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APPROXIMATE SYMMETRIES OF HYPERBOLIC HEAT CONDUCTION EQUATION WITH TEMPERATURE DEPENDENT THERMAL PROPERTIES

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Abstract- Hyperbolic heat conduction equation with temperature dependent thermal properties is considered. The thermal conductivity, specific heat and density are assumed to be functions of temperature. The equation is cast into a non-dimensional form suitable for perturbation analysis. By employing a newly developed approximate symmetry theory, the approximate symmetries of the equation are calculated for the case of small variations in thermal properties. Various similarity solutions corresponding to the symmetries of first order equations are presented. For second order equations, the method of constructing approximate symmetries and similarity solutions are discussed. A linear functional variation is assumed for the thermal properties and a similarity solution is constructed using one of the first order solutions as an example.
Keywords- Approximate Symmetries, Hyperbolic Heat Equation, Similarity Solutions, Perturbation Methods, Variable Thermal Properties

1. INTRODUCTION

The hyperbolic heat conduction equation with temperature dependent thermal properties is given as follows [1]

$$\tau\frac{\partial}{\partial t}\left[\rho(T)C_p(T)\frac{\partial T}{\partial t}\right]+\rho(T)C_p(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{1}$$

where T is the temperature, x and t are the spatial and time variables and $\tau=\alpha_0/c_0^2=k_0/\rho_0C_{p0}c_0^2$ is the relaxation time. α_0 is the reference thermal diffusivity, c_0, k_0, ρ_0 and C_{p0} are the reference propagation speed, thermal conductivity, density and specific heat respectively. This model with relaxation time is better in representing situations involving very low temperatures, very high temperature gradients or extremely short times where the finite speed of heat propagation becomes significant.

Although, there is vast literature on the hyperbolic heat equation with constant thermal properties, research incorporating temperature dependent thermal properties are rare. To mention a few, Glass *et al.* [2] used a predictor-corrector scheme to solve the problem in a semi infinite slab with variable thermal conductivity. Kar *et al.* [3] solved the problem with constant thermal diffusivity, but thermal conductivity, heat capacity and density are temperature dependent. Separation of variables and Laplace transforms are used in finding the solutions. Chen and Lin [1] developed a numerical technique with the hybrid application of the Laplace transform and control volume methods.

A similarity analysis using general Lie Group techniques of the model given in equation (1) is lacking in the literature. A group classification is needed for the equation since the thermal properties are arbitrary functions of the temperature. This analysis might be involved since there are three arbitrary functional dependences on the dependent variable. Instead, an approximate analysis can be done by assuming that the temperature variations of the thermal properties are small compared to the constant reference values. To gain the utmost from perturbation and Lie Group techniques, an approximate symmetry theory which combines both techniques can be used in the analysis.

Tree different approximate symmetry theories have been developed recently. In the first method due to Baikov, Gazizov and Ibragimov[4], the dependent variable is not expanded in a perturbation series as should be done in an ordinary perturbation problem, rather, the infinitesimal generator is expanded in a perturbation series. In this way, an approximate generator is found from which approximate solutions can be retrieved. In the second method due to Fushchich and Shtelen [5] and later followed by Euler *et al.* [6, 7], Euler and Euler [8], the dependent variables are expended in a perturbation series first as also done in usual perturbation analysis. Terms are then separated at each order of approximation and a system of equations to be solved in a hierarchy is obtained. The system of equations is assumed to be coupled and the approximate symmetry of the original equation is defined to be the exact symmetry of the system of equations obtained from perturbations.

In most of the problems, the unperturbed equations are linear and perturbed equations contain nonlinear terms. When expanded in a perturbation series, one thus obtains linear non-homogenous equations to be solved in order. Actually, the system is not coupled and can be solved in hierarchy starting from the first equation. The non-homogenous term is a known function but different at each order of approximation. Requiring this term to be an arbitrary function, the approximate symmetry of the original equation is defined as the exact symmetry of the non-homogenous linear equation [9, 10] in the third method. Since non-homogenous term is considered as an arbitrary function, equation(s) dictating the form of this function arise from the symmetry calculations. Alternatively, equivalence transformation method recently developed by Ibragimov *et al.*[11] may be employed in finding the form of the arbitrary function. For the application of equivalence transformations to the exterior calculus approach, see Pakdemirli and Yürüsoy [12]. For a detailed comparison of the three approximate symmetry methods, one may refer to [9, 10]. For applications of special types of Lie Group transformations, see reference [13] for example.

In this work, the third approximate symmetry method is employed in finding approximate symmetries. The equations are cast into a non-dimensional form first. The thermal property variations are assumed to be small and the approximate symmetries of the equation are calculated. From the general symmetries, the first order degenerate symmetries are retrieved and various similarity solutions are obtained. For the second order equations, the general method to retrieve similarity solutions is discussed and an example solution is presented.

2. PERTURBATION ANALYSIS

First, the equation should be cast into a non-dimensional form suitable for perturbation analysis. Defining the dimensionless quantities and variations in thermal properties

$$\xi = \frac{c_0^2 t}{2\alpha_0}, \qquad \eta = \frac{c_0 x}{2\alpha_0}, \qquad \theta = \frac{T - T_f}{T_i - T_f} \tag{2}$$

$$\rho(T)C_p(T)=\rho_0 C_{p0}\left[1+\overline{A}(\theta)\right],\qquad k(T)=k_0\left[1+\overline{B}(\theta)\right] \tag{3}$$

and inserting into the original equation yields

$$\frac{\partial}{\partial\xi}\left[(1+\overline{A}(\theta))\frac{\partial\theta}{\partial\xi}\right]+2(1+\overline{A}(\theta))\frac{\partial\theta}{\partial\xi}=\frac{\partial}{\partial\eta}\left[(1+\overline{B}(\theta))\frac{\partial\theta}{\partial\eta}\right] \tag{4}$$

Assuming now the dimensionless functional variations to be small compared to one, one can write

$$\overline{A}(\theta)=\varepsilon A(\theta),\qquad \overline{B}(\theta)=\varepsilon B(\theta) \tag{5}$$

where ε is the usual small parameter in perturbations. Assuming an expansion

$$\theta=\theta_0+\varepsilon\,\theta_1+\dots\ , \tag{6}$$

inserting equations (5) and (6) into equation (4) and finally separating at each order of approximation yields

$$O(1):\ \frac{\partial^2\theta_0}{\partial\xi^2}+2\frac{\partial\theta_0}{\partial\xi}=\frac{\partial^2\theta_0}{\partial\eta^2} \tag{7}$$

$$O(\varepsilon):\ \frac{\partial^2\theta_1}{\partial\xi^2}+2\frac{\partial\theta_1}{\partial\xi}=\frac{\partial^2\theta_1}{\partial\eta^2}-\frac{\partial}{\partial\xi}\left[A(\theta_0)\frac{\partial\theta_0}{\partial\xi}\right]-2A(\theta_0)\frac{\partial\theta_0}{\partial\xi}+\frac{\partial}{\partial\eta}\left[B(\theta_0)\frac{\partial\theta_0}{\partial\eta}\right] \tag{8}$$

3. APPROXIMATE SYMMETRIES

To find the symmetries of equations (7) and (8), approximate symmetry theory presented in [9, 10] will be employed. If order 1 solutions are known, then the last three terms in order ε equations are in fact known functions of the independent variables. Therefore, the approximate symmetries of equation (4) are the exact symmetries of the following non-homogenous equation by definition [9, 10]

$$\frac{\partial^2\theta}{\partial\xi^2}+2\frac{\partial\theta}{\partial\xi}=\frac{\partial^2\theta}{\partial\eta^2}+h(\xi,\eta) \tag{9}$$

where the non-homogenous terms are defined at each order of approximation as follows

$$h(\xi,\eta)=0 \qquad \text{at order 1} \tag{10}$$

$$h(\xi,\eta)=-\frac{\partial}{\partial\xi}\left[A(\theta_0)\frac{\partial\theta_0}{\partial\xi}\right]-2A(\theta_0)\frac{\partial\theta_0}{\partial\xi}+\frac{\partial}{\partial\eta}\left[B(\theta_0)\frac{\partial\theta_0}{\partial\eta}\right]\quad \text{at order } \varepsilon \tag{11}$$

Defining the infinitesimal generator

$$X=\xi_1\frac{\partial}{\partial\xi}+\xi_2\frac{\partial}{\partial\eta}+\eta_1\frac{\partial}{\partial\theta} \tag{12}$$

and performing a standard Lie Group analysis to equation (9), one finally gets the infinitesimals

$$\xi_1=a\eta+b,\quad \xi_2=a\xi+c,\quad \eta_1=-a\eta\theta+d\theta+f(\xi,\eta) \tag{13}$$

with the auxiliary equation

$$\frac{\partial^2 f}{\partial\xi^2}+2\frac{\partial f}{\partial\xi}=\frac{\partial^2 f}{\partial\eta^2}+(a\eta-d)h+(a\eta+b)\frac{\partial h}{\partial\xi}+(a\xi+c)\frac{\partial h}{\partial\eta} \tag{14}$$

Parameter a represents the rotational symmetry, parameters b and c are the translational symmetries for the independent coordinates and the structure of function f is determined by equation (14). Note that h=0 corresponds to the symmetries of the first order equation.

4. SIMILARITY SOLUTIONS FOR FIRST ORDER EQUATION

For first order of approximation, taking h=0 in equation (14) yields the symmetries

$$\xi_1 = a\eta + b, \quad \xi_2 = a\xi + c, \quad \eta_1 = -a\eta\theta + d\theta + f(\xi,\eta) \tag{15}$$

$$\frac{\partial^2 f}{\partial \xi^2} + 2\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \eta^2} \tag{16}$$

Various solutions can be retrieved from the symmetries. Some examples will be given.

4.1. Parameter a

This is the rotational symmetry of the equations. Taking parameter "a" and all other parameters zero, the determining equations for similarity transformations are

$$\frac{d\xi}{a\eta} = \frac{d\eta}{a\xi} = \frac{d\theta_0}{-a\eta\theta_0} \tag{17}$$

The similarity variable and function are defined by solving the equation system

$$\mu = \frac{\xi^2 - \eta^2}{2}, \qquad \theta_0 = e^{-\xi} g(\mu) \tag{18}$$

Expressing equation (7) in terms of these new variables yields the ordinary differential equation

$$2\mu g'' + 2g' - g = 0 \tag{19}$$

The solution of the equation is

$$g = c_1 I_0(\sqrt{2\mu}) + c_2 K_0(\sqrt{2\mu}) \tag{20}$$

where I_0 and K_0 are the zero order modified Bessel functions of the first and second kind respectively. Returning back to the original variables, the similarity solution would be as follows

$$\theta_0 = e^{-\xi}[c_1 I_0(\sqrt{\xi^2 - \eta^2}\,) + c_2 K_0(\sqrt{\xi^2 - \eta^2})] \tag{21}$$

4.2. Parameters b and c

Using parameters b and c only and taking all others zero in symmetries (15) and (16) yields the similarity variable and function

$$\mu = \eta - m\xi, \qquad \theta_0 = \theta_0(\mu) \tag{22}$$

where m=c/b is an arbitrary parameter to be selected. Expressing equation (7) in terms of the new variables yields

$$(m^2 - 1)\theta_0'' - 2m\theta_0' = 0 \tag{23}$$

m=1 case is trivial and discarded. The solution in terms of the original variables is

$$\theta_0 = c_1 \exp\left[\frac{2m}{m^2 - 1}(\eta - m\xi)\right] \tag{24}$$

4.3. Parameters b, c and f

If the arbitrary function f is chosen as constant including with the parameters b and c, the similarity variable will be the same with a final solution of the below form

$$\theta_0 = n\eta + c_1 \exp\left[\frac{2m}{m^2 - 1}(\eta - m\xi)\right] \tag{25}$$

where n=f/c. This solution is not much different from the previous one.

4.4. Parameters b, c and d

Taking the parameters b, c and d as nonzero and all others zero yields finally a similarity solution

$$\theta_0 = \exp(n\eta)\{c_1 \exp[\lambda_1(\eta - m\xi)] + c_2 \exp[\lambda_2(\eta - m\xi)]\} \tag{26}$$

where n=d/c and

$$\lambda_{1,2} = \frac{m + n \pm \sqrt{m^2 + 2mn + m^2 n^2}}{m^2 - 1} \tag{27}$$

The above solution as well as the solutions presented in sections 4.2 and 4.3 may represent oscillatory type of solutions depending on the specific values of the arbitrary parameters m and n which may be complex as well.

5. SIMILARITY SOLUTIONS FOR SECOND ORDER EQUATION

At this level of approximation the non-homogenous term is

$$h(\xi,\eta) = -\frac{\partial}{\partial\xi}\left[A(\theta_0)\frac{\partial\theta_0}{\partial\xi}\right] - 2A(\theta_0)\frac{\partial\theta_0}{\partial\xi} + \frac{\partial}{\partial\eta}\left[B(\theta_0)\frac{\partial\theta_0}{\partial\eta}\right] \tag{28}$$

Corresponding to different first order solutions, various h functions can be calculated which leads to different symmetries and hence, different solutions corresponding to the symmetries. Only one of the solutions will be considered to outline the algorithm.

First, for the variation of thermal properties, an assumption is needed. For simplicity, a linear variation which is used extensively in the literature is taken

$$A(\theta_0) = l_1\theta_0, \qquad B(\theta_0) = l_2\theta_0 \tag{29}$$

Solution (24) is taken as our base solution. New coefficients are defined as follows

$$\alpha = \frac{2m^2}{m^2 - 1}, \qquad \beta = -\frac{2m}{m^2 - 1} \tag{30}$$

With these definitions, the direct relation between α and β is

$$\alpha^2 - 2\alpha = \beta^2 \tag{31}$$

The order one solution in terms of these coefficients is

$$\theta_0 = c_1 \exp[-\beta\eta - \alpha\xi] \tag{32}$$

Substituting (29) and (32) into equation (28) yields

$$h = c_1^2\gamma_1 \exp[-2\beta\eta - 2\alpha\xi] \tag{33}$$

where

$$\gamma_1 = 2[l_1(\alpha - \alpha^2) + l_2\beta^2] \tag{34}$$

Having calculated the specific form of h function, one can determine the symmetries at this level of approximation now. To generate a similarity solution, inspired by the order one solution, one again selects a=d=f=0. This yields for equation (14)

$$b\frac{dh}{d\xi} + c\frac{dh}{d\eta} = 0 \tag{35}$$

Substituting equation (33) into (35) yields $2\alpha b + 2\beta c = 0$ or $c/b = -\alpha/\beta = m$. Therefore m parameter should be selected the same as in the first order solution. The determining equations for similarity transformations are

$$\frac{d\xi}{b} = \frac{d\eta}{c} = \frac{d\theta_1}{0} \tag{36}$$

Solving the system, the similarity variable and function are

$$\mu=\eta-m\xi, \qquad \theta_1=\theta_1(\mu) \tag{37}$$

Inserting (33) and (37) into equation (9) and solving yields

$$\theta_1 = c_1^2 \frac{\gamma_1}{4\alpha}\exp(-2\beta\eta - 2\alpha\xi) + c_2 \exp(-\beta\eta - \alpha\xi) \tag{38}$$

Finally, combining the first and second order solutions together, the approximate solution is

$$\theta = c_1 \exp(-\beta\eta - \alpha\xi) + \varepsilon\left\{c_1^2 \frac{\gamma_1}{4\alpha}\exp(-2\beta\eta - 2\alpha\xi) + c_2 \exp(-\beta\eta - \alpha\xi)\right\} + \ldots \tag{39}$$

In generating the above solution, no specific boundary condition is considered. It is well known that boundary conditions restrict much the available similarity solutions. For nonlinear problems, the generator should be applied to the boundaries and boundary conditions also which reduces the number of available Lie Group transformations. For linear problems, the restrictions arising from the boundary conditions are not as strict as those of nonlinear ones. Since the approximate symmetry theory used here makes advantage of the linearity property, it is estimated that the available approximate similarity solutions might be more compared to other approximate symmetry theories.

6. CONCLUDING REMARKS

Hyperbolic heat conduction equation with variable thermal properties is considered. The equations are cast into a non-dimensional form suitable for perturbations. Assuming the variations in thermal properties to be small, the equations are separated at each level of approximation. An approximate symmetry theory newly developed is applied to the resulting equations. The approximate symmetries valid for each level are calculated once. A detailed similarity analysis and possible solutions are discussed at the first level of approximation. At the second level of approximation, only one example first order solution is used to outline the algorithm for retrieving symmetries and corresponding solution.

The emphasis in this work is on the approximate symmetries and how similarity solutions can be constructed using these symmetries. Solution of a specific boundary value problem is beyond the scope of this treatment.

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