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RELATION BETWEEN DARBOUX INSTANTANEOUS ROTATION VECTORS OF CURVES ON A TIME-LIKE SURFACE

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ABSTRACT

In this study, a fundamental relation , as a base for the geometry of the time-like surfaces, among the Darboux vectors of an arbitrary time-like curve (c) on a time-like surface and the parameter curves (c_1) and (c_2) in the Minkowski 3-space R_1^3 was founded.

1. INTRODUCTION

In Euclidean 3-space, the Frenet and Darboux instantaneous rotation vectors for a curve (c) on the surface which the parameter curves are perpendicular to each other are known. Let us consider an arbitrary curve (c) and the parameter curves (c_1) , (c_2) passing through a point P on surface. If the Darboux instantaneous rotation vectors of these curves are shown by **w**, $\mathbf{w}_1$ and $\mathbf{w}_2$ respectively, then the following formula is valid [1]:

$$\mathbf{w} = \mathbf{w}_1 \cos\varphi + \mathbf{w}_2 \sin\varphi + N\frac{d\varphi}{ds}$$

Instead of the space R^3, let us consider the Minkowski 3-space R_1^3 provided with Lorentzian inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3 \tag{1.1}$$

with $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3) \in R^3$. In this case, a vector **a** is said to be space-like if $\langle \mathbf{a}, \mathbf{a} \rangle > 0$, time-like if $\langle \mathbf{a}, \mathbf{a} \rangle < 0$, and light-like (null) if $\langle \mathbf{a}, \mathbf{a} \rangle = 0$. The norm of a vector **a** is defined as $|\mathbf{a}| = \sqrt{|\langle a, a \rangle|}$. Let e=(0,0,1). A time-like vector **a** = (a_1, a_2, a_3) is future pointing (resp., past pointing) if $\langle a, e \rangle < 0$ (resp., $\langle a, e \rangle > 0$). So a vector **a** = (a_1, a_2, a_3) is future pointing time-like if $a_1^2 + a_2^2 - a_3^2 < 0$ and $a_3 > 0$, in other words, if $\sqrt{a_1^2 + a_2^2} < a_3$ [2].

Let a solid perpendicular trihedron in space R_1^3 be $[e_1, e_2, e_3]$. In this condition, the following theorem can be given.

Theorem 1.1. For the unit vectors e_1, e_2 e_3 of the edges of a solid perpendicular trihedron that changes according to the real parameter t, the below formulae is valid:

$$\frac{de_i}{dt} = w \wedge e_i \quad , \quad i = 1,2,3 \tag{1.2}$$

where e_1 and e_2 are space-like vectors and e_3 is a time-like vector and $\wedge$ is Lorentzian vectoral product [3]. Then the Darboux instantaneous rotation vector is given by

$$w = \langle e_3', e_2 \rangle e_1 - \langle e_1', e_3 \rangle e_2 - \langle e_2', e_1 \rangle e_3. \tag{1.3}$$

[4].

2. THE INSTANTANEOUS ROTATION VECTOR FOR THE DARBOUX TRIHEDRON OF A TIME-LIKE CURVE

Let us consider the time-like surface $y = y(u,v)$. At every point of a time-like curve (c) on this surface there exists Frenet trihedron $[t, n, b]$. Since curve (c) is on the surface, another trihedron can be mentioned. Let us show the curves' unit tangent vector as t and the surfaces' space-like normal unit vector as N at the point P on surface. In this case, if we take space-like vector g, which is defined as $t \wedge N = g$, then we construct a new trihedron as $[t, g, N]$. To compare this trihedron with Frenets' let us show the angle between the vectors n and N as φ. In this situation, the formulae

$$g = n \sin\varphi - b \cos\varphi \quad , \quad N = n \cos\varphi + b \sin\varphi \tag{2.1}$$

can be written. If we take the derivatives of vectors t, N and g with respect to arc s of curve (c) then we obtain the formulae

$$\begin{aligned}
\frac{dt}{ds} &= \rho \sin\varphi g + \rho \cos\varphi N \\
\frac{dg}{ds} &= \rho \sin\varphi t - \left(\tau - \frac{d\varphi}{ds}\right) N \\
\frac{dN}{ds} &= \rho \cos\varphi t + \left(\tau - \frac{d\varphi}{ds}\right) g .
\end{aligned} \tag{2.2}$$

Here, if we say

$$\rho\cos\varphi = \frac{\cos\varphi}{R} = \frac{1}{R_n} = \rho_n \ , \ \rho\sin\varphi = \frac{\sin\varphi}{R} = \frac{1}{R_g} = \rho_g \ , \ \tau - \frac{d\varphi}{ds} = \frac{1}{T} - \frac{d\varphi}{ds} = \frac{1}{T_g} = \tau_g$$

then the formulae (2.2) can be written as follows:

$$\frac{d\mathbf{t}}{ds} = \rho_g \mathbf{g} + \rho_n \mathbf{N} \quad , \quad \frac{d\mathbf{g}}{ds} = \rho_g \mathbf{t} - \tau_g \mathbf{N} \quad , \quad \frac{d\mathbf{N}}{ds} = \rho_n \mathbf{t} + \tau_g \mathbf{g} \tag{2.3}$$

where ρ_n is normal curvature, ρ_g is geodesic curvature and τ_g is geodesic torsion.

For this, the Darboux instantaneous rotation vector of the Darboux trihedron can be written as below:

$$\mathbf{w} = \frac{\mathbf{t}}{T_g} + \frac{\mathbf{g}}{R_n} - \frac{\mathbf{N}}{R_g} \tag{2.4}$$

Then the formulae

$$\frac{d\mathbf{t}}{ds} = \mathbf{w} \wedge \mathbf{t} \quad , \quad \frac{d\mathbf{g}}{ds} = \mathbf{w} \wedge \mathbf{g} \quad , \quad \frac{d\mathbf{N}}{ds} = \mathbf{w} \wedge \mathbf{N} \tag{2.5}$$

are satisfied.

3. THE DARBOUX INSTANTANEOUS ROTATION VECTORS OF CURVES ON A TIME-LIKE SURFACE

a) The tangent and geodesic normal of time-like curve (c)

Let us assume the parameter curves u=const. and v=const.which are constant on a time-like surface $\mathbf{y} = \mathbf{y}(u,v)$, as perpendicular to each other. Let any time-like curve that is passing through a point P on the surface be (c). Let us show time-like and space-like parameter curves v=const.and u=const passing through same point P as (c_1) and (c_2). Let the unit tangent vectors of curves (c), (c_1) and (c_2) at the point P as **t**, $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively. For the space-like normal unit vector **N** of the surface at the point P, the followings are satisfied.

$$\mathbf{N} \wedge \mathbf{t} = -\mathbf{g} \quad , \mathbf{N} \wedge \mathbf{t}_1 = -\mathbf{g}_1 \quad , \mathbf{N} \wedge \mathbf{t}_2 = \mathbf{g}_2$$

In this condition, three Darboux trihedron are obtained as below:

$$[\,\mathbf{t}, \mathbf{g}, \mathbf{N}\,] \quad , \quad [\,\mathbf{t}_1, \mathbf{g}_1, \mathbf{N}\,] \quad , \quad [\,\mathbf{t}_2, \mathbf{g}_2, \mathbf{N}\,]$$

The Darboux vectors corresponding to these are in the following form:

$$w = \frac{t}{T_g} + \frac{g}{R_n} - \frac{N}{R_g}$$

$$w_1 = \frac{t_1}{(T_g)_1} + \frac{g_1}{(R_n)_1} - \frac{N}{(R_g)_1} \quad , \quad w_2 = -\frac{t_2}{(T_g)_2} - \frac{g_2}{(R_n)_2} + \frac{N}{(R_g)_2} \tag{3.1}$$

Here, let us show the arc lengths of the curves (c), (c_1) and (c_2) measured in the certain direction from P as s, s_1 and s_2, respectively. In this case, the following formulae are written:

$$t_1 = \frac{y_u}{\|y_u\|} = \frac{y_u}{\sqrt{E}} \quad , \quad t_2 = \frac{y_v}{\|y_v\|} = \frac{y_v}{\sqrt{G}} \quad , \quad t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} \tag{3.2}$$

If we write the two initial terms of (3.2) in the third term, then

$$t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} = \sqrt{E}\, t_1 \frac{du}{ds} + \sqrt{G}\, t_2 \frac{dv}{ds} \tag{3.3}$$

is obtained . Let us show the hyperbolic angle [2] between **t** and t_1 as θ and if we multiply the both sides of the equation (3.3) with t_1 and t_2 scalarly, then

$$\langle t \ , \ t_1 \rangle = -\cosh\theta = -\sqrt{E}\frac{du}{ds} \quad , \quad \langle t \ , \ t_2 \rangle = \sinh\theta = \sqrt{G}\frac{dv}{ds} \tag{3.4}$$

$$t = t_1 \cosh\theta + t_2 \sinh\theta \tag{3.5}$$

are obtained.

For arcs ds, ds_1 and ds_2

$ds^2 = Edu^2 + Gdv^2$; $F = 0$

$ds_1 = Edu^2$; v=const. , dv=0 ; $ds_2 = Gdv^2$; u=const. , du=0 (3.6)

can be written. If the formulae (3.4) and the last two formulae of (3.6) are compared, then

$$\cosh\theta = \frac{\sqrt{E}du}{ds} = \frac{ds_1}{ds} \quad , \quad \sinh\theta = \frac{\sqrt{G}dv}{ds} = \frac{ds_2}{ds} \tag{3.7}$$

are found. Since vector **g** is $t \wedge N = g$, we can write

$$g = \cosh\theta\, g_1 - \sinh\theta\, g_2. \tag{3.8}$$

b) The fundamental theorems connected with the Darboux trihedron

Theorem 3.1. The Darboux trihedrons [$\mathbf{t}_1$, $\mathbf{g}_1$, $\mathbf{N}$] and [$\mathbf{t}_2$, $\mathbf{g}_2$, $\mathbf{N}$] of parameter curves of the time-like surface are such that the directions and the orders are different and they are always coincident. The darboux derivative formulae for these trihedrons are in the below form:

$$\frac{\partial \mathbf{t}_i}{\partial s_j} = \mathbf{w}_j \wedge \mathbf{t}_i \ , \ \frac{\partial \mathbf{g}_i}{\partial s_j} = \mathbf{w}_j \wedge \mathbf{g}_i \ , \ \frac{\partial \mathbf{N}}{\partial s_j} = \mathbf{w}_j \wedge \mathbf{N} \quad (\mathbf{i}, \mathbf{j} = 1, 2) \tag{3.9}$$

Proof. $\mathbf{N}$ is coincident at the Darboux trihedrons of parameter curves on the time-like surface. In equations $-\mathbf{g}_1 = \mathbf{N} \wedge \mathbf{t}_1$ and $\mathbf{g}_2 = \mathbf{N} \wedge \mathbf{t}_2$. If we write $\mathbf{N} = \mathbf{t}_1 \wedge \mathbf{t}_2$, then

$$\mathbf{g}_1 = -\mathbf{t}_2 \quad , \quad \mathbf{g}_2 = \mathbf{t}_1 \tag{3.10}$$

are found. From (2.5),

$$\begin{aligned} &\frac{\partial \mathbf{t}_1}{\partial s_1} = \mathbf{w}_1 \wedge \mathbf{t}_1 \ , \ \frac{\partial \mathbf{g}_1}{\partial s_1} = \mathbf{w}_1 \wedge \mathbf{g}_1 \ , \ \frac{\partial \mathbf{N}}{\partial s_1} = \mathbf{w}_1 \wedge \mathbf{N} \\ &\frac{\partial \mathbf{t}_2}{\partial s_2} = \mathbf{w}_2 \wedge \mathbf{t}_2 \ , \ \frac{\partial \mathbf{g}_2}{\partial s_2} = \mathbf{w}_2 \wedge \mathbf{g}_2 \ , \ \frac{\partial \mathbf{N}}{\partial s_2} = \mathbf{w}_2 \wedge \mathbf{N} \end{aligned} \tag{3.11}$$

are obtained. From (3.10) , we have $\frac{\partial \mathbf{g}_2}{\partial s_2} = \mathbf{w}_2 \wedge \mathbf{g}_2$ and $\frac{\partial \mathbf{t}_1}{\partial s_2} = \mathbf{w}_2 \wedge \mathbf{t}_1$. This shows that the derivative of $\mathbf{t}_1$ with respect to s_2 is equal to the Lorentzian vectoral product of $\mathbf{t}_1$ and $\mathbf{w}_2$. For the other vectors of the Darboux trihedron the same conditions are satisfied.

Corollary 3.2. When (3.10) is considered, the Darboux vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are obtained by using the vectors $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{N}$ in the following form:

$$\mathbf{w}_1 = \frac{\mathbf{t}_1}{(T_g)_1} - \frac{\mathbf{t}_2}{(R_n)_1} - \frac{\mathbf{N}}{(R_g)_1} \ , \ \mathbf{w}_2 = -\frac{\mathbf{t}_2}{(T_g)_2} - \frac{\mathbf{t}_1}{(R_n)_2} + \frac{\mathbf{N}}{(R_g)_2} \tag{3.12}$$

Theorem 3.3. When the tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ of the parameter curves (c_1) and (c_2) on the time-like surface are considered, then the relations

$$\text{i)} \quad \begin{aligned} &\langle \mathbf{t}_1 , \frac{\partial \mathbf{t}_2}{\partial s_1} \rangle = \langle -\mathbf{t}_2 , \frac{\partial \mathbf{t}_1}{\partial s_1} \rangle = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \\ &\langle \mathbf{t}_2 , \frac{\partial \mathbf{t}_1}{\partial s_2} \rangle = \langle -\mathbf{t}_1 , \frac{\partial \mathbf{t}_2}{\partial s_2} \rangle = \frac{(\sqrt{G})_u}{\sqrt{EG}} \end{aligned} \tag{3.13}$$

ii) $$\langle t_1 , dt_2 \rangle = \langle -t_2 , dt_1 \rangle = -\frac{(\sqrt{E})_v}{\sqrt{G}}\, du - \frac{(\sqrt{G})_u}{\sqrt{E}}\, dv \qquad (3.14)$$

are satisfied.

Proof. i) From (3.2) , $t_1 = \frac{y_u}{\sqrt{E}}$ and $t_2 = \frac{y_v}{\sqrt{G}}$ are written. From here, since $\langle y_u , y_u \rangle = \left(t_1\sqrt{E}\right)^2 = -E$ and $\langle y_v , y_v \rangle = \left(t_2\sqrt{G}\right)^2 = G$ we have

$$y_{uv} = \left(t_1\right)_v \sqrt{E} + t_1\left(\sqrt{E}\right)_v \quad , \quad y_{vu} = \left(t_2\right)_u \sqrt{G} + t_2\left(\sqrt{G}\right)_u .$$

From last equations , $\langle y_{uv} , y_u \rangle = -\sqrt{E}\,\left(\sqrt{E}\right)$ and $\langle y_{vu} , y_v \rangle = \sqrt{G}\,\left(\sqrt{G}\right)_u$ are found. From

$$\frac{\partial t_1}{\partial v} = \frac{y_{uv}\sqrt{E} - y_u(\sqrt{E})_v}{E} \quad , \quad \frac{\partial t_2}{\partial u} = \frac{y_{vu}\sqrt{G} - y_v\left(\sqrt{G}\right)_u}{G}$$

the followings are found:

$$\langle t_2 , \frac{\partial t_1}{\partial v} \rangle = \frac{\langle y_{uv} , y_v \rangle\sqrt{E}}{E\sqrt{G}} = \frac{\sqrt{G}\left(\sqrt{G}\right)_u\sqrt{E}}{E\sqrt{G}} = \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}} \qquad (3.15)$$

$$\langle t_1 , \frac{\partial t_2}{\partial u} \rangle = \frac{\langle y_u , y_{vu} \rangle\sqrt{G}}{G\sqrt{E}} = -\frac{\sqrt{E}\left(\sqrt{E}\right)_v\sqrt{G}}{G\sqrt{E}} = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}} \qquad (3.16)$$

From (3.6)

$$\langle t_2 , \frac{\partial t_1}{\partial s_2} \rangle = \frac{1}{\sqrt{G}}\langle t_2 , \frac{\partial t_1}{\partial v} \rangle = \frac{\left(\sqrt{G}\right)_u}{\sqrt{EG}} \qquad (3.17)$$

$$\langle t_1 , \frac{\partial t_2}{\partial s_1} \rangle = \frac{1}{\sqrt{E}}\langle t_1 , \frac{\partial t_2}{\partial u} \rangle = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{EG}} \qquad (3.18)$$

are found. On the other hand, since the parameter curves are perpendicular , if we take derivatives with respect to s_1 and s_2 the proof is completed.

ii) If we take differential from equality $\langle t_1 , t_2 \rangle = 0$, we obtain

$$\langle t_1 , dt_2 \rangle = \langle t_1 , \left(\frac{\partial t_2}{\partial s_1}.ds_1 + \frac{\partial t_2}{\partial s_2}.ds_2\right) \rangle = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{G}}\, du - \frac{\left(\sqrt{G}\right)_u}{\sqrt{E}}\, dv.$$

Corollary 3.4. If the geodesic curvatures of parameter curves (c_1) and (c_2) are $\frac{1}{(R_g)_1}$ and $\frac{1}{(R_g)_2}$ respectively, then

$$\frac{1}{(R_g)_1} = -\frac{\left(\sqrt{E}\right)_v}{\sqrt{EG}} \;,\; \frac{1}{(R_g)_2} = \frac{\left(\sqrt{G}\right)_u}{\sqrt{EG}} \tag{3.19}$$

are valid.

Theorem 3.5. Let us take any curve (c) on the time-like surface and the arc elements of curves (c), (c_1) and (c_2) as s, s_1 and s_2, respectively. Let the Darboux instantaneous rotation vectors of (c_1) and (c_2) be $\mathbf{w}_1$, $\mathbf{w}_2$ and if the hyperbolic angle between the tangent t of curve (c) and $\mathbf{t}_1$ is θ then

$$\mathbf{a} = \cosh\theta\, \mathbf{w}_1 + \sinh\theta\, \mathbf{w}_2 \tag{3.20}$$

and

$$\frac{d\mathbf{t}_1}{ds} = \mathbf{a} \wedge \mathbf{t}_1 \;,\; \frac{d\mathbf{t}_2}{ds} = \mathbf{a} \wedge \mathbf{t}_2 \;,\; \frac{d\mathbf{N}}{ds} = \mathbf{a} \wedge \mathbf{N} \tag{3.21}$$

are valid.

Proof. We saw that **t** , $\mathbf{t}_1$, $\mathbf{t}_2$ and **N** vectors are functions of arcs s_1 and s_2 For this,

$$\frac{d\mathbf{t}_1}{ds} = \frac{\partial \mathbf{t}_1}{\partial s_1}\frac{ds_1}{ds} + \frac{\partial \mathbf{t}_1}{\partial s_2}\frac{ds_2}{ds}$$

is obvious. If (3.7) and (3.9) are considered, then

$$\begin{aligned} \frac{d\mathbf{t}_1}{ds} &= (\mathbf{w}_1 \wedge \mathbf{t}_1)\cosh\theta + (\mathbf{w}_2 \wedge \mathbf{t}_1)\sinh\theta \\ &= (\mathbf{w}_1 \cosh\theta + \mathbf{w}_2 \sinh\theta) \wedge \mathbf{t}_1 \\ &= \mathbf{a} \wedge \mathbf{t}_1 \end{aligned}$$

are obtained. The others are shown, similarly.

Corollary 3.6. The equality $\langle \mathbf{t}_2 , \frac{d\mathbf{t}_1}{ds} \rangle = -\langle \mathbf{t}_1 , \frac{d\mathbf{t}_2}{ds} \rangle = \langle \mathbf{a} , \mathbf{N} \rangle$ is valid.

Theorem 3.7. Let us consider the curves (c), (c_1) and (c_2) passing through a point P of time-like surface. Let the Darboux instantaneous rotation vectors corresponding to these curves at the point P be **w** , $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively.

In this case, the equation

$$\mathbf{w} = \mathbf{w}_1 \cosh\theta + \mathbf{w}_2 \sinh\theta + \mathbf{N}\frac{d\theta}{ds} \tag{3.22}$$

is satisfied.

Proof. From (3.5)

$$\mathbf{t} = \mathbf{t}_1 \cosh\theta + \mathbf{t}_2 \sinh\theta$$

can be written. If the derivatives of both sides of this equation are taken with respect to s then

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}_1}{ds}\cosh\theta + \frac{d\mathbf{t}_2}{ds}\sinh\theta + \mathbf{t}_1 \sinh\theta\frac{d\theta}{ds} + \mathbf{t}_2 \cosh\theta\frac{d\theta}{ds}$$

is obtained. Since in the Darboux trihedrons $[\mathbf{t}_1, \mathbf{g}_1, \mathbf{N}]$ and $[\mathbf{t}_2, \mathbf{g}_2, \mathbf{N}]$,

$-\mathbf{t}_1 = \mathbf{N} \wedge \mathbf{g}_1$ and $\mathbf{t}_2 = \mathbf{N} \wedge \mathbf{g}_2$ we have

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}_1}{ds}\cosh\theta + \frac{d\mathbf{t}_2}{ds}\sinh\theta + (\mathbf{g}_1 \wedge \mathbf{N})\sinh\theta\frac{d\theta}{ds} - (\mathbf{g}_2 \wedge \mathbf{N})\cosh\theta\frac{d\theta}{ds}.$$

If the equations (3.10) and (3.21) are considered

$$\frac{d\mathbf{t}}{ds} = \left(\mathbf{a} + \mathbf{N}\frac{d\theta}{ds}\right) \wedge \mathbf{t}$$

is obtained. If we take $\mathbf{a} + \mathbf{N}\frac{d\theta}{ds} = \mathbf{b}$, then we have

$$\frac{d\mathbf{t}}{ds} = \mathbf{b} \wedge \mathbf{t} \quad , \quad \frac{d\mathbf{N}}{ds} = \mathbf{b} \wedge \mathbf{N}. \tag{3.23}$$

If (2.5) and (3.23) are considered then

$$\mathbf{b} - \mathbf{w} = \lambda\mathbf{t} \quad , \quad \lambda \in \mathbb{R} \tag{3.24}$$

and

$$\mathbf{b} - \mathbf{w} = \mu\mathbf{N} \quad , \quad \mu \in \mathbb{R} \tag{3.25}$$

are obtained. From (3.24) and (3.25) it is easily seen that $\lambda = \mu = 0$. This completes the proof.

Corollary 3.8. If the curve (c) is taken as space-like , the formula (3.22) is given by

$$w = w_1 \sinh\theta + w_2 \cosh\theta + N\frac{d\theta}{ds}.$$

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