Mathematical & Computational Applications, Vol. 1, No. 2, pp. 100-110, 1996

ON THE PARTIAL GENERALIZATION OF THE MEASURE OF TRANSCENDENCE OF SOME FORMAL LAURENT SERİES

Ahmet Ş. ÖZDEMİR
Atatürk Education Faculty , Department of Mathematics
Marmara University , İstanbul / TURKEY

Abstract: In this work , we determine the transcendence measure of the formal Laurent series "that"

$$\xi = \psi(r) = \sum_{k=0}^{\infty} \frac{(-1)^k r^{q^k}}{F_k}$$

whose transcendence has been established by L.I.Wade. Using the methods and lemmas in P.Bundschuh's article, measure of the transcendence for the above ξ is determined as

$$T(n,H) = H^{-aq^d(q\frac{1-k}{k}-d-1)}.$$

On the other hand , it was proven that the transcendence series ξ is not a U but is a S or T-number according to the Mahler's classification.

INTRODUCTION

Let p a prime number and $u \geq 1$ an integer. Let F be a finite field with $q = p^u$ elements. We denote the ring of the polynomials with one variable over F by $F[x]$ and its quotient field by $F(x)$. If $a \in F[x]$ is a non-zero polynomial, denote its degree by ∂a . If $a = 0$, then its degree is defined as $\partial 0 := -\infty$. Let a and b ($b \neq 0$) two polinomials from $F[x]$ and define a discrete valuation of $F(x)$ as follows

$$|\frac{a}{b}| = q^{\partial a - \partial b}.$$

Let K be the completion of $F(x)$ with respect to this valuation. Every element ω of K can be uniquely represented by

$$\omega = \sum_{n=k}^{\infty} c_n x^{-n}, c_n \in F.$$

If $\omega = 0$, then all c_n are zero.If $\omega \neq 0$, then there exist an $k \in Z$ for wich $c_k \neq 0$. If $\omega \neq 0$, then we have

$$|\omega| = q^{-k}.$$

Therefore K is the field of all formal Laurent series. The classical theory of transcendence over complex numbers has a similar version over K . Elements of $F[x]$ and $F(x)$ correspond to integers and fractions of the classical theory, respectively.

If $\omega \in K$ is one of the roots of a non-zero polynomial with coefficients in $F[x]$, then $\omega \in K$ is said to be algebraic over $F(x)$. Otherwise, ω is called transcendental over $F(x)$. The studies of transcendental numbers in K were initiated first by Wade [1-4] . Also Geijsel [5-8] did similar studies. As it is the case in the classical theory of transcendental numbers, it is possible to define a measure of transcendence

The measure of transcendence is thoroughly studied in the classical theory. For example, the transcendence measure of e has been widely investigated by Mahler [9] and Fel'dman [10] . Examples for the transcendence measure in the field K have been given for the first time by Bundschuh [12].

In this work, we determine a transcendence measure of some formal Laurent series whose transcendence has ben established by L.I.Wade [2]. If r^{q^k} (where $r \in RealNumbers)and$ $F_k \in F[x]$ is a fixed non-zero polynomial of degree $\partial(r^{q^k}) = 0$ and $\partial(F_k) = kq^k$ then the series

$$\xi = \psi(r) = \sum_{k=0}^{\infty} \frac{(-1)^k r^{q^k}}{F_k} \tag{1}$$

is an element of K , and L.I.Wade showed its transcendence in [2] . (see Theorem 3.1 and 3.2) Using the methods and lemmas in Bundschuh's article [12] , we determine a transcendence measure of ξ . We take an arbitrary non-zero polynomial

$$P(y) = \sum_{\nu=0}^{n} a_\nu y^\nu, (a_\nu \in F[x]; \nu = 0, 1, ..., n) \tag{2}$$

whose degree $\partial(P)$ is less than or equal to n. The height of P is denoted by

$$h(P) = \max_{\nu=0}^{n} |a_\nu| = q^{\max_{\nu=0}^{n} \partial(a_\nu)}$$

For the transcendental element ξ of K , we define the positive quantity

$$\Gamma_n(H, \xi) = min|P(\xi)| \quad .$$

where $P \neq 0, \partial(P) \leq n, h(P) \leq H$.

If $T(n, H)$ is a function of the variables n, H of $\Gamma_n(H, \xi)$ which satisfies the inequality

$$\Gamma_n(H, \xi) \geq T(n, H) \tag{3}$$

for all sufficiently large values of n and H , then $T(n, H)$ is said to be a transcendence measure of ξ.

Theorem 1 :

We take an arbitrary , non-zero polynomial

$$P(y) = \sum_{\nu=0}^{d} a_\nu y^\nu, (a_\nu \in F[x]; \nu = 0, 1, ..., n) \quad ; \tag{4}$$

further let $\partial(P) = d$, h(P)=h and a=$\max_{\nu=0}^{d} \partial a_\nu$. We assume that

$$dq^d logh > \frac{kq^k logq}{q} \tag{5}$$

we have

$$|P(\xi)| \geq h^{-aq^d(\frac{q^{1-k}}{k}-d-1)} \tag{6}$$

and a transcendence measure of ξ is

$$T(n, H) = H^{-aq^d(\frac{q^{1-k}}{k}-d-1)}. \tag{7}$$

As in the classical theory of transcendental number theory (see Schneider [13], page 6), it is possible to define Mahler's classification on K. Let $\xi \in K$ be transcendental, and define :

$$\Gamma_n(\xi) := \lim_{H\to\infty} sup \ \frac{-log \ \Gamma_n(H, \xi)}{log \ H}$$

$$\Gamma(\xi) = \lim_{n\to\infty} sup \ \frac{1}{n}\Gamma_n(\xi) \tag{8}$$

Hence $\Gamma_n(\xi) \geq n$ for every $n \in N$ and so $\Gamma(\xi) \geq 1$. For every n ,$H \in N$,

$$\Gamma_n(H, \xi) < H^{-n} q^n max(1, |\xi|^n) \tag{9}$$

is satisfied (see Bundschuh [12],Lemma 3) .

On the other hand, let the least naturel number n satisfying $\Gamma_n(\xi) = \infty$ be denoted by $\mu(\xi)$. If there is no such n , then one may define $\mu(\xi)$ as ∞. In this case, the transcendental number $\xi \in R$ is called
S-Laurent series if $1 \leq \Gamma(\infty) < \infty$ and $\mu(\xi) = \infty$,
T-Laurent series if $\Gamma(\xi) = \infty$ and $\mu(\infty) = \infty$,
U-Laurent series if $\Gamma(\xi) = \infty$ and $\mu(\infty) < \infty$.

Moreover, the U-class may be divided into subclasses.If $\mu(\xi) = m(m > 0)$ then ξ is called a U_m-Laurent series. Leveque [11] was the first to show that for all m, U_m is non-empty in the classical theory but the honour goes to Oryan [14] if the ground field is K.

According to the above classification, the series defined in (1) can not be a U-Laurent series. This fact may be proved by the help of the Theorem 1.

Theorem 2 : The ξ Laurent series defined by (1) doesn't belong to the class U so that it belongs to the class S or to the class T.

Preliminary

We will use the following lemmas in proof of the theorem.

Lemma 1: Let

$$P(y) = \sum_{\nu=0}^{d} a_\nu y^\nu \tag{10}$$

$(a_\nu \in F[x],\ a_d \neq 0\ (d \geq 1)\quad , a = \max_{\nu=0}^{d}, \partial a_\nu \quad .$

Then there are some elements $A_0, A_1, ..., A_d \in F[x]$,not all zero satisfying. $\partial A_j \leq ad(q^d - d + 1)$ for $0 \leq j \leq d$ and

$$\sum_{j=0}^{d} A_j y^{q^j} = p(y) \sum_{j=0,q^j \geq d}^{d} A_j \sum_{k=0}^{q^j-d} b_k a_d^{-k-1} y^{q^j-d-k} =: P(y)Q(y) \quad , \tag{11}$$

where $b_0 := 1$ and b_k ,for $k \geq 1$ is the sum of product of exactly k terms from $a_0, a_1, ..., a_d$, multiplied by $\pm$.

Proof: See [12], lemma 4, page 416

Lemma 2: Let $\xi \in K$ and $|\xi| = q^\lambda$. Under the hypotheses of Lemma 1 we have

$$|Q(\xi)| \leq q^{ad(q^d-d+1)+(q^d-d)max(a,\lambda)}. \tag{12}$$

Proof : See [12], lemma 5, page 417

PROOF OF THE THEOREMS

Proof of the Theorem 1 :

Consider the polynomial defined by (4). With $\partial(p) = d$, $a_d \neq 0$. Let $d \geq 1$. By Lemma 1 there are some elements the $A_0, A_1, ..., A_d \in F[x]$ not all zero ,such that

$$\sum_{j=0}^{d} A_j y^{q^j} = P(y) \sum_{j=0, q^J \geq d}^{d} A_j \sum_{k=0}^{q^j - d} b_k a_d^{-k-1} y^{q^j - d - k} =: P(y)Q(y) \tag{13}$$

$$\partial(A_j) \leq ad(q^d - d + 1) \leq adq^d \quad (0 \leq j \leq d) \quad . \tag{14}$$

In (13) we put ξ instead of y :

$$P(\xi)Q(\xi) = \sum_{j=0}^{d} A_j \xi^{q^j} = \sum_{j=0}^{d} A_j \sum_{k=0}^{\infty} (-1)^{r q^{k+j}} F_k^{-q^j} \ . \tag{15}$$

Furthermore let $D_k = \sum_{k=i+j}^{\infty} \frac{(-1)^i A_j r^{q^{k+j}} F_k}{F_j q^j}$

Seperate in (12) sum as $T_1 + T_2$,where

$$T_1 = F_\beta \sum_{k=0}^{\beta} \frac{D_k}{F_k}, \qquad T_2 = F_\beta \sum_{k=\beta+1}^{\infty} \frac{D_k}{F_k} \tag{16}$$

where β ,which is not a negative integer will be chosen later.

1) First, we prove that $|T_1| \geq 1$. That is, we prove T_1 is a polynomial but not equal zero. By the definition of F_k ,obviously T_1 is polynomial. Furthermore,

$$\begin{aligned} T_1 &\equiv D_\beta(mod[\beta - l]) \\ &\equiv (-1)^{\beta - l} A_l [\beta] ... [\beta - l +]^{q^{l-1}} \\ &\equiv (-1)^{\beta - l} A_l F_l \not\equiv 0 (mod[\beta - l]) \end{aligned}$$

for β sufficiently large. Therefore , for all sufficiently β T_1 is not identically zero. So T_1 is non-zero polynomial. So it shown that $|T_1| \geq 1$. (where $degT_1 > 0 \Longrightarrow |T_1| \geq 1 = q^{degT_1} > q^0 = 1$)

2) we wil show $|T_2| < 1$. Let T_2^* be any term of T_2 . Note that ,

$$
\begin{aligned}
degD_\beta &= degF_\beta + degA_j - degF^{q^j}_{\beta-j} + degr^{q^{\beta+j}} \\
&= degF_\beta + degA_j - degF^{q^j}_{\beta-j} (where\ degr^{q^{\beta+j}} = 0) \\
degT_2^* &= degF_\beta + degF_{\beta+1} + degD_\beta \\
&= degF_\beta + degF_{\beta+1} - degF^{q^j}_{\beta-j} + \underbrace{(degA_j)}_{=a^*(constant)} \\
&= rdegF_\beta - degF_{\beta+1} - degF^{q^j}_{\beta_j} + a^* \\
&= r^\beta - (\beta+1)q^{\beta+1} - (\beta-j)q^\beta + a^*
\end{aligned}
$$

where $r\beta q^\beta - (\beta+1)q^{\beta+1} - (\beta-d)q^\beta < 0$. Because ; since $0 \leq j \leq d$ $\beta - j \geq \beta - d \Longrightarrow -(\beta-j) \geq -(\beta-d)$ we have hence $r\beta q^\beta < q^\beta(\beta^q + q + \beta - d)$ so we obtain $\beta + d < (\beta+1)q$. This inequality is true everytime.

Therefore $\beta \longrightarrow +$, $r\beta q^\beta - (\beta)q^{\beta+1} - (\beta-d)q^\beta \longrightarrow -$. hence $(-\infty + a^*) \longrightarrow -\infty$. Therefore we may chose β so large that every term of T_2 is negative. That is ; $|T_2| = q^{degT_2} < q^0 = 1 \Longrightarrow |T_2| < 1$. 3) We will prove the claim of the theorem . By the definition of T_1 and T_2 , we can write

$$T_1 + T_2 = F_\beta P(\xi)Q(\xi).$$

Hence we obtain

$$|T_1 + T_2| = |F_\beta||P(\xi)||Q(\xi)| \ . \tag{17}$$

Since $|T_1| \geq 1$ and $|T_2| < 1$, we get

$$|T_1 + T_2| = max\,(|T_1|, |T_2|) = |T_1| \ . \tag{18}$$

By (17) and (18), we obtain

$$|P(\xi)||Q(\xi)| = |T_1||F_\beta|^{-1} \ . \tag{19}$$

Let $|\xi| = q^\lambda$. By (1) and since

$$|r^{q^k} F_k^{-1}| = q^{deg(r^{q^k} F_k^{-1})} = q^{degr^{q^k} - degF_k} = q^{0-kq^k} = q^{-kq^k}.$$

we get $|\xi| = q^{-kq^k} = q^{-0q^0} = q^0$. Therefore $\lambda = 0$ .Since $Max(a, \lambda) = Max(a, 0) = a$ and by Lemma 2 ,we find

$$
\begin{aligned}
|Q(\xi)| &\leq q^{ad(q^d-d+1)+(q^d-d)max(a,\lambda)} \\
&\leq q^{adq^d + aq^d}
\end{aligned}
$$

$$|Q(\xi)| = q^{a(d+1)q^d} \ . \tag{20}$$

Since $h = h(P) = q^a$,

$$a = \frac{logh}{logq}. \tag{21}$$

By (6) and (21) we find

$$adq^d > \frac{kq^k}{q}. \tag{22}$$

Consider the sequence

$$\{q^{-1}, q^0, q^1, q^2, ...\}$$

There are β non-negative integers such that

$$\beta q^{\beta-1} \leq \frac{adq^d}{kq^k} < \beta q^\beta. \tag{23}$$

Because , by (22)

$$\frac{1}{q} \leq \frac{adq^d}{kq^k}.$$

From (21) we obtain the following statement for the above β

$$\frac{adq^d}{kq^k} < q^\beta \leq \frac{adq^{d+1}}{kq^k}. \tag{24}$$

Further ,by (24)

$$|F_\beta| = q^{degF_\beta} = q^{\beta q^\beta}. \tag{25}$$

By (19),(20),(23),(25) and since $|T_1| \geq 1$ we get

$$\begin{aligned} |P(\xi)| &= |T_1||F_\beta|^{-1}|Q(\xi)|^{-1} \\ &\geq |F_\beta|^{-1}|Q(\xi)|^{-1} \\ &\geq q^{-\frac{1}{k}(adq^{d-k+1})-a(d+1)q^d} \\ &= q^{-aq^d(\frac{q^{1-k}}{k}-d-1)}. \end{aligned} \tag{26}$$

By (26) and since $h = q^a$ we have

$$|P(\xi)| \geq h^{-aq^d(\frac{q^{1-k}}{k}-d-1)}.$$

This is the claim of the theorem 1.

Proof of the Theorem 2 :

Let the degree of the polynomial P in Theorem 1 be $\partial(P) = d \leq n$ and let its height be $h(P) = h \leq H$ By (4),

$$|P(\xi)| \geq H^{-aq^d(\frac{q^{1-k}}{k}-d-1)}. \tag{27}$$

(27) and (6) and by the definition of Mahler's classification

$$\Gamma_n(H,\xi) \geq H^{-aq^d(\frac{q^{1-k}}{k}-d-1)}$$

for all sufficiently large natural numbers n and H . Hence consequently

$$log\Gamma_n(H,\xi) \geq \left[-aq^d(\frac{q^{1-k}}{k} - d - 1)\right] logH$$

$$\frac{-log\Gamma_n(H,\xi)}{logH} \leq aq^d(\frac{q^{1-k}}{k} - d - 1) \tag{28}$$

$$\Gamma_n(\xi) = \lim_{H\to\infty} sup\ \frac{-log\ \Gamma_n(H,\xi)}{log\ H} \leq aq^d(\frac{q^{1-k}}{k} - d - 1). \tag{29}$$

That is, for every index n

$$\Gamma_n(\xi) < \infty.$$

By the definition of Mahler's classification, $\mu(\xi) = \infty$. This shows ξ can never belong to the class U so that it belongs to the class S or to the class T.

REFERENCES

[1] WADE , L . I . ,Certain quantities transcendental over $GF(p^n, x)$, Duke Math. J . **8**, 701 - 702 (1941).

[2] WADE , L . I .,Certain quantities transcendental over $GF(p^n, x)$ II , Duke Math. J .**10** , 587-594(1943).

[3] WADE , L . I . ,Transcendence properties of the Carlitz Ψ - functions , Duke Math . J.**13**, 79-85(1946).

[4] WADE , L. I.,Two types of function field transcendental numbers , Duke Math . J . 755-758 (1944).

[5] GEIJSEL , J.M., Transcendence properties of $Carlitz - Besselfunctions$, Math. Centre Report ZW 2/71 Amsterdam , **19**, (1971).

[6] GEIJSEL , J.M., Schneider's method in fields of characteristic $p \neq 2$. Math. Centre Report ZW 17/73 Amsterdam ,**12** , (1973).

[7] GEIJSEL , J.M., Transcendence properties of certain quantities over the quotient field of $F_q[x]$, Math. Centre Report ZN 58/74 , Amsterdam ,**62** ,(1974).

[8] GEIJSEL , J.M., Transcendence in fields of positive characteristic, Matem atical Centre Tracts 91. Amsterdam : Mathematisch Centrum. X ,not consecutively paged (1979).

[9] MAHLER , K. , Zur Approximation der Exponentialfunktion und des Logarithmus , J. reine angew Math.**166** , 118-150 (1932).

[10] FEL'DMAN , N.I. , On the problem of the measure of transcendence of e (russ) Uspekhi Math. Navk **18** 207-213 (1963).

[11] LE VEQUE W . J ., On Mahler's U- numbers J. London Math. Soc.**28** , 220-229 (1953).

[12] BUNDSCHUH , P. , Transzendenzmasse in Körpern formaler Laurentrei hen Journal für die reine und angewandte Mathematik, **299/300** 411-432 (1978).

[13] SCHNEIDER, Th. , Eunführung in die transzendenten zahlen Berlin -Göttingen-Heidelberg (1957).

[14] ORYAN, M. H. , Über die Unterklassen U_m der Mahlerschen Klassen einteilung der transzendenten formalen Laurentreihen. İstanbul Üniv. Fen. Fak. Mec.**Seri A, 45** 43-63 (1980).

[15] ÖZDEMİR , A. , Ş. , On the Measure of Transcendence of Certain formal Laurent Series. Ph.D.Thesis , Technical University of İstanbul, The Enstitue of Science. No : 34490,(1993)

[16] ÖZDEMİR , A. , Ş., On the Measure of Transcendence of $\xi = \psi(2) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{q^k}}{F_k}$ On the Measure of Transcendence of Certain formal Laurent Series , Marmara University ,The Journal of Science. No : 10 , (1995)