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DYNAMIC BEHAVIOR OF A STIFFENED LAMINATED CONICAL SHELL UNDER HYGROTHERMAL LOADS

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Abstract - The dynamic behavior of a stiffened composite laminated conical thin shell under hygrothermal loads is studied numerically. The governing equations of truncated conical shell are based on the Donnell-Mushtari theory of thin shells including the transverse shear deformation and rotary inertia. The extension-bending coupling is taken into account in the equations. The composite laminated conical shell is also reinforced at uniform intervals by elastic rings and/or stringers. The stiffening elements are relatively closely spaced, and therefore the stiffeners are "smeared out" along the conical shell. The inhomogeneity of material properties because of moisture and temperature is taken into account in the constitutive equations. The equations of laminated composite conical shell are solved by the use of finite difference method. The effect of moisture is examined on the dynamic behavior. The accuracy of results is tested by certain earlier results and an agreement is found.

1. INTRODUCTION

A stiffened laminated conical shell is one of the common structural elements used in modern airplane, missile, booster and other space vehicles. Composite material laminates inherently absorb moisture from the surrounding environment. The degradation of the mechanical properties of newly developed advanced composite materials because of moisture may have an effect on the dynamic behavior of the composite structures.

Studies on the vibrations of cylindrical and truncated conical shells made of isotropic and anisotropic materials were referenced by Leissa in his monograph [1]. Martin [2] studied the free vibrations of anisotropic conical shells. Mecitoglu and Dökmeci [3] developed a shell finite element that includes smeared stringers and rings, and examined the uniqueness on the solutions. Inhomogeneity of the material properties due to temperature was studied by Mecitoglu[4] for a truncated isotropic conical shell. Mecitoglu [5] also derived the governing equations for a laminated composite conical shell considering geometric nonlinearities, initial stresses as well structural and material inhomogeneties. Structural inhomogeneties are because of stringer and ring stiffeners; material inhomogeneties are because of temperature, moisture or manufacturing processes. The uniqueness of solution for the governing equations of the dynamic behavior is proved in the study. Kayran and Vinson [6] examined the effect of transverse shear and rotary inertia on the free vibrations of truncated conical shells. Tong [7] studied the free vibration of laminated conical shells using a simple and exact solution technique. Sai Ram and Sinha [8] investigated hygrothermal effects on the free vibration of laminated composite plates by finite element method. Lyrintsiz and Bofilios [9] studied the moisture effect on the dynamic response of orthotropic stiffened panel structures. Eslami and Maerz [10] considered thermally induced vibration of a symmetric cross-ply plate with hygrothermal effect. Heyliger and Jilani [11] studied the free vibrations of inhomogeneous elastic cylinders and spheres. Örenel [12] examined the moisture effect on the dynamic behavior of a stiffened laminated conical shell.

In this paper, the moisture effects on the dynamic behavior of a composite laminated conical thin shell reinforced by stringers and rings are studied. The dynamic equations of a stiffened composite laminated conical shell are derived within the frame of the Donnell-Mushtari theory of elastic thin shells. The rings and/or stringers are "smeared out" along the conical shell. The inhomogeneity of material properties because of moisture and temperature is taken into account in the constitutive equations. The dynamic equations of the stiffened laminated conical shell are solved by the finite difference method to obtain the vibration characteristics and an agreement is obtained with certain earlier results. Influence of the moisture on the vibration characteristics is examined.

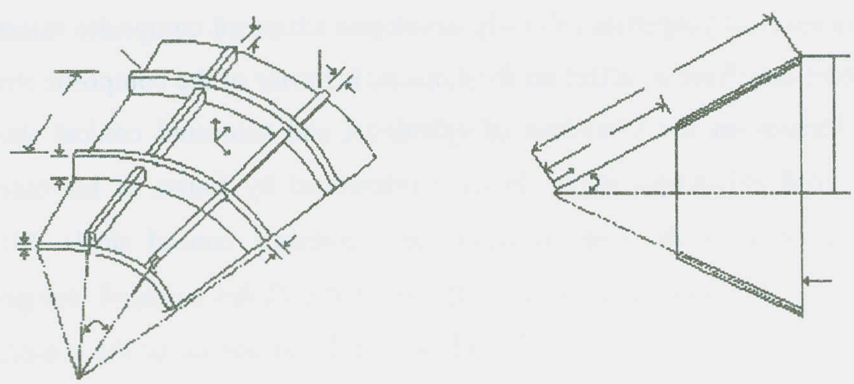

Figure 1. Geometry and Coordinates

2. GOVERNING EQUATIONS

Consider a conical shell as shown in Figure 1, where R indicates the radius of the cone at the large end, α denotes the semivertex angle of the cone and L is the cone length along its generator. h_S and h_R are the height of the stringers and rings, respectively. b_S and b_R are the width of the stringers and rings, respectively. Θ is the angular spacing of the stringers and S is the spacing of the rings. We use the s-θ coordinate system; s is measured along the cone's generator starting at the cone vertex and θ is the circumferential coordinate.

Displacement Field

The displacement field in the shell can be approximated by the use of the Weierstrass theorem and the Kirchhoff-Love hypothesis for linear elastic thin shells [5]. Then, the displacements in a thin shell can be stated in the form

$$U(s,\theta,z) = u(s,\theta) + z\beta_s(s,\theta) \quad V(s,\theta,z) = v(s,\theta) + z\beta_\theta(s,\theta) \quad W(s,\theta,z) = w(s,\theta) \tag{1}$$

where u, v, and w are the special values of the displacement field U, V, and W at the middle surface; β_s and β_θ are the rotations of the normal to the middle surface during deformation about the s and θ axes, respectively.

Kinematical Relations

Kinematical relations of a thin truncated conical shell can be approximated by the use of preceding representation of displacement field. The kinematical relations are given as follows:

$$\begin{aligned}
&\varepsilon_s = \lambda\frac{\partial u}{\partial \bar{s}} \qquad \varepsilon_\theta = \lambda\left(\frac{u}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial v}{\partial \bar{\theta}} + \cot\alpha\frac{w}{\bar{s}}\right) \qquad \varepsilon_{s\theta} = \lambda\left(\frac{1}{\bar{s}}\frac{\partial u}{\partial \bar{\theta}} + \frac{\partial v}{\partial \bar{s}} - \frac{v}{\bar{s}}\right)\\
&\varepsilon_{sz} = \lambda\left(\frac{\beta_s}{\lambda} + \frac{\partial w}{\partial \bar{s}}\right) \qquad \varepsilon_{\theta z} = \lambda\left(\frac{\beta_\theta}{\lambda} - \cot\alpha\frac{v}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial w}{\partial \bar{\theta}}\right)\\
&\kappa_s = \lambda\frac{\partial \beta_s}{\partial \bar{s}} \qquad \kappa_\theta = \lambda\left(\frac{1}{\bar{s}}\frac{\partial \beta_\theta}{\partial \bar{\theta}} + \frac{\beta_s}{\bar{s}}\right) \qquad \kappa_{s\theta} = \lambda\left(\frac{\partial \beta_\theta}{\partial \bar{s}} - \frac{\beta_\theta}{\bar{s}} + \frac{1}{\bar{s}}\frac{\partial \beta_s}{\partial \bar{\theta}}\right)
\end{aligned} \tag{2}$$

where, ε_s, ε_θ, and $\varepsilon_{s\theta}$ are the membrane strains of the middle surface; ε_{sz} and $\varepsilon_{\theta z}$ are the transverse shear strains; and κ_s, κ_θ, and $\kappa_{s\theta}$ the bending strains. The characteristic parameter λ is $\sin\alpha/R$. The coordinates $\bar{s}$ and $\bar{\theta}$ are defined as $\bar{s} = s\lambda$ and $\bar{\theta} = \theta\sin\alpha$, respectively.

Constitutive Equations

The constitutive equations of the laminated conical shell relate the force and moment resultants to the membrane, transverse shear and bending strains. Here, the bending-stretching coupling is considered in the constitutive equations. The stiffening elements are taken into account by the orthotropic material approach of stiffened shells. The constitutive equations of the laminated conical shell can be written as follows:

$$
\begin{aligned}
N_s &= A_{11}\varepsilon_s + A_{12}\varepsilon_\theta + B_{11}\kappa_s + B_{12}\kappa_\theta & \quad M_s &= B_{11}\varepsilon_s + B_{12}\varepsilon_\theta + D_{11}\kappa_s + D_{12}\kappa_\theta \\
N_\theta &= A_{12}\varepsilon_s + A_{22}\varepsilon_\theta + B_{12}\kappa_s + B_{22}\kappa_\theta & \quad M_\theta &= B_{12}\varepsilon_s + B_{22}\varepsilon_\theta + D_{12}\kappa_s + D_{22}\kappa_\theta \\
N_{s\theta} &= A_{33}\varepsilon_{s\theta} + B_{33}\kappa_{s\theta} & \quad M_{s\theta} &= B_{33}\varepsilon_{s\theta} + D_{33}\kappa_{s\theta} \\
Q_s &= A_{55}\varepsilon_{sz} & \quad Q_\theta &= A_{44}\varepsilon_{\theta z}
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
A_{ij} &= \sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)}\left(h_k - h_{k-1}\right) + E_{ij} \qquad B_{ij} = \frac{1}{2}\sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)}\left(h_k^2 - h_{k-1}^2\right) - F_{ij} \\
D_{ij} &= \frac{1}{3}\sum_{k=1}^{N_L} \overline{Q}_{ij}^{(k)}\left(h_k^3 - h_{k-1}^3\right) + G_{ij} \qquad i,j = 1,2,3 \\
\left(A_{44}, A_{55}\right) &= \sum_{k=1}^{N_L} \int_{h_{k-1}}^{h_k} \left(\overline{Q}_{44}^{(k)}, \overline{Q}_{55}^{(k)}\right) f(z)\,dz
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\left\{E_{11}, E_{22}, E_{12}, E_{33}\right\} &= \lambda\left\{\frac{E_S A_S}{\overline{\Theta}}, \frac{E_R A_R}{\overline{S}}, 0, 0\right\} \\
\left\{F_{11}, F_{22}, F_{12}, F_{33}\right\} &= 2\lambda\left\{\frac{E_S A_S e_S}{\overline{\Theta}}, \frac{E_R A_R e_R}{\overline{S}}, 0, 0\right\} \\
\left\{G_{11}, G_{22}, G_{12}, G_{33}\right\} &= \lambda\left\{\frac{E_S A_S\left(i_{\eta S}{}^2 + e_S{}^2\right)}{\overline{\Theta}}, \frac{E_R A_R\left(i_{\xi R}{}^2 + e_R{}^2\right)}{\overline{S}}, 0, \left(\frac{G_S J_S}{\overline{\Theta}} + \frac{G_R J_R}{\overline{S}}\right)\right\}
\end{aligned} \tag{5}
$$

and

$$
f(z) = \frac{5}{4}\left[1 - 4\left(\frac{z}{h}\right)^2\right] \tag{6}
$$

where h is the total wall thickness of the conical shell. E_S, and $G_S J_S$ are the modulus of elasticity, and torsional stiffness of a stringer, respectively. A_S is the cross-sectional area of stringer and i_S is the radius of gyration of the stringer cross-sectional area about a centroidal axis parallel to the θ axis, respectively. e_S is the distance to the centroid of the stringer cross-section from the shell middle surface (eccentricity). The subscript R indicates the properties of a ring. In Eqs. (6), $\overline{\Theta}=\Theta\sin\alpha$ and $\overline{S}=S\lambda$. The parameters $\overline{Q}_{ij}$ can be obtained from the lamina properties in the form

$$\left\{Q^{(k)}{}_{11},Q^{(k)}{}_{12},Q^{(k)}{}_{22}\right\}=\frac{1}{1-\nu^{(k)}{}_{s\theta}\nu^{(k)}{}_{\theta s}}\left\{E^{(k)}{}_{s},\nu^{(k)}{}_{s\theta}E^{(k)}{}_{\theta},E^{(k)}{}_{\theta}\right\}$$
$$\left\{Q^{(k)}{}_{33},Q^{(k)}{}_{44},Q^{(k)}{}_{55}\right\}=E^{(k)}{}_{\theta}\left\{\frac{1}{2},\frac{1}{2\left(1+\nu^{(k)}{}_{\theta z}\right)},\frac{1}{2\left(1+\nu^{(k)}{}_{xz}\right)}\right\} \tag{7}$$

by the use of well known transformations.

Investigations have shown the deleterious effects of hygrothermal environments of structural performance of anisotropic materials [9]. The deleterious effects consist of degradation of the mechanical properties and hygroscopic expansions. The hygrothermal effects are two. One is associated with the temperature, and the second with moisture ingestion. At any temperature there is a shift in the stiffness and strength curves. These curves are lowered by the absorbed moisture. Because of the swelling significant residual stresses are induced by moisture absorption. Ingestion of moisture has an effect similar to the thermal effect.

The material properties of the shell may have a spatial change because of moisture and temperature variation in the shell. Therefore, the material of the conical shell may have an inhomogeneity and it should be considered in the constitutive relations. A simple linear variation in the Young modules of a laminate with moisture concentration rise can be considered in the form

$$E_s(c)=E_{s_0}+E_{s_1}c \qquad E_\theta(c)=E_{\theta_0}+E_{\theta_1}c \tag{8}$$

where c is the moisture concentration; E_{s_0} and E_{θ_0} are the Young modules without moisture effect; E_{s_1} and E_{θ_1} are the parameters. If the shell is exposed to the moisture from the both surfaces, the change of the moisture through the shell thickness can be demonstrated by a parabolic variation in the form

$$c(z)=\alpha_0+\alpha_1 z+\alpha_2 z^2 \tag{9}$$

where α_0, α_1 ve α_2 are the coefficients. If the moisture ingestion is through the outer surface of the shell only, equation (9) can be simplified in a linear form.

Equilibrium Equations

Now, the equilibrium equations, boundary and initial conditions for the vibration of stiffened composite conical shell can be derived from the virtual work principle [5]. The equilibrium equations can be written in terms of the displacements and rotations in the form

$$
\begin{aligned}
&L_{11}u + L_{12}v + L_{13}w + L_{14}\beta_s + L_{15}\beta_\theta + \frac{m_1}{\lambda^2}u + \frac{m_2}{2\lambda^2}\beta_s = 0 \\
&L_{21}u + L_{22}v + L_{23}w + L_{24}\beta_s + L_{25}\beta_\theta + \frac{m_1}{\lambda^2}v + \frac{m_2}{2\lambda^2}\beta_\theta = 0 \\
&L_{31}u + L_{32}v + L_{33}w + L_{34}\beta_s + L_{35}\beta_\theta + \frac{m_1}{\lambda^2}w = 0 \\
&L_{41}u + L_{42}v + L_{43}w + L_{44}\beta_s + L_{45}\beta_\theta + \frac{m_2}{2\lambda^2}u + \frac{m_{12}}{\lambda^2}\beta_s = 0 \\
&L_{51}u + L_{52}v + L_{53}w + L_{54}\beta_s + L_{55}\beta_\theta + \frac{m_2}{2\lambda^2}v + \frac{m_{21}}{\lambda^2}\beta_\theta = 0
\end{aligned}
\tag{10}
$$

where the parameters related with the mass of the conical shell are

$$
\begin{aligned}
&m_1 = \rho_c h + \frac{\lambda\rho_S A_S}{\overline{\Theta}} + \frac{\lambda\rho_R A_R}{\overline{S}} \qquad m_2 = \frac{2\lambda\rho_S A_S e_S}{\overline{\Theta}} + \frac{2\lambda\rho_R A_R e_R}{\overline{S}} \\
&m_{12} = \frac{\rho_c h^3}{12} + \frac{\lambda\rho_S A_S}{\overline{\Theta}}\left(e_S{}^2 + i_S{}^2\right) + \frac{\lambda\rho_R A_R}{\overline{S}}\left(e_R{}^2 + i_{pR}{}^2\right) \\
&m_{21} = \frac{\rho_c h^3}{12} + \frac{\lambda\rho_S A_S}{\overline{\Theta}}\left(e_S{}^2 + i_{pS}{}^2\right) + \frac{\lambda\rho_R A_R}{\overline{S}}\left(e_R{}^2 + i_R{}^2\right)
\end{aligned}
\tag{11}
$$

where ρ_C, ρ_S, and ρ_R are the material densities of the shell, stringer and ring, respectively. i_{pS} and i_{pR} are the polar radius of gyration of the stringer and the ring about the centroidal axes, respectively. The linear operators are given in the appendix.

Boundary and Initial Conditions

The boundary conditions considered in this study are expressed as follows:

$$
\begin{aligned}
&u(\bar{s}_1,\bar{\theta},t) = v(\bar{s}_1,\bar{\theta},t) = w(\bar{s}_1,\bar{\theta},t) = \beta_s(\bar{s}_1,\bar{\theta},t) = M_{s\theta}(\bar{s}_1,\bar{\theta},t) = 0 \\
&N_s(1,\bar{\theta},t) = v(1,\bar{\theta},t) = w(1,\bar{\theta},t) = M(1,\bar{\theta},t)_s = M_{s\theta}(1,\bar{\theta},t) = 0
\end{aligned}
\tag{12}
$$

The initial conditions are given as follows:

$$
\begin{aligned}
&u(\bar{s},\bar{\theta},0) = v(\bar{s},\bar{\theta},0) = w(\bar{s},\bar{\theta},0) = \beta_s(\bar{s},\bar{\theta},0) = \beta_\theta(\bar{s},\bar{\theta},0) = 0 \\
&\dot{u}(\bar{s},\bar{\theta},0) = \dot{v}(\bar{s},\bar{\theta},0) = \dot{w}(\bar{s},\bar{\theta},0) = \dot{\beta}_s(\bar{s},\bar{\theta},0) = \dot{\beta}_\theta(\bar{s},\bar{\theta},0) = 0
\end{aligned}
\tag{13}
$$

3. METHOD OF SOLUTION

The dynamic equations of a stiffened composite laminated conical thin shell which are given Eqs. (10) are solved by finite difference method using a computer program written in FORTRAN language. Assuming the harmonic motion, we assume displacement functions of the form

$$u(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{u}(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \qquad v(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{v}(\bar{s})\sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t$$

$$w(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{w}(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \tag{14}$$

$$\beta_s(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{\beta}_s(\bar{s})\cos\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t \qquad \beta_\theta(\bar{s},\bar{\theta},t)=\sum_{n=1}^{\infty}\hat{\beta}_\theta(\bar{s})\sin\frac{n\bar{\theta}}{\sin\alpha}\cos\omega t$$

where n is the circumferential wave number. Since the shell is axisymmetric (e.g., no cutouts) and has axisymmetric boundary conditions, then the vibration modes uncouple with respect to θ and the summations on the n is not considered in Eqs. (14). Substituting these equations into (10) we obtain five ordinary differential equations.

The finite difference method is applied to solve resulting ordinary differential equations [5] together with the boundary conditions given by equations (12). The central difference formulas are used to obtain the derivatives with respect to $\bar{s}$.

4. NUMERICAL RESULTS

The axisymmetric vibration of antisymmetric cross-plied laminated cones is considered in the examples. The length to radius ratio of the truncated conical shell is taken to be 0.5. The material properties of each layer without moisture effect are taken to be E_s = 130 Gpa, E_θ = 9.5 Gpa, $G_{s\theta} = G_{sz}$ = 6 Gpa, $G_{\theta z} = 0.5G_{s\theta}$, $\nu_{s\theta} = \nu_{xz} = \nu_{\theta z} = 0.3$. Following stiffening parameters are used for the stiffened shell case: $E_S = E_R = 0.52E_s$, $\rho_S = \rho_R = 1.73\rho_c$, $h_S = h_R = 5h$, $b_S = b_R = h$, $N_S = N_R = 6$. Here, N_S and N_R are the number of stringers and rings, respectively.

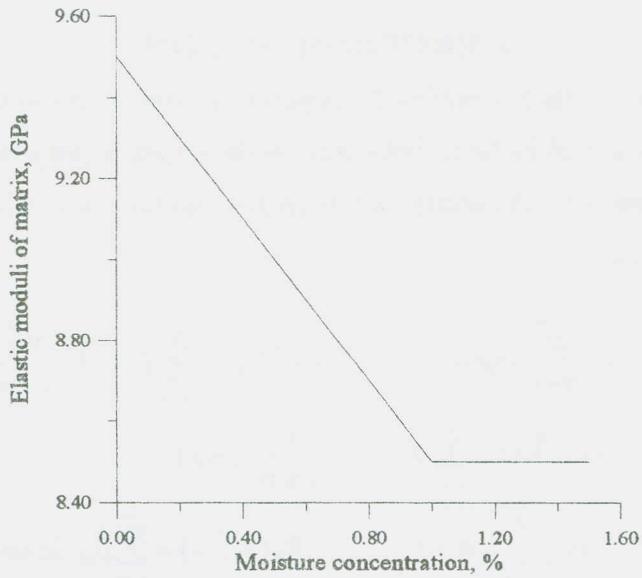

Figure 2. Elastic moduli of graphite/epoxy lamina at different moisture concentrations.

The variation of the E_θ with the moisture concentration is given in the Figure 2. The other material properties of the lamina have no significant variation with the moisture concentration [8].

The frequencies used in comparison are calculated for the first circumferential wave number. The 30 grid points give good convergence. Before presenting the results, let us introduce a frequency parameter defined as $\Omega = \omega R\sqrt{\rho_c h/A_{11}}$, where A_{11} is determined for the unstiffened conical shell without moisture effect.

The variation of the frequency ratio, Ω/Ω_0, with the thickness ratio of the shell is given in Figure 3 for the semivertex angles of 60 degree. The results obtained from this analysis are in an agreement with Tong's results [7]. Here, Ω and Ω_0 are the fundamental frequencies of stiffened and unstiffened conical shells in Hertz, respectively.

Figure 4 shows the variation of the frequency parameter of the unstiffened shell with the thickness for the different semivertex angles. An increase in the thickness results in an increase of frequency ratio for all semivertex angles.

The stiffening effects on the frequencies are shown in Figure 5. In the figure, the frequency parameter vs. the thickness of the shell is plotted for the unstiffened and stiffened cones with semivertex angle of 45 deg.

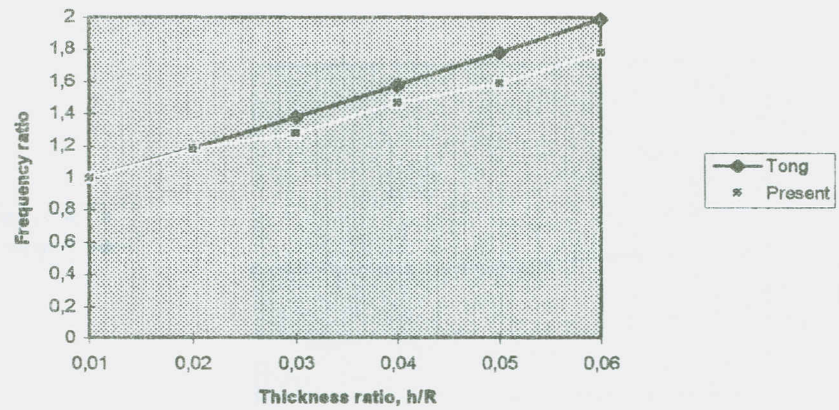

Figure 3 Frequency ratio vs. the thickness ratio ($\alpha = 60^0$, L / R=0.5).

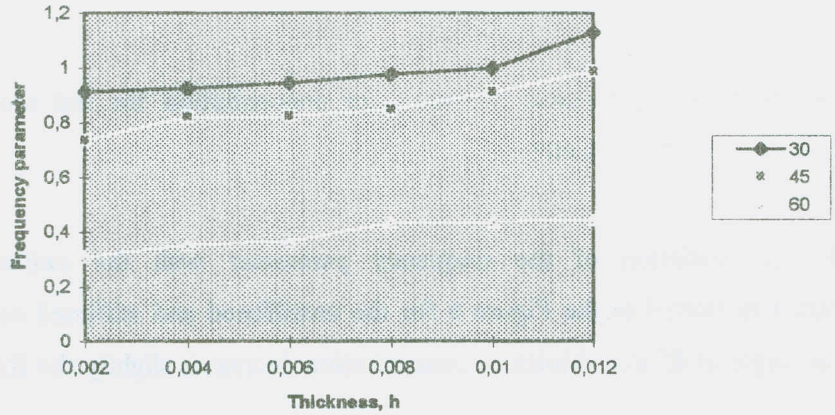

Figure 4 Frequency parameter vs. the thickness of the shell for different semivertex angles.

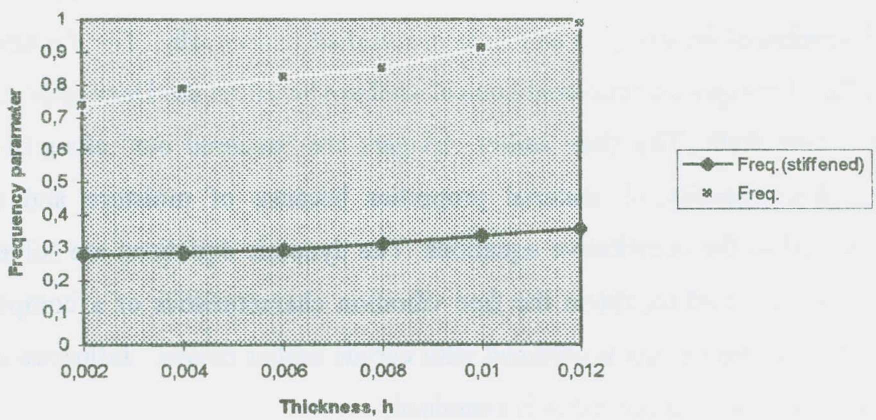

Figure 5 Frequency parameter vs. the shell thickness for the stiffened and unstiffened cases (α=45^0 , L/R=0.5).

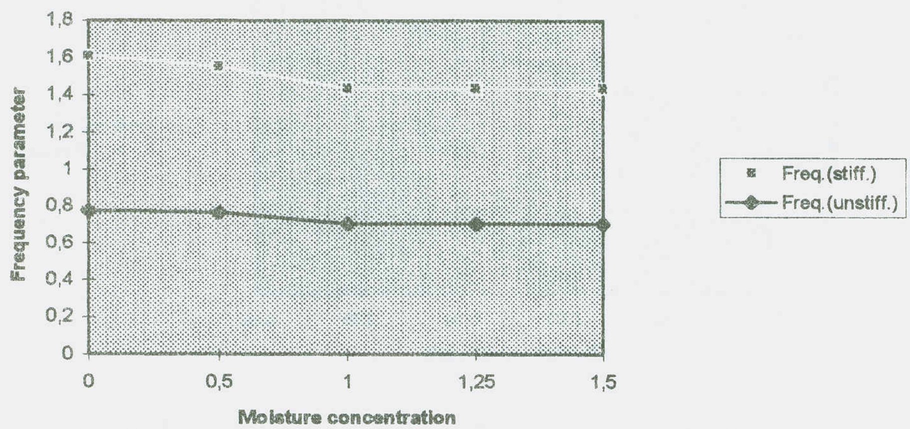

Figure 6 Frequency parameter vs. moisture concentration for the unstiffened and stiffened cases ($\alpha = 45^0$, h=0.002)

Finally, the variation of the frequency parameter with the uniform moisture concentration is plotted in the Figure 6 for the unstiffened and stiffened cones with the semivertex angle of 45 deg. Moisture concentration decreases slightly the frequencies for both cases.

5. CONCLUSION

The moisture effects on the dynamic behavior of a composite laminated conical thin shell reinforced by stringers and rings are studied numerically. The dynamic equations of a stiffened composite laminated conical shell are based on the Donnell-Mushtari theory of elastic thin shells. The rings and/or stringers are "smeared out" along the conical shell. The inhomogeneity of material properties because of moisture and temperature is considered in the constitutive equations. The dynamic equations are solved by the finite difference method to obtain the free vibration characteristics of a composite laminated shell and an agreement is obtained with certain earlier results. Influence of the moisture on the vibration characteristics is examined.

The isotropic stiffeners reduce the fundamental frequencies of the laminated conical shell with all the semivertex angles. Because the mass contribution of the stiffeners is greater than the stiffness contribution of them. Also the effect of moisture on the composite laminated shell is investigated and found that the natural frequency reduces with the increase in uniform moisture concentration.

The present study can be extended to examine the effect of the geometric and material parameters on the dynamic behavior of the laminated conical shell under hygrothermal load. A numerical and parametric study can be performed by accounting the complicating effects such as initial stresses and nonlinearity.

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APPENDIX

The linear differential operators in the eqs. (10) are given as follows

$$L_{11} = -A_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{A_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}}{\bar{s}^2} - \frac{A_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{12} = -\frac{(A_{12}+A_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} + \frac{(A_{22}+A_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{13} = \frac{A_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}}$$

$$L_{14} = -B_{11}\frac{\partial^2}{\partial \bar{s}^2} - \frac{B_{11}}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{B_{22}}{\bar{s}^2} - \frac{B_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2}$$

$$L_{15} = \frac{(B_{22}+B_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} - \frac{(B_{12}+B_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$

$$L_{21} = -\frac{(A_{12}+A_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(A_{22}+A_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{22} = -A_{33}\left[\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{A_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{44}\cot^2\alpha}{\bar{s}^2}$$

$$L_{23} = -\frac{(A_{22}+A_{44})\cot\alpha}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{24} = -\frac{(B_{12}+B_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(B_{22}+B_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{25} = -B_{33}\left[\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right] - \frac{B_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{44}\cot\alpha}{\bar{s}\lambda}$$

$$L_{31} = \frac{A_{12}\cot\alpha}{\bar{s}}\frac{\partial}{\partial \bar{s}} + \frac{A_{22}\cot\alpha}{\bar{s}^2} \qquad L_{32} = -L_{23}$$

$$L_{33} = -A_{55}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}}\right) - \frac{A_{44}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} - \frac{A_{22}\cot^2\alpha}{\bar{s}^2}$$

$$L_{34} = \left(\frac{B_{12}\cot\alpha}{\bar{s}} - \frac{A_{55}}{\lambda}\right)\frac{\partial}{\partial \bar{s}} - \frac{A_{55}}{\lambda\bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{35} = \left(\frac{B_{22}\cot\alpha}{\bar{s}^2} - \frac{A_{44}}{\lambda\bar{s}}\right)\frac{\partial}{\partial \bar{\theta}}$$

$$L_{41} = L_{14} \qquad L_{42} = L_{15}$$

$$L_{43} = \left(\frac{A_{55}}{\lambda} - \frac{B_{12}\cot\alpha}{\bar{s}}\right)\frac{\partial}{\partial \bar{s}} + \frac{B_{22}\cot\alpha}{\bar{s}^2}$$

$$L_{44} = -D_{11}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}}\right) + \frac{D_{22}}{\bar{s}^2} - \frac{D_{33}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{55}}{\lambda^2}$$

$$L_{45} = \frac{(D_{22}+D_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}} - \frac{(D_{12}+D_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}}$$

$$L_{51} = L_{24} \qquad L_{52} = L_{25} \qquad L_{53} = -L_{35}$$

$$L_{54} = -\frac{(D_{12}+D_{33})}{\bar{s}}\frac{\partial^2}{\partial \bar{s}\partial \bar{\theta}} - \frac{(D_{22}+D_{33})}{\bar{s}^2}\frac{\partial}{\partial \bar{\theta}}$$

$$L_{55} = -D_{33}\left(\frac{\partial^2}{\partial \bar{s}^2} + \frac{1}{\bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{1}{\bar{s}^2}\right) - \frac{D_{22}}{\bar{s}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{A_{44}}{\lambda^2}$$