RELATION BETWEEN DARBOUX INSTANTANEOUS ROTATION VECTORS OF CURVES ON A TIME-LIKE SURFACE

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ABSTRACT

In this study, a fundamental relation, as a base for the geometry of the time-like surfaces, among the Darboux vectors of an arbitrary time-like curve \( c \) on a time-like surface and the parameter curves \( (c_1) \) and \( (c_2) \) in the Minkowski 3-space \( \mathbb{R}_1^3 \) was founded.

1. INTRODUCTION

In Euclidean 3-space, the Frenet and Darboux instantaneous rotation vectors for a curve \( c \) on the surface which the parameter curves are perpendicular to each other are known. Let us consider an arbitrary curve \( c \) and the parameter curves \( (c_1), (c_2) \) passing through a point \( P \) on surface. If the Darboux instantaneous rotation vectors of these curves are shown by \( w, w_1 \) and \( w_2 \) respectively, then the following formula is valid [1]:

\[
w = w_1 \cos \phi + w_2 \sin \phi + N \frac{d\phi}{ds}
\]

Instead of the space \( \mathbb{R}^3 \), let us consider the Minkowski 3-space \( \mathbb{R}_1^3 \) provided with Lorentzian inner product

\[
\langle a, b \rangle = a_1b_1 + a_2b_2 - a_3b_3
\]

with \( a = (a_1, a_2, a_3), b=(b_1, b_2, b_3) \in \mathbb{R}_1^3 \). In this case, a vector \( a \) is said to be space-like if \( \langle a, a \rangle > 0 \), time-like if \( \langle a, a \rangle < 0 \), and light-like (null) if \( \langle a, a \rangle = 0 \). The norm of a vector \( a \) is defined as \( |a| = \sqrt{\langle a, a \rangle} \). Let \( e=(0,0,1) \). A time-like vector \( a = (a_1, a_2, a_3) \) is future pointing (resp., past pointing) if \( \langle a, e \rangle < 0 \) (resp., \( \langle a, e \rangle > 0 \)).

So a vector \( a = (a_1, a_2, a_3) \) is future pointing time-like if \( a_1^2 + a_2^2 - a_3^2 < 0 \) and \( a_3 > 0 \), in other words, if \( \sqrt{a_1^2 + a_2^2} < a_3 \) [2].
Let a solid perpendicular trihedron in space $\mathbb{R}^3$ be $[e_1,e_2,e_3]$. In this condition, the following theorem can be given.

**Theorem 1.1.** For the unit vectors $e_1$, $e_2$, $e_3$ of the edges of a solid perpendicular trihedron that changes according to the real parameter $t$, the below formulae is valid:

$$\frac{de_i}{dt} = w \wedge e_i, \quad i = 1,2,3$$  \hspace{1cm} (1.2)

where $e_1$ and $e_2$ are space-like vectors and $e_3$ is a time-like vector and $\wedge$ is Lorentzian vectoral product [3]. Then the Darboux instantaneous rotation vector is given by

$$w = \langle e'_1, e'_2 \rangle e_1 - \langle e'_1, e_3 \rangle e_2 - \langle e'_2, e_1 \rangle e_3.$$  \hspace{1cm} (1.3)

[4].

**2. THE INSTANTANEOUS ROTATION VECTOR FOR THE DARBOUX TRIHEDRON OF A TIME-LIKE CURVE**

Let us consider the time-like surface $y = y(u,v)$. At every point of a time-like curve $(c)$ on this surface there exists Frenet trihedron $[t, n, b]$. Since curve $(c)$ is on the surface, another trihedron can be mentioned. Let us show the curves’ unit tangent vector as $t$ and the surfaces’ space-like normal unit vector as $N$ at the point $P$ on surface. In this case, if we take space-like vector $g$, which is defined as $t \wedge N = g$, then we construct a new trihedron as $[t, g, N]$. To compare this trihedron with Frenets’ let us show the angle between the vectors $n$ and $N$ as $\varphi$. In this situation, the formulae

$$g = n \sin \varphi - b \cos \varphi, \quad N = n \cos \varphi + b \sin \varphi$$  \hspace{1cm} (2.1)

can be written. If we take the derivatives of vectors $t$, $N$ and $g$ with respect to arc $s$ of curve $(c)$ then we obtain the formulae

$$\frac{dt}{ds} = \rho \sin \varphi g + \rho \cos \varphi N$$
$$\frac{dg}{ds} = \rho \sin \varphi t - (\tau - \frac{d\varphi}{ds})N$$  \hspace{1cm} (2.2)
$$\frac{dN}{ds} = \rho \cos \varphi t + (\tau - \frac{d\varphi}{ds})g.$$  

Here, if we say
\[ \rho \cos \varphi = \frac{1}{R_n} = \rho_n, \quad \rho \sin \varphi = \frac{1}{R_g} = \rho_g, \quad \tau = \frac{1}{T} = \frac{1}{T_n} = \tau_n, \quad \tau = \frac{1}{T_g} = \tau_g \]

then the formulae (2.2) can be written as follows:

\[ \frac{dt}{ds} = \rho_t g + \rho_n N, \quad \frac{dg}{ds} = \rho_t t - \tau_g N, \quad \frac{dN}{ds} = \rho_n t + \tau_g g \]

(2.3)

where \( \rho_n \) is normal curvature, \( \rho_g \) is geodesic curvature and \( \tau_g \) is geodesic torsion.

For this, the Darboux instantaneous rotation vector of the Darboux trihedron can be written as below:

\[ \mathbf{w} = \frac{t}{T_t} + \frac{g}{R_n} - \frac{N}{R_g} \]

(2.4)

Then the formulae

\[ \frac{dt}{ds} = \mathbf{w} \times \mathbf{t}, \quad \frac{dg}{ds} = \mathbf{w} \times \mathbf{g}, \quad \frac{dN}{ds} = \mathbf{w} \times \mathbf{N} \]

(2.5)

are satisfied.

3. THE DARBOUX INSTANTANEOUS ROTATION VECTORS OF CURVES ON A TIME-LIKE SURFACE

a) The tangent and geodesic normal of time-like curve \( (c) \)

Let us assume the parameter curves \( u = \text{const.} \) and \( v = \text{const.} \) which are constant on a time-like surface \( y = y(u,v) \), as perpendicular to each other. Let any time-like curve that is passing through a point \( P \) on the surface be \( (c) \). Let us show time-like and space-like parameter curves \( v = \text{const.} \) and \( u = \text{const.} \) passing through same point \( P \) as \( (c_1) \) and \( (c_2) \). Let the unit tangent vectors of curves \( (c) \), \( (c_1) \) and \( (c_2) \) at the point \( P \) be \( \mathbf{t}, \mathbf{t}_1 \) and \( \mathbf{t}_2 \), respectively. For the space-like normal unit vector \( \mathbf{N} \) of the surface at the point \( P \), the followings are satisfied.

\[ \mathbf{N} \times \mathbf{t} = -\mathbf{g}, \quad \mathbf{N} \times \mathbf{t}_1 = -\mathbf{g}_1, \quad \mathbf{N} \times \mathbf{t}_2 = \mathbf{g}_2 \]

In this condition, three Darboux trihedron are obtained as below:

\[ [\mathbf{t}, \mathbf{g}, \mathbf{N}] \quad , \quad [\mathbf{t}_1, \mathbf{g}_1, \mathbf{N}] \quad , \quad [\mathbf{t}_2, \mathbf{g}_2, \mathbf{N}] \]

The Darboux vectors corresponding to these are in the following form:
Here, let us show the arc lengths of the curves \((c), (c_1)\) and \((c_2)\) measured in the certain direction from \(P\) as \(s, s_1\) and \(s_2\), respectively. In this case, the following formulae are written:

\[
\begin{align*}
&w_1 = \frac{t}{R_n} + \frac{g_1}{(R_n)_1} - \frac{N}{(R_n)_1} , \quad w_2 = -\frac{t}{(R_n)_2} - \frac{g_2}{(R_n)_2} + \frac{N}{(R_n)_2} \\
&w = \frac{t}{R_n} + \frac{g}{R_n} , \quad w_1 = \frac{t}{(R_n)_1} + \frac{g_1}{(R_n)_1} - \frac{N}{(R_n)_1} \quad \text{and} \quad w_2 = -\frac{t}{(R_n)_2} - \frac{g_2}{(R_n)_2} + \frac{N}{(R_n)_2}
\end{align*}
\] (3.1)

If we write the two initial terms of (3.2) in the third term, then

\[
t = y_u \frac{du}{ds} + y_v \frac{dv}{ds} = \sqrt{E} t_1 \frac{du}{ds} + \sqrt{G} t_2 \frac{dv}{ds}
\] (3.3)

is obtained. Let us show the hyperbolic angle \([2]\) between \(t\) and \(t_1\) as \(\theta\) and if we multiply the both sides of the equation (3.3) with \(t_1\) and \(t_2\) scalarly, then

\[
\langle t, t_1 \rangle = -\cosh \theta = -\sqrt{E} \frac{du}{ds} , \quad \langle t, t_2 \rangle = \sinh \theta = \sqrt{G} \frac{dv}{ds}
\] (3.4)

\[
t = t_1 \cosh \theta + t_2 \sinh \theta
\] (3.5)

are obtained.

For arcs \(ds, ds_1\) and \(ds_2\)

\[
ds^2 = Edu^2 + Gdv^2 ; F = 0
\]

\[
ds_1 = Edu^2 ; v = \text{const.} , dv = 0 \quad ; \quad ds_2 = Gdv^2 ; u = \text{const.} , du = 0 \quad (3.6)
\]
can be written. If the formulae (3.4) and the last two formulae of (3.6) are compared, then

\[
\cosh \theta = \frac{\sqrt{E}du}{ds} = \frac{ds_1}{ds} , \quad \sinh \theta = \frac{\sqrt{G}dv}{ds} = \frac{ds_2}{ds}
\] (3.7)

are found. Since vector \(g\) is \(t \wedge N = g\), we can write

\[
g = \cosh \theta \, g_1 - \sinh \theta \, g_2.
\] (3.8)

b) The fundamental theorems connected with the Darboux trihedron
Theorem 3.1. The Darboux trihedrons \([ t_1 , g_1 , N ]\) and \([ t_2 , g_2 , N ]\) of parameter curves of the time-like surface are such that the directions and the orders are different and they are always coincident. The darboux derivative formulae for these trihedrons are in the below form:

\[
\frac{\partial t_1}{\partial s_j} = w_j \wedge t_1, \quad \frac{\partial g_1}{\partial s_j} = w_j \wedge g_1, \quad \frac{\partial N}{\partial s_j} = w_j \wedge N \quad (i, j = 1, 2) \quad (3.9)
\]

Proof. \(N\) is coincident at the Darboux trihedrons of parameter curves on the time-like surface. In equations \(-g_1 = N \wedge t_1\) and \(g_2 = N \wedge t_2\). If we write \(N = t_1 \wedge t_2\), then

\[
g_1 = -t_2, \quad g_2 = t_1 \quad (3.10)
\]

are found. From (2.5),

\[
\frac{\partial t_1}{\partial s_1} = w_1 \wedge t_1, \quad \frac{\partial g_1}{\partial s_1} = w_1 \wedge g_1, \quad \frac{\partial N}{\partial s_1} = w_1 \wedge N
\]

\[
\frac{\partial t_2}{\partial s_2} = w_2 \wedge t_2, \quad \frac{\partial g_2}{\partial s_2} = w_2 \wedge g_2, \quad \frac{\partial N}{\partial s_2} = w_2 \wedge N
\]

are obtained. From (3.10), we have \(\frac{\partial g_2}{\partial s_2} = w_2 \wedge g_2\) and \(\frac{\partial t_1}{\partial s_1} = w_2 \wedge t_1\). This shows that the derivative of \(t_1\) with respect to \(s_2\) is equal to the Lorentzian vectoral product of \(t_1\) and \(w_2\). For the other vectors of the Darboux trihedron the same conditions are satisfied.

Corollary 3.2. When (3.10) is considered, the Darboux vectors \(w_1\) and \(w_2\) are obtained by using the vectors \(t_1\), \(t_2\) and \(N\) in the following form:

\[
w_1 = \frac{t_1}{(T_1)_1} - \frac{t_2}{(R_1)_1} - \frac{N}{(R_1)_1}, \quad w_2 = -\frac{t_2}{(T_2)_2} + \frac{t_1}{(R_2)_2} + \frac{N}{(R_2)_2} \quad (3.12)
\]

Theorem 3.3. When the tangent vectors \(t_1\) and \(t_2\) of the parameter curves \((c_1)\) and \((c_2)\) on the time-like surface are considered, then the relations

\[
\langle t_1, \frac{\partial t_2}{\partial s_1} \rangle = \langle -t_2, \frac{\partial t_1}{\partial s_1} \rangle = -\frac{(\sqrt{E})_2}{\sqrt{EG}} \quad (3.13)
\]

\[
\langle t_2, \frac{\partial t_1}{\partial s_2} \rangle = \langle -t_1, \frac{\partial t_2}{\partial s_2} \rangle = \frac{(\sqrt{G})_2}{\sqrt{EG}}
\]
ii) \[ \langle t_1, dt_2 \rangle = \langle -t_2, dt_1 \rangle = -\frac{(\sqrt{E})_v}{\sqrt{G}} du - \frac{(\sqrt{G})_u}{\sqrt{E}} dv \quad (3.14) \]

are satisfied.

**Proof.** i) From (3.2), \( t_1 = \frac{y_u}{\sqrt{E}} \) and \( t_2 = \frac{y_v}{\sqrt{G}} \) are written. From here, since \( \langle y_u, y_u \rangle = (t_1 \sqrt{E})^2 = -E \) and \( \langle y_v, y_v \rangle = (t_2 \sqrt{G})^2 = G \) we have

\[
y_{uv} = (t_1)^v \sqrt{E} + t_1 (\sqrt{E})_v, \quad y_{vu} = (t_2)^u \sqrt{G} + t_2 (\sqrt{G})_u.
\]

From last equations, \( \langle y_{uv}, y_u \rangle = -\sqrt{E} (\sqrt{E}) \) and \( \langle y_{vu}, y_v \rangle = \sqrt{G} (\sqrt{G})_u \) are found. From

\[
\frac{\partial t_1}{\partial v} = \frac{y_{uv} \sqrt{E} - y_u (\sqrt{E})_v}{E}, \quad \frac{\partial t_2}{\partial u} = \frac{y_{vu} \sqrt{G} - y_v (\sqrt{G})_u}{G}
\]

the followings are found:

\[
\langle t_2, \frac{\partial t_1}{\partial v} \rangle = \frac{\langle y_{uv}, y_v \rangle}{E} = \frac{\sqrt{G} (\sqrt{G})_u \sqrt{E}}{E \sqrt{G}} = \frac{(\sqrt{G})_u}{\sqrt{E}} \quad (3.15)
\]

\[
\langle t_1, \frac{\partial t_2}{\partial u} \rangle = \frac{\langle y_u, y_{vu} \rangle \sqrt{G}}{G \sqrt{E}} = -\frac{\sqrt{E} (\sqrt{E})_v \sqrt{G}}{G \sqrt{E}} = -\frac{(\sqrt{E})_v}{\sqrt{G}} \quad (3.16)
\]

From (3.6)

\[
\langle t_2, \frac{\partial t_1}{\partial s_2} \rangle = \frac{1}{\sqrt{G}} \langle t_2, \frac{\partial t_1}{\partial v} \rangle = \frac{(\sqrt{G})_u}{\sqrt{EG}} \quad (3.17)
\]

\[
\langle t_1, \frac{\partial t_2}{\partial s_1} \rangle = \frac{1}{\sqrt{E}} \langle t_1, \frac{\partial t_2}{\partial u} \rangle = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \quad (3.18)
\]

are found. On the other hand, since the parameter curves are perpendicular, if we take derivatives with respect to \( s_1 \) and \( s_2 \) the proof is completed.

ii) If we take differential from equality \( \langle t_1, t_2 \rangle = 0 \), we obtain

\[
\langle t_1, dt_2 \rangle = \langle t_1, \left( \frac{\partial t_2}{\partial s_1} ds_1 + \frac{\partial t_2}{\partial s_2} ds_2 \right) \rangle = -\frac{(\sqrt{E})_v}{\sqrt{G}} du - \frac{(\sqrt{G})_u}{\sqrt{E}} dv.
\]
Corollary 3.4. If the geodesic curvatures of parameter curves \((c_1)\) and \((c_2)\) are \(\frac{1}{(R_g)_1}\) and \(\frac{1}{(R_g)_2}\) respectively, then

\[
\frac{1}{(R_g)_1} = -\frac{(\sqrt{E})\gamma}{\sqrt{EG}}, \quad \frac{1}{(R_g)_2} = \frac{(\sqrt{G})\mu}{\sqrt{EG}}
\]  

(3.19)

are valid.

Theorem 3.5. Let us take any curve \((c)\) on the time-like surface and the arc elements of curves \((c), (c_1)\) and \((c_2)\) as \(s, s_1\) and \(s_2\), respectively. Let the Darboux instantaneous rotation vectors of \((c_1)\) and \((c_2)\) be \(w_1, w_2\) and if the hyperbolic angle between the tangent \(t\) of curve \((c)\) and \(t_1\) is \(\theta\) then

\[
a = \cosh\theta \ w_1 + \sinh\theta \ w_2
\]  

(3.20)

and

\[
\frac{dt_1}{ds} = a \wedge t_1, \quad \frac{dt_2}{ds} = a \wedge t_2, \quad \frac{dN}{ds} = a \wedge N
\]  

(3.21)

are valid.

Proof. We saw that \(t, t_1, t_2\) and \(N\) vectors are functions of arcs \(s_1\) and \(s_2\).

For this,

\[
\frac{dt_1}{ds} = \frac{\partial t_1}{\partial s_1} \frac{ds_1}{ds} + \frac{\partial t_1}{\partial s_2} \frac{ds_2}{ds}
\]

is obvious. If (3.7) and (3.9) are considered, then

\[
\frac{dt_1}{ds} = (w_1 \wedge t_1) \cosh\theta + (w_2 \wedge t_1) \sinh\theta
\]

\[
= (w_1 \cosh\theta + w_2 \sinh\theta) \wedge t_1
\]

\[
= a \wedge t_1
\]

are obtained. The others are shown, similarly.

Corollary 3.6. The equality \(\langle t_2, \frac{dt_1}{ds}\rangle = -\langle t_1, \frac{dt_2}{ds}\rangle = \langle a, N\rangle\) is valid.

Theorem 3.7. Let us consider the curves \((c), (c_1)\) and \((c_2)\) passing through a point \(P\) of time-like surface. Let the Darboux instantaneous rotation vectors corresponding to these curves at the point \(P\) be \(w, w_1\) and \(w_2\), respectively.
In this case, the equation

\[ w = w_1 \cosh \theta + w_2 \sinh \theta + N \frac{d\theta}{ds} \quad (3.22) \]

is satisfied.

**Proof.** From (3.5)

\[ t = t_1 \cosh \theta + t_2 \sinh \theta \]

can be written. If the derivatives of both sides of this equation are taken with respect to \( s \) then

\[
\frac{dt}{ds} = \frac{dt_1}{ds} \cosh \theta + \frac{dt_2}{ds} \sinh \theta + t_1 \sinh \theta \frac{d\theta}{ds} + t_2 \cosh \theta \frac{d\theta}{ds}
\]

is obtained. Since in the Darboux trihedrons \([t_1, g_1, N]\) and \([t_2, g_2, N]\),

\[-t_1 = N \wedge g_1 \text{ and } t_2 = N \wedge g_2\]

we have

\[
\frac{dt}{ds} = \frac{dt_1}{ds} \cosh \theta + \frac{dt_2}{ds} \sinh \theta + (g_1 \wedge N) \sinh \theta \frac{d\theta}{ds} - (g_2 \wedge N) \cosh \theta \frac{d\theta}{ds}.
\]

If the equations (3.10) and (3.21) are considered

\[
\frac{dt}{ds} = \left( a + N \frac{d\theta}{ds} \right) \wedge t
\]

is obtained. If we take \( a + N \frac{d\theta}{ds} = b \), then we have

\[
\frac{dt}{ds} = b \wedge t \quad , \quad \frac{dN}{ds} = b \wedge N. \quad (3.23)
\]

If (2.5) and (3.23) are considered then

\[
b - w = \lambda t \quad , \quad \lambda \in \mathbb{R} \quad (3.24)
\]

and

\[
b - w = \mu N \quad , \quad \mu \in \mathbb{R} \quad (3.25)
\]

are obtained. From (3.24) and (3.25) it is easily seen that \( \lambda = \mu = 0 \). This completes the proof.

**Corollary 3.8.** If the curve (c) is taken as space-like, the formula (3.22) is given by
\[ w = w_1 \sinh \theta + w_2 \cosh \theta + N \frac{d\theta}{ds}. \]

REFERENCES


