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THE IDENTIFICATION OF NONDOMINATED AND EFFICIENT PATHS ON A NETWORK

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Abstract-In this paper, we present an application of 0-1 linear programming problem in the indentification of the nondominated paths on a network. To find efficient paths, an adaptationof the Additive model, which called Additive model without output, is used.
Key Words- Data Envelopment Analysis, 0-1 Linear Programming, Efficiency

1. INTRODUCTION

Data Envelopment Analysis (DEA) originating from Farell's [9] seminal work and popularized by Charnes et al. [6], provides a flexible nonparametric doctrine for empirical production analysis. In recent decades, DEA has rapidly expanded towards new application areas (see e.g. Seiford [11] for survey). The technique aroused great interest as the development of several variant of the CCR model [6] and their applications demonstrate. There are many alternative DEA models with different characteristics such as the BCC model [2], the Multiplicative models [7], the Additive model [4] and the FDH model [14,8]. Since the choice of a particular DEA model determines consequences on the study, careful considerations shoulde precede the selection of the model to solve any evaluation problem. Overviews on the subject are presented in Refs. [13,3,1] and some applications are cited in Ref. [12]. Other important references on DEA include some books such as [5,10].

This paper deals with a technique based on 0-1 Linear Programming problem that identifies nondominated paths on a network. Then, these paths are evaluated by an adaptation of the Additive model. The rest of the paper unfolds as follows. Section 2, presents a method for determining of nondominated paths. In section 3, we specify the efficient paths among nondominated paths. Finally, section 4 draws our conclusive remarks.

2. APPOINTMENT OF NONDOMINATED PATHS

Suppose we have n paths in a network characterized by m costs on each arc, the costs are independent and noncommensurate. Each of them represents a factor that characterizes paths quality. We denote the costs vectors related to path j as $X_j = (x_{1j},...,x_{mj}),\ j = 1,...,n$. Let $S = \{X_j \mid j = 1,...,n\}$ be a set of m-dimensional vectors related to all paths.

Definition 2.1. We say that $X_k = (x_{1k},...,x_{mk})$ is nondominated in *S* if and only if there is not $X_l \in S$ such that, $X_k \geq X_l$ with at least one strict inequality. Otherwise, we say that X_l dominates X_k in *S*.

Definition 2.2. The vector $\overline{X} = (\overline{x}_1,...,\overline{x}_m)$ where

$$\overline{x}_i = \min_{1\leq j\leq n} \{ x_{ij} \}, \qquad i = 1,...,m \tag{1}$$

is called the ideal of paths.

Theorem 2.1. *If there are i and k such that k is unique and* $x_{ik} = \overline{x}_i$ *then* k^{th} *path is nondominated.*

Proof: Since $\overline{x}_i = x_{ik} = \min_{1\leq j\leq n} \{ x_{ij} \}$ and x_{ik} unique,so

$$\overline{x}_i = x_{ik} < x_{ij}, \quad j = 1,...,n,\ j \neq k.$$

Therefore, there is not $X_j \in S\,(j \neq k)$ so that $X_j \leq X_k$ and $X_j \neq X_k$. Consequently, based on definition 2.1, k^{th} path is nondominated.□

Suppose that $S_o = \{X_{i_1},...,X_{i_k}\}$ is the set of vectors corresponding to the paths which have been distinguished nondominated by using Theorem 2.1. The distance of each vector $X \in S_o^{'} = S - S_o$ from $\overline{X}$ by using l_1 – norm is as $L(X,\overline{X}) = \sum_{i=1}^{m} |\overline{x}_i - x_i|$.

Since $\overline{x}_i \leq x_i (i = 1,...,m)$ so, $L(X,\overline{X}) = \sum_{i=1}^{m}(x_i - \overline{x}_i) = \sum_{i=1}^{m} x_i - \sum_{i=1}^{m}\overline{x}_i$.

To find other nondominated paths, we specify a path from $S_o^{'}$ that is dominated by none members of S_o and it has the shortest distance from ideal. To do so, consider the following problem in which $\alpha = -\sum_{i=1}^{m}\overline{x}_i$ is constant.

$$\begin{array}{lll}
Min & L(X,\overline{X}) = \sum_{i=1}^{m} x_i + \alpha & \\
s.t & x_{ip} > \sum_{j\in S_o^{'}} t_j x_{ij} - Mw_{ip}, & i = 1,..., m, \quad p \in S_o \\
& \sum_{i=1}^{m} w_{ip} \leq m - 1, & p \in S_o \\
& \sum_{j\in S_o^{'}} t_j = 1 & \\
& w_{ip},\ t_j \in \{0,1\}, & i = 1,..., m, P \in S_o,\ j \in S_o^{'}
\end{array} \tag{2}$$

where *M* is a positive large number and constraints $\sum_{i=1}^{m} w_{ip} \leq m-1,\ (p \in S_o)$ imply that for each $p \in S_o$, at least one of constraints be active and other constraints are redundant. Since $t_j \in \{0,1\}$ and $\sum_{j\in S_o^{'}} t_j = 1$, the vectore $\sum_{j\in S_o^{'}} t_j X_j$ is one of the members $S_o^{'}$. Therefore, the model (2) is converted in the following model:

$$
\begin{array}{llll}
Min & \sum_{j\in S_o^{'}} \sum_{i=1}^{m} x_{ij} t_j & & \\
s.t & x_{ip} > \sum_{j\in S'_o} t_j x_{ij} - Mw_{ip}, & i=1,\dots,m \quad , p\in S_o & \\
 & \sum_{i=1}^{m} w_{ip} \le m-1, & p\in S_o & (3)\\
 & \sum_{j\in S_o^{'}} t_j = 1 & & \\
 & w_{ip}, t_j \in \{0,1\}, & i=1,\dots,m,\ p\in S_o,\ j\in S_o^{'}. &
\end{array}
$$

Theorem 2.2. *If in optimal solution of problem (3) $t_h^* = 1$, then $X_h \in S_o^{'}$ is nondominated.*

Proof: We show that X_h is dominated by none of members S_o and $S_o^{'}$. Assume that (T^*, W^*) is an optimal solution of the problem (3) with components $t_j^*(j\in s_o^{'})$ and $w_{ip}^*\ (p\in s_o, i=1,\dots,m)$. Since $\sum_{i=1}^{m} w_{ip}^* \le m-1$, there is at least one index i which $w_{ip}^* = 0$. Hence, for each $p\in S_o$ we have $x_{ip} > x_{ih}$. Consequently, p^{th} path does not dominate h^{th} path. By contradiction, suppose that $X_h \in S_o^{'}$ dominates X_h that is $X_k \le X_h$ and $X_k \ne X_h$ .Therefore

$$x_{ik} \le x_{ih}, \quad i=1,\dots,m \tag{4}$$

and at least one of inequalities (4) strictly hold. By summing the inequalities (4), we will have, $\sum_{i=1}^{m} x_{ik} < \sum_{i=1}^{m} x_{ih}$. This shows $L(X_k, \overline{X}) < L(X_h, \overline{X})$. Since X_h is dominated by none of members S_o, any member of S_o will not dominate X_k. Therefore (T^*, W^*) will not be optimal solution of the problem (3), which is a contradiction. □

In order to develop the algorithm for finding nondominated paths, the sets S_k and $S_k^{'}$ are defined as follows:

S_k : S_k is a subset of S that denotes the set of paths which has been specified nondominated until k^{th} iteration.

$S_k^{'}$: $S_k^{'} = S\text{-}S_k$.

An Algorithm for Appointment of Nondominated Paths

Stage 0: Initialization

Step 0: Identify the members S_o by using (1),

Stage 1: Identification

Step 1-1: Solve the problem (3),

Step 1-2: If there is not h where $t_h = 1$, stop. Otherwise, put $S_{k+1} = S_k \cup \{X_h\}$ and go to step 1-1,

Stage 2: End.

3. DETERMINIG EFFCIENT PATHS

Suppose that we have n DMUs with input vectors $X_j\ (j=1,...,n)$ and output vectors $Y_j\ (j=1,...,n)$. To identify the efficient DMUs, we can apply the Additive model [5] which is as follows:

$$\begin{aligned}
P_o^* = Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \qquad r=1,...,s \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \qquad i=1,...,m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0, s_i^- \ge 0, s_r^+ \ge 0, \qquad j=1,...,n, \quad i=1,...,m, \quad r=1,...,s.
\end{aligned} \tag{5}$$

It has been proved that a DMU is efficient if and only if $P_o^* = 0$ in the model (5). An interesting property of Additive model is the translation invariance proved by Ali and Seiford [1]. Feasibility And boundedness are the other properties of this model.

Using DEA terminology, we observe that the paths in the network are units to compare, each of then is described the set of the costs (inputs). In DEA, this units are considered as DMUs without output.

Definition 3.1. A unit k is efficient if and only if there is not a convex combination of the units such that every component of the convex combination is less than or equal X_k and at least one component is not equal.

Since the units corresponding to the paths in the network have not the output vectors, we can evaluate these units by the following model which is called the Additive model without outputs (envelopment form).

$$\begin{aligned}
P_o^{'*} = Max \quad & \sum_{i=1}^{m} s_i^- \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \qquad i=1,...,m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0, s_i^- \ge 0, \qquad j=1,...,n, \quad i=1,...,m.
\end{aligned} \tag{6}$$

It is evident that if $P_o^{'*} = 0$ then, unit o is surely efficient, otherwise it is inefficient.

Theorem 3.1. *The Additive model without output is feasible and bounded.*

Proof: $\lambda_o = 1,\ \lambda_j = 0,\ (j \ne o),\ s_i^- = 0,\ (i=1,...,m)$ is a feasible solution of problem (6). Consider the dual form (multiplier form) of the model (6) which is as follows:

$$
\begin{aligned}
D_o^{'*} = Min \quad & \sum_{i=1}^{m} v_{io} x_{io} + u_o \\
s.t. \quad & \sum_{j=1}^{n} v_{io} x_{ij} + u_o \geq 0, \qquad j = 1,\ldots,n \\
& v_{io} \geq 1, \qquad i = 1,\ldots,m.
\end{aligned} \tag{7}
$$

Since $(v_{1o},\ldots,v_{mo},u_o) = (1,\ldots,1,u_o)$ is feasible solution for (7), where $u_o = \max_{1\leq j\leq n}\{\sum_{i=1}^{m} x_{ij}\}$, therefore the model (6) is bounded.□

Theorem 3.2. *The Additive model without output is translation invariance.*

The proof is straightforward.

We know that each efficient unit(path) is a nondominated path. Hence, if $K = \{X_{i_1},\ldots,X_{i_l}\}$ be the set of vectors corresponding to the paths which have been distinguished nondominated by means of algorithm of the section 2, then the efficient paths are determined by evaluating the members of K by model (5).

Note that the efficient paths can be determined by evaluating units corresponding to all the paths by model (6).

4. EXAMPLE

Suppose there exist eight paths which each path has two attributes (the costs). The costs of these paths have been reported in Table 1.

Table 1: the costs of the paths

No.	1	2	3	4	5	6	7	8
Attribute1	124	263	82	338	434	140	130	438
Attribute2	25	26	26	35	53	14	34	50

We first identify the nondominated paths by using the presented algorithm.

Step 0: We have $\bar{x}_1 = \min_{1\leq j\leq 8}\{x_{1j}\} = 82$ and $\bar{x}_2 = \min_{1\leq j\leq 8}\{x_{2j}\} = 14$. Hence, $S_o = \{X_3, X_6\} = \{(82,26),(140,14)\}$.

Iteration 1 - Stage 1:

Step 1-1: In this step, we solve the following problem:

$$
\begin{aligned}
Min \quad & 149\,t_1 + 289\,t_2 + 373\,t_4 + 487\,t_5 + 164\,t_7 + 484\,t_8 \\
S.t \quad & 124\,t_1 + 263\,t_2 + 338\,t_4 + 434\,t_5 + 130\,t_7 + 434\,t_8 - 1000\,w_{13} < 82 \\
& 25\,t_1 + 26\,t_2 + 35\,t_4 + 53\,t_5 + 34\,t_7 + 50\,t_8 - 1000\,w_{23} < 26 \\
& 124\,t_1 + 263\,t_2 + 338\,t_4 + 434\,t_5 + 130\,t_7 + 434\,t_8 - 1000\,w_{16} < 140 \\
& 25\,t_1 + 26\,t_2 + 35\,t_4 + 53\,t_5 + 34\,t_7 + 50\,t_8 - 1000\,w_{26} < 14 \\
& t_1 + t_2 + t_4 + t_5 + t_7 + t_8 = 1 \\
& w_{13} + w_{23} \leq 1 \\
& w_{16} + w_{26} \leq 1 \\
& t_1, t_2, t_4, t_5, t_7, t_8, w_{13}, w_{23}, w_{16}, w_{26} \in \{0,1\}.
\end{aligned}
$$

The optimal solution of the above problem is as follows:

$$(t_1^*, t_2^*, t_4^*, t_5^*, t_7^*, t_8^*, w_{13}^*, w_{23}^*, w_{16}^*, w_{26}^*) = (1,0,0,0,0,0,1,0,0,1)$$

Step 1-2: Since $t_1^* = 1$, so $S_1 = \{X_1, X_3, X_6\} = \{(124,25),(106,33),(140,14)\}$.

Iteration 2 - Stage 1:

Step 1-1: In this step, we solve the following problem:

$$
\begin{array}{lll}
Min & 289\,t_2 + 373\,t_4 + 487\,t_5 + 164\,t_7 + 484\,t_8 & \\
S.t & 263\,t_2 + 338\,t_4 + 434\,t_5 + 130\,t_7 + 434\,t_8 - 1000\,w_{13} & < 82 \\
 & 26\,t_2 + 35\,t_4 + 53\,t_5 + 34\,t_7 + 50\,t_8 - 1000\,w_{23} & < 26 \\
 & 263\,t_2 + 338\,t_4 + 434\,t_5 + 130\,t_7 + 434\,t_8 - 1000\,w_{16} & < 140 \\
 & 26\,t_2 + 35\,t_4 + 53\,t_5 + 34\,t_7 + 50\,t_8 - 1000\,w_{26} & < 14 \\
 & 263\,t_2 + 338\,t_4 + 434\,t_5 + 130\,t_7 + 434\,t_8 - 1000\,w_{16} & < 124 \\
 & 26\,t_2 + 35\,t_4 + 53\,t_5 + 34\,t_7 + 50\,t_8 - 1000\,w_{26} & < 14 \\
 & t_2 + t_4 + t_5 + t_7 + t_8 = 1 & \\
 & w_{13} + w_{21} \le 1 & \\
 & w_{13} + w_{23} \le 1 & \\
 & w_{16} + w_{26} \le 1 & \\
 & t_2, t_4, t_5, t_7, t_8, w_{11}, w_{21}, w_{13}, w_{23}, w_{16}, w_{26} \in \{0,1\}. &
\end{array}
$$

The above problem is infeasible. Therefore, the set of nondominated paths is as follows:

$$S_1 = \{X_1, X_3, X_6\} = \{(124,25),(106,33),(140,14)\}.$$

In order to determine the efficient paths among nondominated paths, we evaluate the corresponding units of the members S_1 by model (6). The results of evaluation has been reported in Table 2.

Table 2: the results of evaluation

	$P_o^{'*}$	λ_1^*	λ_3^*	λ_6^*	s_1^{-*}	s_2^{-*}
X_1	3.6842	0	0.5789	0.4211	3.6842	0
X_3	0	0	1	0	0	0
X_6	0	0	0	1	0	0

Table 2 shows that the paths X_3 and X_6 are efficient but the path X_1 is not efficient.

5. CONCLUSION

This paper presents a method for determining nondominated paths by using 0-1 linear programing problem. To construct 0-1 linear programing problem has been used from l_1 norm and concept of nondominance. The efficient paths are specified by evaluating nondominated paths or all paths by means of the Additive model without output.

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