has a bounded inverse, $\Gamma_0=\left[P'(u_0^{*})\right]^{-1}$, on the space $H_{\delta}[a,b]$ for any fixed point u_0^{*} and arbitrary element $h(x)$ belong to $H_{\delta}[a,b]$, such that :

$$h^{(q)}(a)=h^{(q)}(b),\ (q=\overline{1,n-1}) \qquad (4.6)$$

where $f(x)$ is an arbitrary continuous function in $H_{\delta}[a,b]$ and $A_{\omega}=G_{u^{(\omega)}}(s_0,u_0^{(0)},\ldots,u_0^{(n)})$ are nonzero constants.

Proof

Consider the holomorphic function

$$\Phi(t)=\frac{1}{2\pi i}\int_a^b\frac{ih(s)}{s-t}ds \quad , \ i=\sqrt{-1}. \qquad (4.7)$$

Then, by using condition (4.6), we can show that :

$$\Phi^{(q)}(t)=\frac{1}{2\pi i}\int_a^b \frac{ih^{(q)}(s)}{s-t}ds \quad , \quad i=\sqrt{-1} \quad \text{and} \quad q=\overline{0,n}\ . \tag{4.8}$$

Hence , the Sokhotski formulae , [6], are given by :

$$\Phi^{(q)\pm}(z)=\pm\frac{1}{2}ih^{(q)}(z)+\Phi^{(q)}(z) \quad , \quad q=\overline{0,n} \tag{4.9}$$

Therefore ,

$$\left.\begin{aligned} &\Phi^{(q)+}(z)+\Phi^{(q)-}(z)=2\Phi^{(q)}(z) \\ &\text{and} \\ &\Phi^{(q)+}(z)-\Phi^{(q)}(z)=ih^{(q)}(z) \quad , \ i=\sqrt{-1}\ , \quad q=\overline{0,n} \end{aligned}\right\} \tag{4.10}$$

From (4.8) - (4.10) and equation (4.5) ,we have :

$$-f(z)=-h(z)+\frac{1}{2\pi}\sum_{q=0}^{n}A_q\left(2\pi\,\Phi^{(q)}(z)\right)$$

$$=i\left(\Phi^{+}(z)-\Phi^{-}(z)\right)+\sum_{q=0}^{n}\frac{A_q}{2}\left(\Phi^{(q)+}(z)+\Phi^{(q)-}(z)\right)$$

$$=i\,\Phi^{+}(z)-i\,\Phi^{-}(z)+\frac{A_0}{2}\Phi^{+}(z)+\frac{A_0}{2}\Phi^{-}(z)+\sum_{q=1}^{n}\frac{A_q}{2}\Phi^{(q)+}(z)+\sum_{q=1}^{n}\frac{A_q}{2}\Phi^{(q)-}(z)$$

$$=\left(\frac{A_0+2i}{2}\right)\Phi^{+}(z)+\left(\frac{A_0-2i}{2}\right)\Phi^{-}(z)+\frac{A_n}{2}\Phi^{(n)+}(z)+\frac{A_n}{2}\Phi^{(n)-}(z)+\sum_{q=1}^{n-1}\frac{A_q}{2}\Phi^{(q)+}(z)+$$

$$+\sum_{q=1}^{n-1}\frac{A_q}{2}\Phi^{(q)-}(z)\ ,$$

hence,

$$\begin{aligned}\frac{-2f(z)}{A_n}=&\left(\Phi^{+(n)}(z)+\sum_{q=1}^{n-1}\left(\frac{A_q}{A_n}\right)\Phi^{+(q)}(z)+\left(\frac{A_0+2i}{A_n}\right)\Phi^{+}(z)\right)\\ &+\left(\Phi^{-(n)}(z)+\sum_{q=1}^{n-1}\left(\frac{A_q}{A_n}\right)\Phi^{-(q)}(z)+\left(\frac{A_0-2i}{A_n}\right)\Phi^{-}(z)\right)\ .\end{aligned} \tag{4.11}$$

Equation (4.11) can be rewritten in the form:

$$F^{+}(z)+F^{-}(z)=\frac{-2f(z)}{A_n}$$

where

$$F^{\pm}(z)=\Phi^{\pm(n)}(z)+\sum_{q=1}^{n-1}\left(\frac{A_q}{A_n}\right)\Phi^{\pm(q)}(z)+\left(\frac{A_0\pm 2i}{A_n}\right)\Phi^{\pm}(z). \tag{4.12}$$

According to Lemma 2.3 ,[14], equation (4.12) has exactly one solution hence, equation (4.5) is solvable and has a unique solution $\Gamma_0 = \left[P'\left(u_0^*\right)\right]^{-1}$. Moreover,

$$\| \Gamma_0 \|_\delta \leq N^* \equiv \text{constant} \ , \ \left\| \Gamma_0 P(u_0^*) \right\|_\delta \leq M^* \equiv \text{constant}$$

and the lemma is proved .

Thus , all the conditions of applicability and convergence of the modified Newton's method are satisfied .Therefore , the following theorem is valid.

Theorem 4.1

Let the conditions of lemma (4 .2) be satisfied and let $u_0^*(s) \in H_\delta[a,b]$ be the initial approximation for the solution of the NSIDE (1.2) so, if :

$$\left\| \left[P'(u_0^*)\right]^{-1} P(u_0^*) \right\|_\delta \leq M^* \ , \ \left\| \left[P'(u_0^*)\right]^{-1} \right\|_\delta \leq N^* \text{ and } \wp^* = M^* N^* \xi^* < \frac{1}{2} \ .$$

Then , equation (1.2) has a unique solution v^* in the sphere

$$\left\| u - \mathrm{v}^* \right\|_\delta \leq r_0^* \quad , \ r^* > r_o^* = M^*(1 - \sqrt{1 - 2\wp^*}\ )/\wp^*$$

to which the successive approximations

$$u_{n+1} = u_n - \left[P(u_0^*)\right]^{-1} P(u_n)$$

of the modified Newton's method converges and the rate of convergence is given by the inequality

$$\left\| u_n - v^* \right\|_\delta \ \leq \ M^* \ \frac{\left[\ 1 - \sqrt{\ 1 - 2\wp^*}\ \right]^n}{\sqrt{\ 1 - 2\wp^*}} \ .$$

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