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HYPERBOLIC SINE AND COSINE RULES FOR GEODESIC TRIANGLES ON THE HYPERBOLIC UNIT SPHERE H_0^2

A. Özdemir and M. Kazaz
Department of Mathematics, Celal Bayar University,
Muradiye Campus, 45047, Manisa, Turkey. m_kazaz@hotmail.com

Abstract- In this work, we proved the sine and cosine rules for a geodesic triangle on the hyperbolic unit sphere H_0^2 by means of timelike unit vectors. We also obtained some useful results.
Keywords- Geodesic triangle, Lorentzian space, Timelike vector.

1. INTRODUCTION

In plane Lorentzian geometry it is studied points, timelike, spacelike and lightlike lines, triangles, etc [4]. On the hyperbolic sphere, there are points, but there are no straight lines, at least not in the usual sense. However, straight timelike lines in the Lorentzian plane are characterized by the fact that they are the shortest paths between points. The curves on the hyperbolic sphere with the same property are hyperbolic circles. Thus it is natural to use these circles as replacements for timelike lines.

The formulas for the sine and cosine rules are given for the Euclidean sphere S^2 [2, 3, 6] and hyperbolic sphere [5]. In this study, we obtained formulas related with the spacelike angles and hyperbolic angles corresponding to the sides of geodesic triangles on hyperbolic unit sphere H_0^2.

2. BASIC CONCEPTS

In this section, we give a brief summary of the theory of Lorentzian concepts.

Let Lorentzian 3-space L^3 be the vector space IR^3 provide with Lorentzian inner product $< , >$ given by

$$< a,b >= a_1b_1 + a_2b_2 - a_3b_3 \ ,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in IR^3$.

A vector $a = (a_1, a_2, a_3) \in IR^3$ is said to be *timelike* if $< a, a >< 0$, *spacelike* if $< a, a >> 0$ and *lightlike* (or *null*) if $< a, a >= 0$. The *norm* of a vector a is defined by $|a| = \sqrt{|< a, a >|}$.

Let $e = (0,0,1) \in IR^3$. A timelike vector $a = (a_1, a_2, a_3)$ is called *future pointing* (resp. *past pointing*) if $< a, e >< 0$ (resp. $< a, e > > 0$). Thus a timelike vector $a = (a_1, a_2, a_3)$ is future pointing if and only if $a_1^2 + a_2^2 < a_3^2$ and $a_3 > 0$.

The set of all timelike unit vectors is called *hyperbolic unit sphere* and denoted by

$$H_0^2 = \{a = (a_1, a_2, a_3) \in L^3 \ \big| < a, a >= -1\} .$$

There are two components of the hyperbolic unit sphere H_0^2. The components of H_0^2 through $(0,0,1)$ and $(0,0,-1)$ are called *the future pointing hyperbolic unit sphere* and *the past pointing hyperbolic unit sphere* and denoted by H_0^{+2} and H_0^{-2}, respectively. Thus we have

$$H_0^{+2}=\{\, a=(a_1,a_2,a_3)\in L^3 \;\big|\; a \text{ is a future pointing vector}\}$$

and

$$H_0^{-2}=\{\, a=(a_1,a_2,a_3)\in L^3 \;\big|\; a \text{ is a past pointing vector}\}.$$

Henceforth, we will use the notation H_0^2 instead of H_0^{+2}.

Now let $a=(a_1,a_2,a_3),\, b=(b_1,\, b_2,\, b_3)\in L^3$. Then the *Lorentzian cross product* of a and b is defined by

$$(a_1,\, a_2,\, a_3)\times(b_1,\, b_2,\, b_3)=(a_3b_2-a_2b_3,\; a_1b_3-a_3b_1,\; a_1b_2-a_2b_1)\,.\ [1]$$

Lemma 1.1. *Let $a,\, b,\, c,\; d\in L^3$. Then we have*

$$< a\times b,\, c >=-\det(a,\, b,\, c),$$
$$a\times b=-\,b\times a,$$
$$(a\times b)\times c=-< a,\, c > b+< b,\, c > a,$$
$$< a\times b,\, c\times d >=-< a,\, c >< b,\, d > + < a,\, d >< b,\, c >,$$
$$< a\times b,\, a >=0,\ \textit{and} < a\times b,\, b >=0.$$

■

2. THE SINE AND COSINE RULES FOR HYPERBOLIC GEODESIC TRIANGLES

Let A and B be two future pointing timelike unit vectors in Lorentzian 3-space L^3, let $|A|$ and $|B|$ be the magnitudes (lengths) of these vectors and let θ be the hyperbolic angle between A and B . Then the *Lorentzian inner product* of A and B is defined by

$$< A,\, B >=-|A||B|\cosh\theta\,.$$

The *Lorentzian cross product* of vectors A and B is defined by

$$K=A\times B=|A||B|\sinh\theta\; n,$$

where n is a spacelike unit vector in the direction of K and $|A||B|\sinh\theta$ is the length of the vector K. Thus the vectors A, B and K constitute the right-system.

In the trihedral $OABC$ (Figure 2.1), let $i,\, j$ and k be the timelike unit vectors in the direction of the vectors $OC,\, OB$ and OA, respectively. Thus $|\,i\,|=|\,j\,|=|\,k\,|=1$. When we apply the definitions to calculate the pair wise inner products and cross products of the vectors $i,\, j$ and k, we find

$$< i,\, j >=-\cosh a;\; < i,\, k >=-\cosh b;\; < j,\, k >=-\cosh c;$$
$$j\times i=\sinh a\, h_{ji};\quad k\times i=\sinh b\, h_{ki};\quad k\times j=\sinh c\, h_{kj},$$

where $h_{ji},\, h_{ki}$ and h_{kj} are unit vectors perpendicular to planes $OCB,\, OAC$ and OAB, respectively.

The *mixed product* of $i,\, j$ and k ,

$$V=< i,\, j\times k >=< j,\, k\times i >=< k,\, i\times j >,$$

is the volume of the parallel piped (parallelogram-sided box) determined by i, j and k.

Now we will state and prove the hyperbolic sine rule, cosine rule I and cosine rule II for a hyperbolic geodesic triangle.

Lemma 2.1. (The Hyperbolic Sine Rule) *Let ABC be a hyperbolic geodesic triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic sine rule is given by*

$$\frac{\sinh a}{\sin\alpha}=\frac{\sinh b}{\sin\beta}=\frac{\sinh c}{\sin\gamma}\ .$$

Proof: Consider the trihedral $OABC$ in Figure 2.1. Draw the tangent lines at the vertex A to the sides AB and AC. Let T_{AB} and T_{AC} be two unit vectors on these tangent lines. Let N be the point of intersection of the line OB and the tangent line drawn the side AB. Then we have the relation

$$OA+AN=ON \tag{1}$$

between the vectors determined by the sides of the right triangle OAN (Figure 2.2). Then $OA=k\left|OA\right|$, $AN=T_{AB}\left|AN\right|$, $ON=j\left|ON\right|$. In the triangle OAN, we get

$$\left|AN\right|=\left|OA\right|\tanh c,\ \left|OA\right|=\cosh c\ \left|ON\right|\ .$$

Thus by the equation (1), we can write

$$k\left|OA\right|+T_{AB}\left|OA\right|\tanh c=j\ \frac{\left|OA\right|}{\cosh c}\ .$$

Then we get

$$j=\cosh c\,k+\sinh c\,T_{AB}\ . \tag{2}$$

Similarly, we can write

$$i=\cosh b\,k+\sinh b\,T_{AC}\ . \tag{3}$$

On the other hand, from the cross product of the vectors j and i we obtain

$$\begin{aligned} j\ x\ i &= (\cosh c\ k+\sinh c\ T_{AB})\times(\cosh b\ k+\sinh b\,T_{AC}\ ) \\ &=\cosh c\cosh b\ k\times k+\cosh c\sinh b\ k\times T_{AB}+ \\ &\qquad \sinh c\cosh b\ T_{AB}\times k+\sinh c\sinh b\,T_{AB}\times T_{AC}\ . \end{aligned}$$

Since $k\times k=0$ and $T_{AB}\times T_{AC}=\sinh a\ k$, we obtain

$$j\ x\ i=\cosh c\sinh c\ k\times T_{AC}+\sinh c\cosh b T_{AB}\times k+\sinh c\sinh b\sin\alpha\ k\ .$$

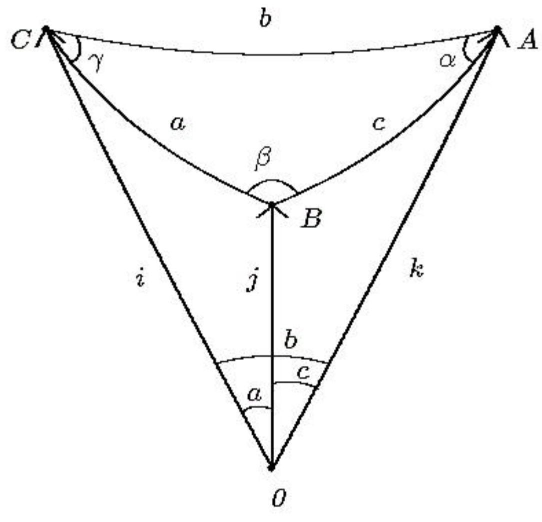

Figure 2.1.

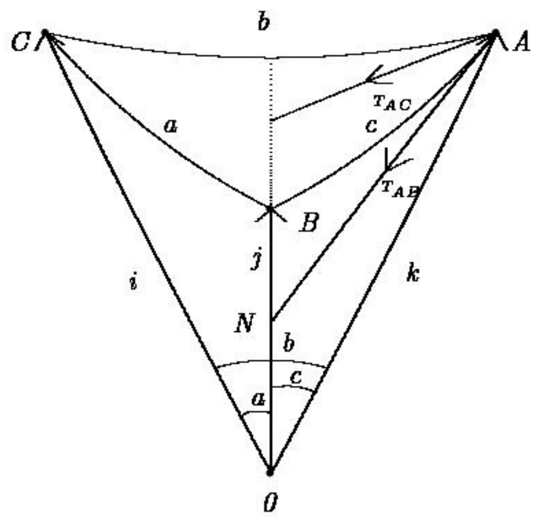

Figure 2.2.

The Lorentzian inner product of k and $j \times i$ gives

$$< k, j\times i > = \cosh c \sinh b < k,\ k\times T_{AC} > + \sinh c \cosh b < k,\ T_{AB}\times k > + \sinh b \sinh c \sin\alpha < k, k > .$$

Then using the facts $< k, k > = -1$ and

$$< k,\ k\times T_{AC} > = < k,\ T_{AC}\times k > = < T_{AC},\ k\times k > = < k, T_{AB}\times k > = < T_{AB}, k\times k > = 0$$

give

$$< k,\ j\times i > = \sinh b \sinh c \sin\alpha. \tag{4}$$

In a similar way, we get

$$< j, i\times k > = \sinh c\ \sinh a\ \sin\beta \tag{5}$$

$$< i, k\times j > = \sinh a \sinh b \sin\gamma. \tag{6}$$

Further, each of the equations (4), (5) and (6) gives the volume of the parallel piped determined by the unit vectors i, j and k. Thus, we have

$$V = \sinh b \sinh c \sin\alpha = \sinh c \sinh a \sin\beta = \sinh a \sinh b \sin\gamma. \tag{7}$$

Dividing the equation (7) by

$$\sinh a \sinh b \sinh c$$

gives the hyperbolic sine rule. ■

Lemma 2.2 (The Hyperbolic Cosine Rule I) *Let ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic cosine rule I is given by*

$$\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}.$$

Proof: The inner product of the vectors j and i is equal to

$$< j, i > = -\cosh a. \tag{8}$$

On the other hand, from the equations (2) and (3) we get

$$< j,\ i\ > = < \cosh c\ k + \sinh c T_{AB},\ \cosh b\ k + \sinh b T_{AC} >$$
$$= \cosh b \cosh c < k,\ k > + \sinh b \sinh c < k, T_{AC} > +$$
$$\cosh b \sinh c < T_{AB}\ ,\ k > + \sinh b \sinh c < T_{AB}\ ,\ T_{AC} >,$$

where $< k, k > = -1$, $< T_{AB}, T_{AC} > = \cos\alpha$ and the spacelike tangent vectors T_{AB} and T_{AC} are perpendicular to k, that is, $< k, T_{AB} > = < k, T_{AC} > = 0.$ Thus

$$< j, i > = -\cosh b \cosh c + \sinh b \sinh c \cos\alpha. \tag{9}$$

From the equations (8) and (9) we obtain

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha. \tag{10}$$

■

Similarly, it is seen that

$$\cosh b = \cosh c \cosh a - \sinh c \sinh a \cos\beta, \tag{11}$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma\ . \tag{12}$$

Lemma 2.3 (The Hyperbolic Cosine Rule II). *Let ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2. Then the hyperbolic cosine rule II is given by*

$$\cosh c = \frac{\cos\alpha\ \cos\beta + \cos\gamma}{\sin\alpha\ \sin\beta}. \tag{13}$$

Proof: For brevity, let X, Y and Z be $\cosh a, \cosh b$ and $\cosh c$, respectively. Then the cosine rule I yields

$$\cos\gamma = \frac{XY - Z}{(X^2-1)^{\frac{1}{2}}\ (Y^2-1)^{\frac{1}{2}}} \tag{14}$$

$$\cos\beta = \frac{XZ - Y}{(X^2-1)^{\frac{1}{2}}\ (Z^2-1)^{\frac{1}{2}}} \tag{15}$$

$$\cos\alpha = \frac{YZ - X}{(Y^2-1)^{\frac{1}{2}}\ (Z^2-1)^{\frac{1}{2}}} \tag{16}$$

On the other hand, since $\cos^2\alpha + \sin^2\alpha = 1,$ it follows that

$$\sin^2\alpha = \frac{D}{(Y^2-1)\ (Z^2-1)},$$

where $D = 1 + 2XYZ - (X^2 + Y^2 + Z^2).$ We note that D is positive and symmetric in X, Y and Z. Then we obtain

$$\sin\alpha = \frac{\sqrt{D}}{(Y^2-1)^{\frac{1}{2}}\ (Z^2-1)^{\frac{1}{2}}}, \tag{17}$$

$$\sin\beta = \frac{\sqrt{D}}{(X^2-1)^{\frac{1}{2}}\ (Z^2-1)^{\frac{1}{2}}}, \tag{18}$$

$$\sin\gamma = \frac{\sqrt{D}}{(X^2-1)^{\frac{1}{2}}\ (Y^2-1)^{\frac{1}{2}}}.$$

If we write the formulas (14)-(18) in the right side of the formula (13), then the equality is satisfied:

$$\frac{\cos\alpha\ \cos\beta + \cos\gamma}{\sin\alpha\ \sin\beta} = \frac{(YZ-X)(XZ-Y)+(XY-Z)(Z^2-1)}{(Y^2-1)^{\frac{1}{2}}(X^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}} \cdot \frac{(X^2-1)^{\frac{1}{2}}(Y^2-1)^{\frac{1}{2}}(Z^2-1)^{\frac{1}{2}}}{D}$$

$$= \frac{XYZ^2 - Y^2Z - X^2Z + XY + XYZ^2 - XY - Z^3 + Z}{D}$$

$$= \frac{Z(1+2XYZ - X^2 - Y^2 - Z^2)}{D}$$

$$= Z$$

$$= \cosh c.$$

■

By the same way, we can give the similar formulas for $\cosh b$ and $\cosh a$ as follows:

$$\cosh b = \frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma}, \quad \cosh a = \frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\beta\sin\gamma}.$$

3. SOME INEQUALITIES BETWEEN THE SIDES AND THE HYPERBOLIC ANGLES

In this chapter, we give some inequalities related with the angles and sides of a geodesic hyperbolic triangle.

Lemma 3.1. Let α, β and γ be the angles of a hyperbolic geodesic triangle ABC. Then $\pi > \alpha + \beta + \gamma$.

Proof: Let α, β and γ be the angles of a hyperbolic geodesic triangle ABC. Since $\cosh a > 1$, from the cosine rule II

$$\cosh a = \frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\beta\sin\gamma},$$

we get

$$\cos\alpha > -\cos\beta\cos\gamma + \sin\beta\ \sin\gamma,$$

or equivalently

$$\cos\alpha > -\cos(\beta+\gamma). \tag{21}$$

Then it follows that

$$-\cos\alpha < \cos(\beta+\gamma) \text{ or } \cos(\pi-\alpha) < \cos(\beta+\gamma).$$

Then we get

$$\pi - \alpha > \beta + \gamma.$$

■

Inequalities related with the hyperbolic angles and sides can be given with the aid of the hyperbolic cosine rules.

Lemma 3.2. *If two sides of a hyperbolic geodesic triangle ABC are equal to each other, then the opposite angles are equal each other, and conversely.*
Proof: From the equations (10) and (11), we have

$$\cos\alpha = \frac{-\cosh a + \cosh b \cosh c}{\sinh b \sinh c}, \qquad \cos\beta = \frac{-\cosh b + \cosh a \cosh c}{\sinh a \sinh c}.$$

Putting $a = b$ in these equations gives $\cos\alpha = \cos\beta$. Since $0 < \alpha, \beta < \pi$, we obtain $\alpha = \beta$.

Conversely, putting $\alpha = \beta$ in the cosine rules II

$$\cosh b = \frac{\cos\alpha \cos\gamma + \cos\beta}{\sin\alpha \, \sin\gamma},$$

$$\cosh a = \frac{\cos\beta \cos\gamma + \cos\alpha}{\sin\beta \, \sin\gamma},$$

give $\cosh a = \cosh b$, or equivalently $a = b$.

■

Let ABC be a geodesic triangle on the hyperbolic unit sphere H_0^2. Then from the hyperbolic cosine rule I we have

$$\cosh a \leq \cosh b \, \cosh c + \sinh b \, \sinh c = \cosh(b + c).$$

Since $\cosh x$ is an increasing function for positive x, then we get an inequality related with the sides of the geodesic triangle as follows:

$$b \leq a + c, \text{ or } b - c \leq a.$$

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