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THE STUDY MAPPING FOR DIRECTED SPACE -LIKE AND TIME-LIKE LINES IN MINKOWSKI 3-SPACE R_1^3

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ABSTRACT

In this study, the E. Study mapping was defined for the space-like and time-like lines in the Minkowski 3-space R_1^3. Hence, there is one to one correspondence between directed space-like (resp.,time-like) lines of R_1^3 and ordered pair of vectors $(\mathbf{a}, \mathbf{a}_0)$ such that $\langle \mathbf{a},\mathbf{a}\rangle=1$ (resp., $\langle \mathbf{a},\mathbf{a}\rangle=-1$) and

$\langle \mathbf{a},\mathbf{a}_0\rangle=0$.

1. INTRODUCTION

Let R_1^3 be the vector space R^3 provided with Lorentzian inner product of signature (+ , + , -). Let $\mathbf{a} = (a_1, a_2, a_3) \in R_1^3$. In this case , a vector $\mathbf{a}$ is said to be space-like if $\langle \mathbf{a},\mathbf{a}\rangle > 0$, time-like if $\langle \mathbf{a},\mathbf{a}\rangle < 0$, and light-like (null) if $\langle \mathbf{a},\mathbf{a}\rangle = 0$. The set of all vectors such that $\langle \mathbf{a},\mathbf{a}\rangle = 0$ is called the light-like (null) cone . The norm of vector $\mathbf{a}$ is defined to be $|\mathbf{a}| = \sqrt{|\langle \mathbf{a},\mathbf{a}\rangle|}$. We also consider the time orientation as follows: A time-like vector $\mathbf{a} = (a_1, a_2, a_3)$ is future pointing (resp., past- pointing) if $\langle \mathbf{a},\mathbf{e}\rangle < 0$ (resp., $\langle \mathbf{a},\mathbf{e}\rangle > 0$) , with $\mathbf{e} = (0,0,1)$ [1] . So a time-like vector $\mathbf{a} = (a_1, a_2, a_3)$ is future pointing (resp., past- pointing) iff $\sqrt{a_1^2 + a_2^2} < a_3$ (resp. , $\sqrt{a_1^2 + a_2^2} > a_3$) . The Lorentzian and hyperbolic sphere of radius 1 in R_1^3 are defined by

$$S_1^2 = \{\mathbf{a} = (a_1, a_2, a_3) \in R_1^3 \mid \langle \mathbf{a},\mathbf{a}\rangle = 1\}$$

and

$$H_0^2 = \{\mathbf{a} = (a_1, a_2, a_3) \in R_1^3 \mid \langle \mathbf{a},\mathbf{a}\rangle = -1\} ,$$

respectively.

Lemma1.1. Let **a** and **b** be two future pointing (resp., past-pointing) time-like unit vectors in $\mathbf{R}_1^3$. Then

$$\langle \mathbf{a}, \mathbf{b} \rangle = -\cosh\theta \tag{1.1}$$

[1] .

The vectoral product of two vector **a** ,**b** $\in \mathbf{R}_1^3$ is defined by

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{1.2}$$

where $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \wedge \mathbf{e}_3 = -\mathbf{e}_1$, $\mathbf{e}_3 \wedge \mathbf{e}_1 = -\mathbf{e}_2$ [2]. For this, following equalities are satisfied:

$$\langle \mathbf{a} \wedge \mathbf{b}, \mathbf{c} \rangle = -\det(\mathbf{a}, \mathbf{b}, \mathbf{c}), \tag{1.3}$$

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = -\langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} + \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a} \tag{1.4}$$

$$\langle \mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{b} \rangle = -\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle + (\langle \mathbf{a}, \mathbf{b} \rangle)^2. \tag{1.5}$$

2. DUAL LORENTZIAN SPACE $\mathbf{D}_1^3$

Let $\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_0$, $\mathbf{B} = \mathbf{b} + \varepsilon \mathbf{b}_0 \in \mathbf{D}^3$. The *Lorentzian inner product* of **A** and **B** is defined by

$$\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \varepsilon (\langle \mathbf{a}, \mathbf{b}_0 \rangle + \langle \mathbf{a}_0, \mathbf{b} \rangle). \tag{2.1}$$

We call the dual space $\mathbf{D}^3$ together with this Lorentzian inner product as *dual Lorentzian space* and show by $\mathbf{D}_1^3$.

Definition 2.1. Let $\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_0 \in \mathbf{D}_1^3$. The dual vector **A** is said to be *space-like* if the vector **a** is space-like , *time-like* if the vector **a** is time-like, and *light-like* (*dual null*) if the vector **a** is light-like. We also defined the time orientation as follows, A time-like vector $\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_0$ is *future- pointing* (resp., *past- pointing*) if the vector **a** is future-pointing (resp., past- pointing)

The set of all light-like vectors in D_1^3 is called the *dual light-like cone* and shown by Λ.

Definition 2.2. The *norm* of dual vector $\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_\circ$ is a dual number giving by

$$|\mathbf{A}| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \left(|\mathbf{a}|, \varepsilon \frac{\langle \mathbf{a}, \mathbf{a}_\circ \rangle}{|\mathbf{a}|^2} \right), \tag{2.2}$$

where $|\mathbf{a}| \neq 0$.

Definition 2.3. Let $\mathbf{A}$, $\mathbf{B} \in D_1^3$. We define the *Lorentzian vectoral product* of $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A} \wedge \mathbf{B} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & -\mathbf{E}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} , \tag{2.3}$$

where $\mathbf{A}=(A_1,A_2,A_3)$, $\mathbf{B}=(B_1,B_2,B_3)$ and $\mathbf{E}_1 \wedge \mathbf{E}_2 = \mathbf{E}_3$, $\mathbf{E}_2 \wedge \mathbf{E}_3 = -\mathbf{E}_1$, $\mathbf{E}_3 \wedge \mathbf{E}_1 = -\mathbf{E}_2$.

Lemma 2.4. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in D_1^3$. In this case , we have

i) $\langle \mathbf{A} \wedge \mathbf{B} , \mathbf{C} \rangle = -\det(\mathbf{A}, \mathbf{B}, \mathbf{C})$

ii) $(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} = -\langle \mathbf{A}, \mathbf{C} \rangle \mathbf{B} + \langle \mathbf{B}, \mathbf{C} \rangle \mathbf{A}$

iii) $\langle \mathbf{A} \wedge \mathbf{B}, \mathbf{A} \wedge \mathbf{B} \rangle = -\langle \mathbf{A}, \mathbf{A} \rangle \langle \mathbf{B}, \mathbf{B} \rangle + (\langle \mathbf{A}, \mathbf{B} \rangle)^2$

iv) $(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} + (\mathbf{B} \wedge \mathbf{C}) \wedge \mathbf{A} + (\mathbf{C} \wedge \mathbf{A}) \wedge \mathbf{B} = 0$.

Proof. By using the definitions of the Lorentzian inner product and the Lorentzian vectoral product it is easily shown .

Definition 2.5. Let $\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_\circ \in D_1^3$.

i) The set

$$S_1^2 = \left\{ \mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_\circ \;\middle|\; |\mathbf{A}| = (1,0) ;\ \mathbf{a}, \mathbf{a}_\circ \in R_1^3 \text{ and the vector } \mathbf{a} \text{ is space-like} \right\}$$

is called the *dual Lorentzian unit sphere* in D_1^3.

ii) The set

$$H_0^2 = \left\{ \mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}_\circ \;\middle|\; |\mathbf{A}| = (1,0) ;\ \mathbf{a}, \mathbf{a}_\circ \in R_1^3 \text{ and the vector } \mathbf{a} \text{ is time-like} \right\}$$

is called the *dual hyperbolic unit sphere* in D_1^3.

There are two component of the dual hyperbolic unit sphere H_0^2. The components of H_0^2 through (0,0,1) and (0,0,-1) are called the *future dual hyperbolic unit sphere* and the *past dual hyperbolic unit sphere* and shown by $\overset{+}{H}{}_0^2$ and $\overset{-}{H}{}_0^2$, respectively. In this case ,we have

$\overset{+}{H}{}_0^2$ ={**A**=**a** + ε**a**$_\bullet$ | | **A** | = (1,0); **a**, **a**$_\bullet$ ∈ R_1^3 and the vector **a** is future pointing time-like}

and

$\overset{-}{H}{}_0^2$= {**A** = **a** + ε**a**$_\bullet$ | | **A**| = (1,0) ; **a** , **a**$_\bullet$ ∈ R_1^3 and the vector **a** is past pointing time-like}.

Theorem 2.6. There is one to one correspondence between directed space-like (resp., time-like) lines of R_1^3 and ordered pair of vectors (**a** , **a**$_\bullet$) such that

$$\langle \mathbf{a} , \mathbf{a} \rangle = 1 \ \ (\text{resp.}, \langle \mathbf{a} , \mathbf{a} \rangle = -1) \ \text{ and } \ \langle \mathbf{a} , \mathbf{a}_\bullet \rangle = 0 .$$

Proof i) In R_1^3, a directed space-like line can be given by **y** = **x** + λ **a**, where **x** and **a** are position vector and the direction vector of line, respectively . The moment vector **a**$_\bullet$ = **x** ∧ **a** is not depend on chosen of the point on line. For this reason, by the help of ordered pair of vectors (**a** , **a**$_\bullet$) , a directed space-like line was determined as one unique and the following conditions are satisfied:

$$\langle \mathbf{a} , \mathbf{a} \rangle = 1 , \qquad \langle \mathbf{a} , \mathbf{a}_\bullet \rangle = 0 .$$

In D_1^3, let us define a dual space-like unit vector **A** = **a** + ε **a**$_\bullet$ with **a** and **a**$_\bullet$ which are determines a directed space-like line , where ε is a special dual unit with ε^2 =0. In the equation (2,1) if we take **B** = **A** , then we obtain

$$\langle \mathbf{A} , \mathbf{A} \rangle = \langle \mathbf{a} , \mathbf{a} \rangle + 2\varepsilon \ \langle \mathbf{a} , \mathbf{a}_\bullet \rangle = 1,$$

where the dual space-like unit vector **A** represented the directed space-like line $(\mathbf{a}, \mathbf{a}_0)$.

The coordinates of ordered pair of vectors $(\mathbf{a}, \mathbf{a}_0)$ are called the normed plücker coordinates of a directed space-like line **A** in $\mathbf{R}_1^3$.

ii) Let the directed line be time-like. In this case , the moment vector $\mathbf{a}_0$ of **a** is a space-like vector . For this reason , by the help of ordered pair of vectors $(\mathbf{a}, \mathbf{a}_0)$, the directed time-like line is determined as one unique. Similarly to i), we have the dual time-like vector $\mathbf{A} = \mathbf{a} + \varepsilon\mathbf{a}_0$. If we take $\mathbf{B} = \mathbf{A}$ in (2,1) then we obtain

$$\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + \varepsilon \langle \mathbf{a}, \mathbf{a}_0 \rangle = -1 ,$$

where the dual time-like unit vector **A** represents the directed time-like line $(\mathbf{a}, \mathbf{a}_0)$. That is, our directed time-like line will correspond to a dual point of the dual hyperbolic sphere.

The coordinates of ordered pair of vectors $(\mathbf{a}, \mathbf{a}_0)$ are also called the normed plücker coordinates of a directed time-like line **A** in $\mathbf{R}_1^3$.

3. ANGLE IN SPACE D_1^3

Case 1: Let **A** and **B** be dual space-like unit vectors. Let us consider the Lorentzian inner product of **A** and **B** which is given by (2.1). The dual space like unit vectors **A** and **B** determine two directed space-like lines d_1 and d_2 , since the moment vectors of **a** and **b** are $\mathbf{a}_0$ and $\mathbf{b}_0$,respectively .The real part of inner product (2.1) is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \cos\varphi. \qquad 0 \leq \varphi \leq \Pi \qquad \varphi \in \mathbb{R} \quad (3.1)$$

and the dual part is

$$\langle \mathbf{a}, \mathbf{b}_0 \rangle + \langle \mathbf{a}_0, \mathbf{b} \rangle = -\varphi_0 \sin\varphi . \qquad (3.2)$$

Consequently , we have

$$\langle \mathbf{A}, \mathbf{B} \rangle = \cos\varphi - \varepsilon\varphi_0 \sin\varphi = \cos(\varphi + \varepsilon\varphi_0) = \cos\Phi. \qquad (3.3)$$

Where the real part φ and the dual part φ_0 of Φ give the angle and the smallest distance between two directed space-like lines , respectively.

Definition 3.1. We shall call the dual number $\Phi = \varphi + \varepsilon\varphi_0$ the *dual central angle* between dual space-like unit vectors **A** and **B**.

Since endpoints of dual space-like unit vectors **OA** = **A** and **OB** = **B** indicate the dual points **A** and **B** of the dual Lorentzian unit sphere with the center O , the angle $\Phi = \varphi + \varepsilon\varphi_0$ between dual space-like unit vectors **A** and **B** can be considered as arc length $\overset{\frown}{AB}$ of dual curve passing from the dual points A and B of S_1^2.

Case 2. Let **A** and **B** be two future pointing (resp., past-pointing)dual time-like unit vectors. In this case, the real part of the inner product (2.1) is

$$\langle \mathbf{a}, \mathbf{b} \rangle = -\cosh\theta \quad , \quad \theta \in \mathbf{R} \tag{3.4}$$

and the dual part of (2.1) is

$$\langle \mathbf{a}, \mathbf{b_0} \rangle + \mathbf{a_0}, , \mathrm{b} \rangle = -\theta_0 \sinh\theta . \tag{3.5}$$

Consequently, the Lorentzian inner product of **A** and **B** isgiven by

$$\langle \mathbf{A}, \mathbf{B} \rangle = -\cosh\theta - \varepsilon\theta_0 \sinh\theta . = -\cosh(\theta + \varepsilon\theta_0) = -\cosh\Theta .$$
(3.6)

Hence we give the following definition:

Definition 3.2. We shall call the dual number $\Theta=\theta+\varepsilon\theta_0$ the *dual hyperbolic angle* between future pointing (resp., past-pointing) time-like unit vectors **A** and **B**.

The dual hyperbolic angle $\Theta=\theta+\varepsilon\theta_0$ consists of the hyperbolic angle θ between directed time-like lines which are represented in R_1^3 of dual time-like unit vectors A and **B** and the smallest distance θ_0 between two lines.

Since the endpoints of future-pointing (resp., past-pointing) time-like unit vectors **OA** = **A** and **OB** = **B** indicate the dual points A and B of $\overset{+2}{H_0}$ (resp., $\overset{-2}{H_0}$) with the center O, the dual hyperbolic angle $\Theta=\theta+\varepsilon\theta_0$ between the vectors **A** and **B** can be considered as arc length AB of dual space-like curve passing from the dual points A and B of $\overset{+2}{H_0}$ (resp., $\overset{-2}{H_0}$) .

From the formula (3.2) , the cases with respect to each other of future pointing time-like vectors **A** and **B** can be given as follows:

i) $\langle \mathbf{A}, \mathbf{B} \rangle \neq$ pure dual . That is , **A** and **B** can not be orthogonal .

ii) $\langle \mathbf{A}, \mathbf{B} \rangle =$ pure real iff $\theta_0 = 0$. The lines **A** and **B** intersect and the expression

$$\langle a, b_0 \rangle + \langle a_0, b \rangle = 0$$

is the condition of intersection of two lines.

iii) If $\langle \mathbf{A}, \mathbf{B} \rangle = (-1,0)$, the lines **A** and **B** are coincide and same directed.

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